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A GEOMETRIC CHARACTERIZATION OF THE REACHABLE AND THE CONTROLLABLE SUBSPACES OF DESCRIPTOR SYSTEMS*

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Abstract. The concepts of $\{A,E,R(B)\}$ and restricted $\{E,A,R(B)\}$ invariance are introduced. The reachable subspace of a descriptor system is shown to be the supremal $\{A,E,R(B)\}$-invariant subspace contained in the least restricted $\{E,A,R(B)\}$ subspace of R^n. Algorithms to compute the reachable subspace of a descriptor system $E\dot{x}= Ax + Bu$ in terms of E,A and B are given. A new proof of the feedback invariance of the reachable subspace is presented.

1. Introduction

We consider the linear, time-invariant descriptor system

$$D\Sigma : E\dot{x}(t) = Ax(t) + Bu(t) \tag{1.1}$$

where $E,A : C^n \to C^n$ and $B : C^m \to C^n$. We assume that $|\lambda E - A| \neq 0$, i.e., $D\Sigma$ is regular.

It is well known that there exists a basis $\{v_i : i \in \mathbf{n}\}$ for the domain (where $\mathbf{n} = \{1,2, \ldots, n\}$) and a basis $\{w_i : i \in \mathbf{n}\}$ for the codomain of E and A such that in the new coordinates **(1.1)** decomposes into two subsystems (see **[1]**)

$$D\Sigma WF_F : \dot{x}_1(t) = Jx_1(t) + B_1u(t) \tag{1.2a}$$

$$D\Sigma WF_\infty : N\dot{x}_2(t) = x_2(t) + B_2u(t) \tag{1.2b}$$

where $x_1(t) \in C^{n_1}$ and $x_2(t) \in C^{n_2}$. J and N in **(1.2)** are Jordan form matrices and N is nilpotent with index of nilpotency α_1.

It was shown in **[2]** and **[3]** that the trajectory of **(1.2)** starting from an arbitrary initial condition may exhibit impulsive behavior. We shall call a point $x_0 = [x_{01}' \; x_{02}']'$ an *admissible initial condition* for **(1.2)** if there exists an $(\alpha_1 - 1)$ times continuously differentiable input $u_T(t) : [0,\infty) \to C^m$ such that the solution $x(t; 0,x_0,u_T(t))$ is continuously differentiable on $[0,T]$ for

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some $T > 0$. With some minor changes in the analyses of **[3]** and of **[4]**, it was shown in **[5]** that $[x'_{01}\ x'_{02}]'$ is an admissible condition for **(1.2)** if and only if $[x'_{01}\ x'_{02}]' \in C^{n_1} \oplus C_\infty$ where $C_\infty = \Sigma_{i=0}^{\alpha_1 - 1} N^i\, R(B_2) + N(N)$ and $\oplus$ denotes direct sum.

Given a point $x_0 = [x'_{01}\ x'_{02}]'$ in $C^{n_1+n_2}$, we say that $y_0 = [y'_{01}\ y'_{02}]'$ is *reachable* from x_0 if there exists an $(\alpha_1 - 1)$ times continuously differentiable input $u_T(t)$ such that $x(t;\ 0,x_0,u_T(t)) = y_0$ for some $T > 0$. In case the origin is reachable from x_0, x_0 is said to be *controllable*. A slight modification of the analysis of **[3,4]** as carried out in **[5]** shows that any point in $R_F \oplus R_\infty$ is reachable from any other point in $R_F \oplus R_\infty$ where $R_F = \Sigma_{i=0}^{n_1-1} J^i\, R(B_1)$ and $R_\infty = \Sigma_{i=0}^{\alpha_1-1} N^i\, R(B_2)$. It also follows from **[2]** and **[5]** that if $C_F = R_F$ and if C_∞ is as above, then a point $[x'_{01}\ x'_{02}]'$ is controllable if and only if $[x'_{01}\ x'_{02}]' \in C_F \oplus C_\infty$. Consequently, $R_F \oplus R_\infty$ and $C_F \oplus C_\infty$ are called the *reachable* and *controllable subspaces* of **(1.2)**. Equation **(1.2)** is said to be reachable (controllable) if $R_F \oplus R_\infty = C^{n_1+n_2}(C_F \oplus C_\infty = C^{n_1+n_2})$. This definition of reachability is equivalent to the controllability of **[3]**, C-controllability of **[4]**, and the absence of input decoupling zeros (finite or infinite) of $[sE\text{-}A\ B]$ in the sense of **[6]**. On the other hand, controllability, as defined above, is equivalent to modal controllability of **[2]**.

This analysis of reachability and controllability, which is the prevalent approach used in the literature, has a main drawback in that it depends on the decomposition of $D\Sigma$ into two subsystems $D\Sigma WF_F$ and $D\Sigma WF_\infty$. A characterization of the reachable and the controllable subspaces in terms of E,A, and B should be welcome for two main reasons. First of all, the decomposition has a computational cost associated with it. Secondly, decomposing the system into two subsystems (using the Weierstrass decomposition or any other one **[3,7]**) destroys one of the main advantages of the descriptor variable formulation. As it was pointed out in **[6,8]**, the descriptor variable formulation is preferable to the state-space formulation, even when the latter exists, simply because it is "more natural" in the sense of displaying and preserving the information content of the dynamical equations and the physical significance of the variables. It is exactly this advantage which is being subverted by the decomposition of the system into two subsystems.

In Section 3 we circumvent these difficulties by showing that the reachability and the controllability of a descriptor system can be assessed and the reachable and the controllable subspaces can be constructed by using the original system matrices E,A, and B. The results of Section 3 depend on some geometric concepts which are introduced and discussed in Section 2. Section 4 will give an alternative proof of Cobb's result in **[3]** which states that the reachable subspace is feedback invariant.

In what follows, the superscript -1 on a linear operator will denote its pre-image and $\oplus$ will be used to denote the direct sum of subspaces and/or of linear operators.

2. $(A,E,R(B))$ and $(E,A,R(B))_r$ invariance

Let K be a given but otherwise arbitrary subspace of C^n. We define two families of subspaces of K as follows:

$$\mathfrak{F}_{A,E,B}(K) = \{S \subset K : AS \subset ES + R(B)\}$$

$$\mathfrak{F}_{E,A,B}(K) = \{S \subset C^n : S = K \cap E^{-1}(AS + R(B))\}$$

An element S of $\mathfrak{F}_{A,E,B}(K)$ will be said to be an $(A,E,R(B))$ *invariant subspace of* K, and an element S of $\mathfrak{F}_{E,A,B}(K)$ will be said to be a *restricted* $(E,A,R(B))$ *invariant subspace*, or in short, an $(E,A,R(B))_r$ *invariant subspace of* K.

Our definition of $(A,E,R(B))$ invariance is similar to, though not the same as, that of [9]. Compare to [11,12] which used these definitions with $B = 0$.

We first consider $\mathfrak{F}_{A,E,B}(K)$. As it contains the zero vector it is non-empty and if S_1 and S_2 are two elements from $\mathfrak{F}_{A,E,B}(K)$, we have

$$\begin{aligned} A(S_1 + S_2) &= AS_1 + AS_2 \subset ES_1 + R(B) + ES_2 + R(B) \\ &= E(S_1 + S_2) + R(B) \end{aligned}$$

Therefore, $\mathfrak{F}_{A,E,B}(K)$ is closed under the operation of subspace addition. Consequently, it contains a (unique) supremal element. Lemma 2.1 gives an algorithm to compute it.

Lemma 2.1. *Let* $L^*(K)$ *denote the limit of the sequence* $\{L_k\}$ *where*

$$L_{k+1} = K \cap A^{-1}(EL_k + R(B)); \quad L_o = K \tag{2.1}$$

Then $L^*(K) \in \mathfrak{F}_{A,E,B}(K)$. *Furthermore, if* $S \in \mathfrak{F}_{A,E,B}(K)$, *then* $S \subset L^*(K)$.

Proof. First of all, note that $L_1 \subset L_o$. If $L_k \subset L_{k-1}$, then

$$\begin{aligned} L_{k+1} &= K \cap A^{-1}(EL_k + R(B)) \subset K \cap A^{-1}(EL_{k-1} + R(B) \\ &= L_k \end{aligned}$$

Therefore, $\{L_k\}$ is nonincreasing. This fact and the assumption of finite dimensionality imply that $L^*(K)$ exists. Also, $L^*(K) \in \mathfrak{F}_{A,E,B}(K)$ because $L^*(K) \subset K$, and

$$\begin{aligned} AL^*(K) &= A\{K \cap A^{-1}(EL^*(K) + R(B))\} \\ &\subset AK \cap (EL^*(K) + R(B)) \cap R(A) \\ &\subset EL^*(K) + R(B) \end{aligned}$$

To show that $L^*(K)$ is supemal, let $S \in \mathfrak{F}_{A,E,B}(K)$. Then, $S \subset L_0$ and if $S \subset L_k$, then

$$\begin{aligned} L_{k+1} &= K \cap A^{-1}(EL_k + R(B)) \\ &\supset K \cap A^{-1}(ES + R(B)) \\ &\supset K \cap S \\ &= S \end{aligned}$$

As $S \subset L_k$ for all k, clearly $S \subset L^*(K)$. So, $L^*(K)$ is the supremal element of $\mathfrak{F}_{A,E,B}(K)$ and, therefore, it is unique. **Q.E.D.**

In case $K = C^n$, we shall simply write $\mathfrak{F}_{A,E,B}$, $\mathfrak{F}_{E,A,B}$, L^*, etc., rather than $\mathfrak{F}_{A,E,B}(K)$, $\mathfrak{F}_{E,A,B}(K)$, $L^*(K)$, etc.

Since it will not be used in the sequel and since, computationally speaking, it is less attractive than the algorithm given by **(2.1)**, we shall not prove it, but the interested reader can convince himself/herself that if the recursion **(2.1)** is run with $K = C^n$ and with L_0 equal to the supremal $(A,E,0)$ *invariant subspace of* C^n, rather than $L_0 = C^n$, the limit of the subspace sequence still exists and is equal to L^*.

We now consider $\mathfrak{F}_{E,A,B}(K)$. Lemma 2.2 below shows that it is nonempty and has a (unique) infimum.

Lemma 2.2. *Let $S_*(K)$ denote the limit of the sequence $\{S_k\}$ where*

$$S_{k+1} = K \cap E^{-1}(AS_k + R(B)) \; ; \; S_0 = 0 \qquad \textbf{(2.2)}$$

Then $S_*(K) \in \mathfrak{F}_{E,A,B}(K)$, and moreover if $S \in \mathfrak{F}_{E,A,B}(K)$, then $S_*(K) \subset S$.

Proof. Note that $S_0 \subset S_1$. If $S_k \supset S_{k-1}$, then

$$\begin{aligned} S_{k+1} &= K \cap E^{-1}(AS_k + R(B)) \\ &\supset K \cap E^{-1}(AS_{k-1} + R(B)) \\ &= S_k \end{aligned}$$

Therefore, $\{S_k\}$ is nondecreasing. Hence, by the virtue of finite dimensionality, $S_*(K)$ exists. $S_*(K)$ satisfies

$$S_*(K) = K \cap E^{-1}(AS_*(K) + R(B)).$$

So, $S_*(K) \in \mathfrak{F}_{E,A,B}(K)$. Let S be another element of $\mathfrak{F}_{E,A,B}(K)$. Obviously, $S_0 \subset S$, and if $S_k \subset S$, then the argument above shows that $S_{k+1} \subset S$ also. Since $S_k \subset S$ for all k, we have $S_*(K) \subset S$. Moreover, $S^*(K)$ is unique as it is infimal. **Q.E.D.**

The following lemma shows that if K is "large enough," then the initial condition in **(2.2)** can be changed without affecting the limit of the sequence. Let E_∞ be the least $(E,A,0)_r$ invariant subspace of C^n which is given by the solution of the recursion

$$E_\infty^{k+1} = E^{-1}AE_\infty^k; \quad E_\infty^0 = 0 \tag{2.3}$$

Lemma 2.3 *Let* $E_\infty \subset K \subset C^n$. *Define* $\{\tilde{S}_k\}$ by

$$\tilde{S}_{k+1} = K \cap E^{-1}(A\tilde{S}_k + R(B)) \; ; \; \tilde{S}_0 = E_\infty \tag{2.4}$$

Then $lim_k\{\tilde{S}_k\}$ *exists and is equal to* $S_*(K)$.

Proof. Note that $E_\infty = E^{-1} AE_\infty$. Then,

$$\begin{aligned}\tilde{S}_1 &= K \cap E^{-1}(AE_\infty + R(B)) \\ &\supset K \cap E^{-1}AE_\infty \\ &= K \cap E_\infty \\ &= E_\infty \\ &= \tilde{S}_0\end{aligned}$$

If $\tilde{S}_{k-1} \subset \tilde{S}_k$, then it is easy to show that $\tilde{S}_k \subset \tilde{S}_{k+1}$ also. Thus, $\{\tilde{S}_k\}$ is nondecreasing and $lim_k\{\tilde{S}_k\}$ exists. Let $\tilde{S}_*(K)$ denote $lim_k\{\tilde{S}_k\}$.

Let $\{S_k\}$ be given by **(2.2)**. $S_0 \subset \tilde{S}_0$ is clear, and if $S_k \subset \tilde{S}_k$, then

$$\begin{aligned}S_{k+1} &= K \cap E^{-1}(AS_k + R(B)) \\ &\subset K \cap E^{-1}(A\tilde{S}_k + R(B)) \\ &= \tilde{S}_{k+1}\end{aligned}$$

Since $S_k \subset \tilde{S}_k$. for all k, $S_*(K) \subset \tilde{S}_*(K)$. To show the reverse inclusion, let E_∞^k be as in (2.3) and note that $E_\infty^0 \subset S_*(K)$. Assume $E_\infty^k \subset S_*(K)$. Then,

$$\begin{aligned}E_\infty^{k+1} &= E^{-1}AE_\infty^k \\ &\subset E^{-1}(AS_*(K) + R(B)).\end{aligned}$$

Since $E_\infty^{k+1} \subset E_\infty \subset K$, it follows that

$$\begin{aligned}E_\infty^{k+1} &\subset K \cap E^{-1}(AS_*(K) + R(B)) \\ &= S_*(K).\end{aligned}$$

Therefore, $E_\infty = \tilde{S}_0 \subset S_*(K)$, and if $\tilde{S}_k \subset S_*(K)$, then

$$\begin{aligned}\tilde{S}_{k+1} &= K \cap E^{-1}(A\tilde{S}_k + R(B))\\ &\subset K \cap E^{-1}(AS_*(K) + R(B))\\ &= S^*(K).\end{aligned}$$

This proves that $\tilde{S}_k \subset S_*(K)$ for all k. Consequently, $\tilde{S}_*(K) \subset S_*(K)$ also. Thus, $\tilde{S}_*(K) = S_*(K)$ and the proof is complete. **Q.E.D.**

3. Constructing the reachable and the controllable subspaces

We consider $D\Sigma$ given by **(1.1)**. In the light of our discussion in Section 1, we let $\boldsymbol{R}$ denote the subspace of C^n with the property that any $x \in \boldsymbol{R}$ can be reached from any other $y \in \boldsymbol{R}$. $\boldsymbol{R}$ is the reachable subspace of $D\Sigma$. Similarly, we let $\boldsymbol{C}$ denote the subspace of C^n with the property that the origin is reachable from any point x in $\boldsymbol{C}$. $\boldsymbol{C}$ is the controllable subspace of $D\Sigma$. The subspace of admissible initial conditions will be denoted by $\boldsymbol{M}$.

We can now present the following lemma which gives a system theoretic interpretation of $\boldsymbol{L}^*$, the supremal $\{A,E,\boldsymbol{R}(B)\}$ invariant subspace of C^n.

Lemma 3.1. $\boldsymbol{L}^* + N(E) = \boldsymbol{M}$.

Proof. Let V and W be the matrices whose columns form a basis for the domain and the codomain of E respectively, so that

$$W^{-1}EV = I_{n_1} \oplus N: = \bar{E},$$
$$W^{-1}AV = J \oplus I_{n_2}: = \bar{A}$$

and

$$W^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}: = \bar{B}.$$

The sequence $\{L_k\}$ in **(2.1)**, which is defined in the domain of E, becomes

$$L_{k+1} = (W\bar{A}V^{-1})^{-1}\{(W\bar{E}V^{-1})L_k + W\boldsymbol{R}(\bar{B})\} \; ; L_0 = C^n$$

It is trivial to show that $(W\bar{A}V^{-1})^{-1} = V(\bar{A})^{-1}W^{-1}$ (where, as before, the superscript -1 denotes the pre-image of an operator). Letting $\tilde{L}_k = V^{-1}L_k$, we get

$$\tilde{L}_{k+1} = (\bar{A})^{-1}\{\bar{E}\tilde{L}_k + \boldsymbol{R}(\bar{B})\} \; ; \tilde{L}_0 = C^n$$

Now, note that

$$\tilde{L}_1 = (J \oplus I_{n_2})^{-1}\{(I_{n_1} \oplus N)C^n + R\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\}$$
$$= C^{n_1} \oplus (R(N) + R(B_2))$$

Assuming that

$$\tilde{L}_k = C^{n_1} \oplus (R(N^k) + \sum_{i=0}^{k-1} R(N^i B_2))$$

we have

$$\tilde{L}_{k+1} = (J \oplus I_{n_2})^{-1}\{C^{n_1} \oplus R(N^{k+1}) + \sum_{i=1}^{k} R(N^i B_2) + R\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\}$$
$$= C^{n_1} \oplus (R(N^{k+1}) + \sum_{i=0}^{k} R(N^i B_2))$$

Since $N^{\alpha_1} = 0$ for some $\alpha_1 < n_2$, and since $R(N^k) \leq R(N^{k-1})$ for any $k > \alpha_1 - 1$,

$$\tilde{L}^* := lim_k\{\tilde{L}_k\} = C^{n_1} \oplus \sum_{i=0}^{\alpha_1 - 1} R(N^i B_2))$$
$$= C^{n_1} \oplus R_\infty \tag{3.1}$$

Furthermore, $V^{-1}N(E) = V^{-1}N(W\bar{E}V^{-1}) = N(\bar{E}) = 0 \oplus N(N)$ and the proof follows from the fact that the set of admissible conditions for $D\Sigma WF$ is $C^{n_1} \oplus (R_\infty + N(N))$. **Q.E.D.**

It is by construction that L^* satisfies $L^* = A^{-1}(EL^* + R(B))$. Some other properties of L^* are given by lemma 3.2. The proofs are immediate from the characterization **(3.1)** of $V^{-1}L^*$ and will be omitted.

Lemma 3.2.

(1) $AL^* \subset L^*$
(2) $EL^* \subset L^*$
(3) $L^* = EL^* + R(B)$
(4) $L^* = EL^* + AL^*$.

We can now prove our main result which provides us with not only geometrical insight into DΣ but also with a way (indeed, to our best knowledge, the only way) to construct the reachable subspace R using **(1.1)** rather than **(1.2)** or another decomposition of **(1.1)**.

Theorem 3.1.

(1) $R = L^* \cap S_*$
(2) $R = L^*(S_*)$.

Proof. We first prove **(1)**. Let $\{S_k\}$ be as defined in **(2.4)**. Let $V, W, J, N, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ be as in the proof of Lemma 3.1. Then, letting $\tilde{S}_k = V^{-1}S_k$ and noting that in the bases used in Lemma 3.1, $E_\infty = 0 \oplus C^{n_2}$ [4],

$$\tilde{S}_{k+1} = (I_{n_1} \oplus N)^{-1}\{(J \oplus I_{n_1})\tilde{S}_k + R\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\}; \tilde{S}_0 = 0 \oplus C^{n_2}.$$

Note that

$$\begin{aligned}\tilde{S}_1 &= (I_{n_1} \oplus N)^{-1}\{(J \oplus I_{n_2})(0 \oplus C^{n_2}) + R\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\} \\ &= R(B_1) \oplus C^{n_2}\end{aligned}$$

An induction argument similar to the one in the proof of Lemma 3.1 yields

$$\tilde{S}_k = \sum_{i=0}^{k-1} R(J^i B_1) + C^{n_2} \tag{3.2}$$

Then $\lim_k\{\tilde{S}_k\} = R_F \oplus C^{n_2}$. Furthermore, Lemma 2.3 shows that this limit is $V^{-1}S_*$. Then, by **(3.1)**, $V^{-1}L^* \cap V^{-1}S^* = R_F \oplus R_\infty$ which completes the proof of part (1).

Part (2) states that R is the supremal $\{A,E,R(B)\}$ invariant subspace of the least $\{E,A,R(B)\}_r$ C^n. Then it is constructed by the recursion **(2.1)** with $\tilde{K}$ replaced by S_*. Again, letting $\tilde{L}_k = V^{-1}L_k$ where L_k is as in **(2.1)** and letting $\tilde{L}_0 = V^{-1}S_* = R_F \oplus C^{n_2}$, we have

$$\tilde{L}_1 = (R_F \oplus C^{n_2}) \cap (J \oplus I_{n_2})^{-1}\{(I_{n_1} \oplus N)(R_F \oplus C^{n_2}) + R\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\}$$

Noticing that $R_F + R(B_1) = R_F$ and $R_F \subset J^{-1}R_F$, we have

$$\tilde{L}_1 = R_F \oplus (R(N) + R(B_2))$$

Using induction, we conclude that

$$\tilde{L}_k = R_F \oplus (R(N^k) + \sum_{i=0}^{k-1} R(N^i B_2))$$

which, as $N^{\alpha_1} = 0$ for some $\alpha_1 \leq n_2$, implies $\lim_k\{\tilde{L}_k\} = R_F \oplus R_\infty$. Hence the proof. **Q.E.D.**

Remark. In all the examples that we have worked out, the supremal $\{A,E,R(B)\}$ invariant subspace contained in the least $\{E,A,R(B)\}_r$ invariant subspace of C^n turns out to be the same as the least $\{E,A,R(B)\}_r$ invariant subspace contained in the supremal (A,E,B) invariant subspace of C^n. So, we state the following conjecture which remains to be proved (or refuted): $L^*(S_*) = S_*(L^*)$.

Corollary 3.1. *If either (1) or (2) below does not hold true then $D\Sigma$ is not reachable.*

(1) $R(A) \subset R(E) + R(B)$
(2) $R(E) \subset R(A) + R(B)$

Proof. If $R = C^n$, then by (1) of Theorem 3.1 $L^* = S_* = C^n$. Then

$A^{-1}(EC^n + R(B)) = C^n$ implies $R(A) \subset R(E) + R(B)$ and $E^{-1}(AC^n + R(B)) = C^n$ implies $R(E) \subset R(A) + R(B)$. **Q.E.D.**

S_*, the least $\{E,A,R(B)\}_r$ invariant subspace of C^n was seen to be crucial in the construction of R. Lemma 3.3 below summarizes some of the properties enjoyed by S_*. Its proof is trivial using **(3.2)** (with k replaced by n_1) and will be left to the reader.

Lemma 3.3.
(1) $ES_* \subset S_*$
(2) $AS_* \subset S_*$
(3) $S_* = AS_* + R(B)$
(4) $S_* = ES_* + AS_*$.

The following theorem on the computation of the controllable subspace is included for the sake of completeness. Its proof is imminent from the proof of Theorem 3.1 and the fact that $C_\infty = R_\infty + N(N)$.

Theorem 3.2.
(1) $C = S_* \cap (L^* + N(E))$
(2) $C = L^*(S_*) + N(E)$.

Corollary 3.2. *If* $D\Sigma$ *is controllable, then*
(1) $R(E) \subset R(A) + R(B)$, *and*
(2) $R(A) \subset R(E) + R(B) + AN(E)$.

Proof. It follows from the proof of Theorem 3.1 that if $D\Sigma$ is controllable, then $S_* = C^n$, from which **(1)** follows immediately, and $L^* + N(E) = C^n$. Then,

$$\begin{aligned} R(A) &= A(L^* + N(E)) \\ &= AL^* + AN(E) \\ &= AA^{-1}(EL^* + R(B)) + AN(E) \\ &\subset EL^* + R(B) + AN(E) \\ &= E(L^* + N(E)) + R(B) + AN(E) \\ &= R(E) + R(B) + AN(E). \end{aligned}$$

Q.E.D.

1. Feedback invariance of L^* and S_*

Let a feedback $u(t) = Fx(t) + v(t)$ be applied to $D\Sigma$ to yield the closed-loop system

$$CLD\Sigma : E\dot{x}(t) = (A + BF)x(t) + Bv(t)$$

Let the subscript/superscript CL denote closed-loop quantities. We first show that S_* is invariant under the feedback given above.

Lemma 4.1. $S_*^{CL} = S_*$.

Proof. Since S_*^{CL} is the least $\{E,(A + BF),R(B)\}_r$ invariant subspace of C^n, it is given (see Lemma 2.2) by the limit of $\{S_k^{CL}\}$ where

$$S_{k+1}^{CL} = E^{-1}((A + BF)S_k^{CL} + R(B)) \; ; \; S_0^{CL} = 0$$

Since $(A + BF)S_k^{CL} + R(B) = AS_k^{CL} + R(B)$, we have $S_k^{CL} = S_k$ where S_k is as defined in Lemma 2.2 Therefore, $S_*^{CL} = \lim_k\{S_k^{CL}\} = \lim_k\{S_k\} = S_*$.
Q.E.D.

To show $L_{CL}^* = L^*$ we need the following lemma which is given in [9] for the case $E = I$. However, replacing I by E does not affect the lemma or its proof.

Lemma 4.2. *A subspace S belongs to* $\mathfrak{F}_{A,E,B}$ *if and only if there exists a map* $F : C^n \to C^m$ *such that* $(A + BF)S \subset ES$.

We now prove the following lemma which states that the set of admissible initial conditions of $D\Sigma$, i.e., $L^* + N(E)$, is invariant under the feedback given above.

Lemma 4.3. $L_{CL}^* = L^*$.

Proof. We first show that $L^* \subset L_{CL}^*$. Note that, by Lemma 3.2,

$$\begin{aligned}(A + BF)L^* &\subset AL^* + R(B)\\ &\subset EL^* + R(B)\end{aligned}$$

Therefore, L^* is $((A + BF), E, R(B))$ invariant and since L_{CL}^* is the supremal $((A + BF), E, R(B))$ invariant subspace, we have $L^* \subset L_{CL}^*$.

By Lemma 2.1, L_{CL}^* is given by $\lim_k\{L_{CL}^k\}$ where

$$L_{CL}^{k+1} = (A + BF)^{-1}\{EL_{CL}^k + R(B)\} \; ; \; L_{CL}^0 = C^n$$

Then L_{CL}^* satisfies $(A + BF)L_{CL}^* \subset EL_{CL}^* + R(B)$. By Lemma 4.2, there exists a map $\tilde{F} : C^n \to C^m$ such that $(A + BF + B\tilde{F})L_{CL}^* \subset EL_{CL}^*$, i.e., $(A + B(F + \tilde{F}))L_{CL}^* \subset EL_{CL}^*$, which, again by Lemma 4.2, implies that $AL_{CL}^* \subset EL_{CL}^* + R(B)$. That is, $L_{CL}^* \in \mathfrak{F}_{A,E,R(B)}$ and since L^* is the supremal element of $\mathfrak{F}_{A,E,R(B)}$, $L_{CL}^* \subset L^*$. Together with $L^* \subset L_{CL}^*$, this implies that $L^* = L_{CL}^*$.
Q.E.D.

Since, by Theorem 3.1, $R_{CL} = L_{CL}^* \cap S_*^{CL}$, Lemmas 4.1 and 4.3 already prove the following theorem.

Theorem 4.1. $R_{CL} = R$.

Remarks. (1) It is evident that $N(E)$ is not affected by the feedback. So, we have $C_{CL} = (L_{CL}^* + N(E)) \cap S_*^{CL} = (L^* + N(E)) \cap S_* = C$. (2) Although in practice it is required, for the uniqueness of the solutions of $CLD\Sigma$, to guarantee that F preserves regularity, (that is, $|\lambda E - (A + BF)| \neq 0$) the results in this section are valid for any F.

5. Conclusions

The concepts of $\{A,E,R(B)\}$ and restricted $\{E,A,R(B)\}$ invariance were introduced and were shown to be instrumental in manufacturing the reachable and controllable subspaces of descriptor systems.

Aside from the obvious advantage of bypassing the decomposition of the system into subsystems, the geometric analysis of the reachable and controllable subspaces above yields an alternative proof of the feedback invariance of reachability and controllability in descriptor systems.

It is our hope that the results of this paper will be helpful in an attempt to generalize Wonham's reachability subspaces [10] to descriptor systems.

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