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Singular Value Analysis of Deformable Systems*

Edmond A. Jonckheere[1]
and Leonard M. Silverman[1]

Abstract. Singular value analysis, balancing, and approximation of a class of deformable systems are investigated. The deformable systems considered herein include several important cases of flexible aerospace vehicles and are characterized by countably infinitely many poles and zeros on the imaginary axis. The analysis relies completely on the so-called asymptotic singular value decompositon of the Hankel operator associated with the impulse response of the system. A parametric study of a six-dimensional single-input single-output case is performed.

1. Introduction

This paper addresses singular value analysis, balancing, and approximation [16]-[25] of a class of single-input single-output transfer functions having the general form

$$h^0(s) = \frac{1}{s^2} + \sum_{k=1}^{\infty} \frac{c_k}{s^2 + w_k^2}. \tag{1}$$

To take care of several convergence issues, we shall assume throughout the paper that

$$0 < w_1 < \dots < w_k < w_{k+1} < \dots , \tag{2a}$$

$$\sum_{k=1}^{\infty} |c_k| < \infty. \tag{2b}$$

This indeed guarantees that the formal inverse Laplace transform of $h^0(s)$,

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[1] Department of Electrical Engineering – Systems, University of Southern California, Los Angeles, California 90007, USA.

that is, $h^0(t) = t + \sum_{k=1}^{\infty} \frac{c_k}{w_k} \sin w_k t$ exists uniformly, is continuous and differentiable over $[0,\infty)$.

Transfer functions of the type (1), consisting of a double integrator and countably infinitely many oscillators interconnected in a parallel structure as depicted by Figure 1, are common in the description of the linearized dynamics of flexible aerospace vehicles [1]-[15]. The c_k's are real constants, called *modal constants* or *modal gains*; the w_k's are the so-called *global eigenfrequencies.*

The particular example that one will refer to is the case of the linearized roll dynamics of a satellite consisting of a main rigid body and two flexible panels undergoing skew-symmetric bending [12], [13]. In this particular case, the input u is the torque acting on the roll axis divided by the overall roll moment of inertia, and the output y is the roll angle of the main rigid body. The linearized dynamical equations take the form of a second order differential equation, accounting for the dynamics of the main rigid body, and a partial differential equation, accounting for the dynamics of the deformation of the panels, intercoupled via the angular acceleration of the main rigid body and the deformation acceleration of the panels. Eigenmode analysis then yields countably infinitely many differential equations. Finally, taking Laplace transform yields a transfer function of the type (1). For this particular case, it is noteworthy that $c_k > 0$, $\forall k$; further, $c_k \downarrow 0$ and $w_k \uparrow \infty$, in a strict monotone sense as $k \uparrow \infty$; finally, Condition (2b) is satisfied [7].

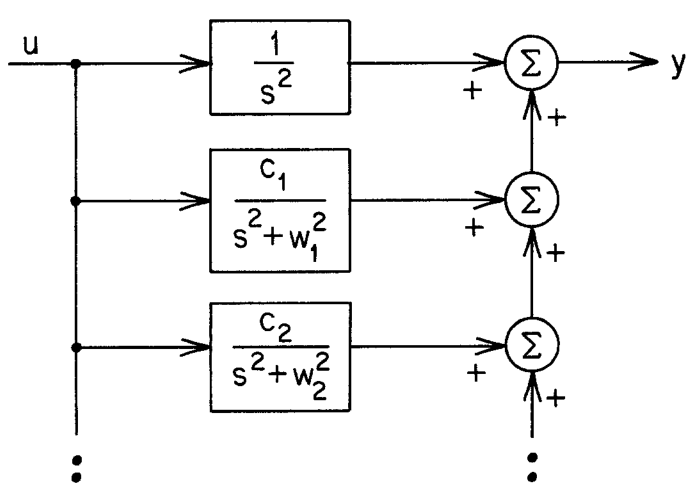

Figure 1. Parallel connection structure of a transfer function of Type (1).

This paper provides a brief case study of applying singular value analysis, balancing, and related approximation procedures [16]-[25] to systems having transfer functions of Type (1). A deeper aim of the paper is to investigate what the rather theoretical machinery of [16]-[25] becomes in the physical world.

An outline of the paper follows. In Section 2, we introduce the so-called *asymptotic singular value decomposition* of the Hankel operator associated with the impulse response of the class of systems considered. This concept, which is the key to the entire paper, provides a nice physical interpretation of the singular values of a deformable system. In Section 3, asymptotic singular value decomposition is used to derive approximations of the transfer function (1) and allows us to prove that several approximations are equivalent in some asymptotic sense. Section 4 provides a parametric study of singular value analysis, balancing, and approximation of a transfer function of Type (1), where only two oscillators have been retained. Section 5 is the conclusion.

2. Asymptotic singular value analysis

Singular value analysis of linear systems is, so far, restricted to *asymptotically stable* systems [17], [18], [21]-[25], so that it is not directly applicable to transfer functions of Type (1). It is, however, simple to circumvent this difficulty. Indeed, deformable systems like flexible aerospace vehicles exhibit the so-called *structural damping*. From a purely technological point of view, structural damping is not easy to assess because very small; here, we choose a more mathematical avenue of approach by shifting all of the poles of $h^0(s)$ to the left by a quantity ϵ. This yields the "damped" transfer function

$$h^\epsilon(s) = \frac{1}{(s+\epsilon)^2} + \sum_{k=1}^{\infty} \frac{c_k}{s_2 + 2\epsilon s + w_k^2 + \epsilon^2}. \qquad (3)$$

Generally, the superscript ϵ will denote the quantity resulting from this pole shift. In this paper, we shall rather perform singular value analysis of $h^\epsilon(s)$, as $\epsilon \downarrow 0$.

Still a difficulty with (3) is that it is infinite dimensional. There are several ways of introducing singular values of a transfer function [25], and some of them are conveying an underlying finite dimensionality assumption. However, singular values of a transfer function can be equally defined as the singular values [19], [26], [27] of the Hankel operator associated with the impulse response $h^\epsilon(t)$ of the system [18], [19], [25], and this definition can be carried over in the infinite dimensional situation, with some additional care.

To be more specific, it is easily seen from (3) that the impulse response is

$$h^\epsilon(t) = te^{-\epsilon t} + \sum_{k=1}^{\infty} \frac{c_k}{w_k} e^{-\epsilon t} \sin w_k t = e^{-\epsilon t} h^0(t), \qquad (4)$$

which is readily checked to exist uniformly and to be continuous and differentiable over $[0,\infty)$. Then the *Hankel operator* $\mathcal{H}^\epsilon$ associated with the impulse response $h^\epsilon(t)$ is, by definition,

$$\mathcal{H}^\epsilon : L^2[0,\infty) \to L^2[0,\infty), \tag{5a}$$

$$u \mapsto \mathcal{H}^\epsilon u = \int_0^\infty h^\epsilon(t+\tau)u(\tau)d\tau. \tag{5b}$$

Lemma 1. With Condition (2), and for any $\epsilon > 0$, the operator $\mathcal{H}^\epsilon$ is *self-adjoint and Hilbert-Schmidt.* Moreover, $\mathcal{H}^{\epsilon *}\mathcal{H}^\epsilon = \mathcal{H}^\epsilon\mathcal{H}^{\epsilon *}$ is *self-adjoint, positive semi-definite, and Hilbert-Schmidt.*

Proof. Self-adjointness of $\mathcal{H}^\epsilon$ is clear from (5). Further, if Condition (2) is verified,

$$f(t) \triangleq \sum_{k=1}^{\infty} \frac{c_k}{w_k} \sin w_k t$$

exists as an *almost periodic* function [28, VI.101]. Further, $h^\epsilon(t) = [t+f(t)]e^{-\epsilon t}$. It is then readily verified that

$$\int_0^\infty \int_0^\infty (h^\epsilon(t+\tau))^2 dt d\tau$$

exists and is finite. Hence $\mathcal{H}^\epsilon$ is Hilbert-Schmidt; see Kato [29, V.2.4].

Finally, the operator $\mathcal{H}^{\epsilon *}\mathcal{H}^\epsilon = \mathcal{H}^\epsilon\mathcal{H}^{\epsilon *} = (\mathcal{H}^\epsilon)^2$ is obviously self-adjoint and positive semi-definite; furthermore, it is also Hilbert-Schmidt, as the product of two Hilbert-Schmidt operators; see Kato [29, V.2.4]. The proof is completed.

$\mathcal{H}^{\epsilon 2}$ is Hilbert-Schmidt and hence compact. Since it is also self-adjoint and positive semi-definite, it has a countably infinite set $\{\sigma_n^2 : n = 1, 2, \ldots\}$ of (repeated) eigenvalues classified in decreasing order, i.e., $0 \leq \ldots \leq \sigma_{n+1} \leq \sigma_n \leq \ldots, \forall n$. Further, by virtue of the Hilbert-Schmidt property of $\mathcal{H}^{\epsilon 2}$, we have

$$\sum_{n=1}^{\infty} \sigma_n^2 < \infty; \tag{6}$$

see Kato [29, V.2.4]. Let $u_n \in L^2[0,\infty)$ be a normalized eigenvector of $\mathcal{H}^{\epsilon 2}$ corresponding to σ_n, i.e.,

$$\mathcal{H}^{\epsilon 2}u_n = \sigma_n^2 u_n, \qquad \|u_n\| = 1. \tag{7a}$$

It follows that $\mathcal{H}^{\epsilon 2}(\mathcal{H}^\epsilon u_n) = \sigma_n^2(\mathcal{H}^\epsilon u_n)$. If $\sigma_n \neq 0$, let $\mathcal{H}^\epsilon u_n = \|\mathcal{H}^\epsilon u_n\| v_n$, with $v_n \in L^2[0,\infty)$, $\|v_n\| = 1$. We have $\|\mathcal{H}^\epsilon u_n\| = ((\mathcal{H}^\epsilon u_n, \mathcal{H}^\epsilon u_n))^{1/2} =$

$((u_n, \mathcal{H}^{\epsilon 2} u_n))^{1/2} = (u_n, \sigma_n^2 u_n)^{1/2} = \sigma_n$. Hence

$$\mathcal{H}^{\epsilon 2} v_n = \sigma_n^2 v_n, \quad \|v_n\| = 1, \tag{7b}$$

and

$$\mathcal{H}^{\epsilon} u_n = \sigma_n v_n, \quad \|u_n\| = \|v_n\| = 1. \tag{8a}$$

From the above, it follows that $\mathcal{H}^{\epsilon 2} u_n = \sigma_n \mathcal{H}^{\epsilon} v_n$. But $\mathcal{H}^{\epsilon 2} u_n = \sigma_n^2 u_n$. Hence

$$\mathcal{H}^{\epsilon} v_n = \sigma_n u_n. \tag{8b}$$

The singular functions v_n associated with $\sigma_n = 0$ are defined as those functions in the kernel of $\mathcal{H}^{\epsilon 2}$ completing the previous singular functions into an orthonormal basis of $L^2[0,\infty)$.

The σ_n's are called the *singular values* of the operator $\mathcal{H}^{\epsilon}$, see Kato [29, V.2.3]. Equations (8) (or (7)) define a so-called *Schmidt pair* (u_n, v_n) for $\mathcal{H}^{\epsilon}$ corresponding to σ_n, see [19]. Obviously, $\{u_n : n = 1, 2, \ldots\}$ and $\{v_n : n = 1, 2, \ldots\}$ are orthonormal bases for $L^2[0,\infty)$. Finally, $\mathcal{H}^{\epsilon}$ admits the so-called *singular value decomposition*

$$\mathcal{H}^{\epsilon} = \sum_{n=1}^{\infty} \sigma_n u_n (v_n, \bullet), \tag{9a}$$

or

$$h^{\epsilon}(t+\tau) = \sum_{n=1}^{\infty} \sigma_n u_n(t) v_n(\tau); \tag{9b}$$

see Kato [29, V.2.3]; from (6), it follows that the convergence of the right-hand side of (9a) is *uniform*; from (8), it follows that the left-hand side and the right-hand side of (9) coincide on the dense subset generated by $\{v_n : n = 1, 2, \ldots\}$, so that the two members in (9) coincide everywhere and are hence equal.

It is no easy task to derive the *exact* singular value decomposition of a system whose impulse response has the form (4). However, an approximate, more precisely, *an asymptotic as* $\epsilon \downarrow 0$, singular value decomposition is easily derived. By direct calculation, one finds that

$$h^{\epsilon}(t+\tau) = \sum_{n=1}^{\infty} \sigma_{a,n} u_{a,n}(t) v_{a,n}(\tau), \tag{10a}$$

where

$$\sigma_{a,1} = \frac{\sqrt{3+2\sqrt{2}}}{4\epsilon^2}, \tag{10b}$$

$$u_{a,1}(t) = \frac{\sqrt{2\epsilon}}{\sqrt{2+2\sqrt{2}}}(1+\sqrt{2}\epsilon t)e^{-\epsilon}t, \tag{10c}$$

$$v_{a,1}(\tau) = \frac{\sqrt{2\epsilon}}{\sqrt{2+2\sqrt{2}}}(1+\sqrt{2}\epsilon\tau)\gamma^{-\epsilon\tau}, \tag{10d}$$

$$\sigma_{a,2} = \frac{\sqrt{3-2\sqrt{2}}}{4\epsilon^2}, \tag{10e}$$

$$u_{a,2}(t) = \frac{\sqrt{2\epsilon}}{\sqrt{2-\sqrt{2}}}(-1+\sqrt{2}\epsilon t)e^{-\epsilon\tau}, \tag{10f}$$

$$v_{a,2}(\tau) = \frac{\sqrt{2\epsilon}}{\sqrt{2-\sqrt{2}}}(1-\sqrt{2}\epsilon\tau)\gamma^{-\epsilon\tau}, \tag{10g}$$

$$\sigma_{a,2k+1} = \sigma_{a,2k+2} = \frac{|c_k|}{4\epsilon w_k}, \quad k = 1, 2, \ldots, \tag{10h}$$

$$u_{a,2k+1}(t) = 2\sqrt{\epsilon}\ \text{sign}\ (c_k)e^{-\epsilon t}\cos w_k t, \quad k = 1, 2, \ldots, \tag{10i}$$

$$v_{a,2k+1}(\tau) = 2\sqrt{\epsilon}\ e^{-\epsilon\tau}\sin w_k\tau, \quad k = 1, 2, \ldots, \tag{10j}$$

$$u_{a,2k+2}(t) = 2\sqrt{\epsilon}\ \text{sign}(c_k)e^{-\epsilon t}\sin w_k t, \quad k = 1, 2, \ldots, \tag{10k}$$

$$v_{a,2k+2}(\tau) = 2\sqrt{\epsilon}\ e^{-\epsilon\tau}\cos w_k\tau, \quad k = 1, 2, \ldots; \tag{10ℓ}$$

further, the above is an *asymptotic* singular value decomposition, in the sense that the orthonormality relationships are not verified exactly, but only *asymptotically as* $\epsilon \downarrow 0$, i.e,

$$\int_0^\infty |u_{a,n}(t)|^2 dt \to 1, \text{ as } \epsilon\downarrow 0, \forall n, \tag{10m}$$

$$\int_0^\infty |v_{a,n}(\tau)|^2 d\tau \to 1, \text{ as } \epsilon\downarrow 0, \forall n, \tag{10n}$$

$$\int_0^\infty u_{a,m}(t)u_{a,n}(t)dt \to 0, \text{ as } \epsilon\downarrow 0, \forall\ m \neq n, \tag{10o}$$

$$\int_0^\infty v_{a,m}(\tau)v_{a,n}(\tau)d\tau \to 0, \text{ as } \epsilon\downarrow 0, \forall\ m \neq n. \tag{10p}$$

The asymptotic singular value decomposition is useful, because it is easily derived in terms of the physical parameters of the system, and also because it gives a rather precise idea as to what the *exact* singular value decomposition is, as asserted by the following theorem.

Theorem 1. Let Condition (2) be verified; then

$$\frac{\sigma_n}{\sigma_{a,n}} \to 1, \text{ as } \epsilon \downarrow 0, \ \forall n, \tag{11a}$$

$$\|u_n - u_{a,n}\| \to 0, \text{ as } \epsilon \downarrow 0, \ \forall n, \tag{11b}$$

$$\|v_n - v_{a,n}\| \to 0, \text{ as } \epsilon \downarrow 0, \ \forall n, \tag{11c}$$

Proof. This is most easily proved by showing that $\sigma^2_{a,n}$, and $u_{a,n}$ and $v_{a,n}$ are "approximate" eigenvalue and eigenvectors of the self-adjoint, positive semi-definite operator $\mathcal{H}^{\epsilon 2}$, whose kernel is given by

$$H(t,\tau) = \int_0^\infty h^\epsilon(t+x)h^\epsilon(x+\tau)dx.$$

After some long – but elementary – calculation, the above and (4) yield

$$\begin{aligned} H(t,\tau) &= e^{-\epsilon(t+\tau)}\left(\frac{t\tau}{2\epsilon} + \frac{1}{4\epsilon^3} + \frac{t+\tau}{4\epsilon^2}\right) \\ &+ \sum_{k=1}^{\infty} \frac{c_k^2}{4\epsilon w_k^2}\, e^{-\epsilon(t+\tau)} \cos w_k(t-\tau) \\ &+ \sum_{k=1}^{\infty} \frac{c_k}{4\epsilon^2 + w_k^2}\, e^{-\epsilon(t+\tau)}(t\cos w_k\tau + \tau\cos w_k t) + \delta(t,\tau), \end{aligned} \tag{12}$$

with

$$\lim_{\epsilon\downarrow 0} \epsilon^2 \|\delta\|_{HS} = 0,$$

where $\|\delta\|^2_{HS} = \int_0^\infty \int_0^\infty |\delta(t,\tau)|^2 dt d\tau$, that is, the *Hilbert-Schmidt* norm; see Kato [29, V.2.4].

For $n = 1, 2$, (12) and (10) yield

$$\int_0^\infty H(t,\tau)u_{a,n}(\tau)d\tau = \sigma^2_{a,n}u_{a,n}(t) + \alpha_n(t), \quad n = 1, 2, \tag{13a}$$

$$\int_0^\infty H(t,\tau)v_{a,n}(\tau)d\tau = \sigma^2_{a,n}v_{a,n}(t) + \beta_n(t), \quad n = 1, 2, \tag{13b}$$

where α_n and β_n are in $L^2[0,\infty)$ and such that

$$\lim_{\epsilon\downarrow 0} \epsilon^4 \|\alpha_n\| = 0, \quad n = 1, 2, \tag{13c}$$

$$\lim_{\epsilon\downarrow 0} \epsilon^4 \|\beta_n\| = 0, \quad n = 1, 2, \tag{13d}$$

Hence (11) is verified for $n = 1, 2$.

For $n = 2k+1$ ($k = 1, 2, \ldots$), the problem is a little more difficult. Observe that

$$\int_0^\infty H(t,\tau) u_{a,n}(\tau) d\tau = \sigma_{a,n}^2 u_{a,n}(t) + \alpha_n'(t) + $$

$$+ te^{-\epsilon t} \frac{1}{2\sqrt{\epsilon}} \frac{c_k}{4\epsilon^2 + w_k^2} + 2\sqrt{\epsilon}\, e^{-\epsilon t} \frac{12\epsilon^2 + w_k^2}{4\epsilon^2(4\epsilon^2 + w_k^2)^2}, \tag{14a}$$

$$\int_0^\infty H(t,\tau) v_{a,n}(\tau) d\tau = \sigma_{a,n}^2 v_{a,n}(t) + \beta_n'(t) + \frac{\sqrt{\epsilon}\, e^{-\epsilon t} w_k (t+1)}{2\epsilon^2(4\epsilon^2 + w_k^2)}, \tag{14b}$$

where α_n' and β_n' are in $L^2[0,\infty)$ and such that

$$\lim_{\epsilon\downarrow 0} \epsilon^2 \|\alpha_n'\| = 0, \quad n = 2k+1, \quad k = 1, 2, \tag{14c}$$

$$\lim_{\epsilon\downarrow 0} \epsilon^2 \|\beta_n'\| = 0, \quad n = 2k+1, \quad k = 1, 2, \tag{14d}$$

Let $\mathcal{H}_0$ be the operator whose kernel is given by

$$H_0(t,\tau) = e^{-\epsilon(t+\tau)} \left(\frac{t\tau}{2\epsilon} + \frac{1}{4\epsilon^3} + \frac{t+\tau}{4\epsilon^2} \right).$$

It is easily seen that $\dfrac{\sqrt{2\epsilon}}{\sqrt{2 \pm 2\sqrt{2}}} (1 \pm \sqrt{2}\epsilon t) e^{-\epsilon\tau}$ are *exact* normalized eigenvectors of $\mathcal{H}_0$ corresponding to the *exact* eigenvalues $\dfrac{3 \pm 2\sqrt{2}}{16\epsilon^4}$. Hence, if we define

$$f_1(t) \triangleq -\mathcal{H}_0^{-1} \left(2\sqrt{\epsilon}\, e^{-\epsilon t} \frac{12\epsilon^2 + w_1^2}{4\epsilon^2(4\epsilon^2 + w_1^2)^2} + te^{-\epsilon t} \frac{1}{2\sqrt{\epsilon}} \frac{c_k}{4\epsilon^2 + w_k^2} \right), \tag{15a}$$

$$f_2(t) \triangleq -\mathcal{H}_0^{-1} \left\{ \frac{\sqrt{\epsilon}\, e^{-\epsilon t} w_k (t+1)}{2\epsilon^2(4\epsilon^2 + w_k^2)} \right\}, \tag{15b}$$

we have

$$\lim_{\epsilon\downarrow 0} \frac{1}{\epsilon^2} \|f_1\| < \infty, \tag{15c}$$

$$\lim_{\epsilon\downarrow 0} \frac{1}{\epsilon} \|f_1\| < \infty, \tag{15d}$$

Combining (14) and (15) yields

$$\int_0^\infty H(t,\tau)[u_{a,n}(\tau)+f_1(\tau)]d\tau = \sigma_{a,n}^2[u_{a,n}(t)+f_1(t)]+\alpha_n(t),$$
$$n = 2k+1, \quad k=1, 2,\ldots, \tag{16a}$$

$$\int_0^\infty H(t,\tau)[v_{a,n}(\tau)+f_2(\tau)]d\tau = \sigma_{a,n}^2[v_{a,n}(t)+f_2(t)]+\beta_n(t),$$
$$n = 2k+1, \quad k=1, 2,\ldots, \tag{16b}$$

with

$$\lim_{\epsilon\downarrow 0} \epsilon^2\|\alpha_n\| = 0, \quad n = 2k=1, \quad k=1,2,\ldots, \tag{16c}$$

$$\lim_{\epsilon\downarrow 0} \epsilon^2\|\beta_n\| = 0, \quad n = 2k+1, \quad k=1, 2,\ldots, \tag{16d}$$

(16) together with (15,c,d) means that (11) is verified for $n = 2k+1\,(k = 1, 2,\ldots)$.

The proof for the case $n = 2k+2$ $(k = 1, 2,\ldots)$ is the same as for the case $n = 2k+1$ $(k = 1, 2, \ldots)$ and is hence omitted. The proof is completed.

The preceding theorem provides the asymptotic singular values of the transfer function (3), consisting of the summation of infinitely many elementary transfer functions. It is of interest to look at the singular value of each elementary transfer function . . .

Theorem 2. We have

$$\sigma_i\left(\frac{1}{(s+\epsilon)^2}\right) = \frac{\sqrt{3\pm 2\sqrt{2}}}{4\epsilon^2}, \tag{17a}$$

$$\frac{\sigma_i\left(\dfrac{c_k}{s^2+2\epsilon s+w_k^2+\epsilon^2}\right)}{\dfrac{|c_k|}{4\epsilon w_k}} \to 1, \text{ as } \epsilon\downarrow 0,\ i = 1, 2. \tag{17b}$$

Proof. The *exact* singular value decomposition of the kernel of the Hankel operator associated with the inverse Laplace transform of $\dfrac{1}{(s+\epsilon)^2}$ is

$$(t+\tau)e^{-\epsilon(t+\tau)} = \frac{\sqrt{3+2\sqrt{2}}}{4\epsilon^2}\left[\frac{\sqrt{2\epsilon}}{\sqrt{2+2\sqrt{2}}}\ (1+\sqrt{2}\,\epsilon t)e^{-\epsilon t}\right]$$

$$\cdot \left[\frac{\sqrt{2\epsilon}}{\sqrt{2+2\sqrt{2}}} \quad (1+\sqrt{2}\,\epsilon\tau)e^{-\epsilon\tau} \right]$$

$$+ \frac{\sqrt{3-2\sqrt{2}}}{4\epsilon^2} \left[\frac{\sqrt{2\epsilon}}{\sqrt{2-\sqrt{2}}} (-1+\sqrt{2}\,\epsilon t)e^{-\epsilon t} \right]$$

$$\cdot \left[\frac{\sqrt{2\epsilon}}{\sqrt{2-\sqrt{2}}} \quad (1-\sqrt{2}\,\epsilon\tau)e^{-\epsilon\tau} \right].$$

From this, (17a) follows.

To prove (17b), consider the kernel of the product of the Hankel operator associated with the inverse Laplace transform of $\frac{c_k}{s^2+2\epsilon s+w_k^2+\epsilon^2}$ by itself:

$$K(t,\tau) \triangleq \int_0^\infty \frac{c_k^2}{w_k^2} e^{-\epsilon(t+x)} \sin w_k(t+x) e^{-\epsilon(x+\tau)} \sin w_k(x+\tau)\,dx$$

$$= \frac{c_k^2}{4\epsilon w_k^2} e^{-\epsilon(t+\tau)} \cos w_k(t-\tau)$$

$$- \frac{c_k^2}{2w_k^2} e^{-\epsilon(t+\tau)} \frac{\epsilon \cos w_k(t+\tau) - w_k \sin w_k(t+\tau)}{2(\epsilon^2+w_k^2)}.$$

Then observe that

$$\int_0^\infty K(t,\tau) 2\sqrt{\epsilon}\, e^{-\epsilon\tau} \sin w_k\tau\,d\tau = \frac{c_k^2}{16\epsilon^2 w_k^2} 2\sqrt{\epsilon}\, e^{-\epsilon t} \sin w_k t + \mu(t),$$

$$\lim_{\epsilon\downarrow 0} \epsilon^2 \|\mu\| = 0, \qquad \lim_{\epsilon\downarrow 0} \|2\sqrt{\epsilon}\, e^{-\epsilon\tau} \sin w_k\tau\| = 1,$$

$$\int_0^\infty K(t,\tau) 2\sqrt{\epsilon}\, e^{-\epsilon\tau} \cos w_k\tau\,d\tau = \frac{c_k^2}{16\epsilon^2 w_k^2} 2\sqrt{\epsilon}\, e^{-\epsilon t} \cos w_k t + \upsilon(t),$$

$$\lim_{\epsilon\downarrow 0} \epsilon^2 \|\upsilon\| = 0, \qquad \lim_{\epsilon\downarrow 0} \|2\sqrt{\epsilon}\, e^{-\epsilon\tau} \cos w_k\tau\| = 1,$$

From this, (17b) follows. This completes the proof.

We finally have

Theorem 3. The set of singular values of the transfer function (3), consisting in the summation of infinitely many elementary transfer functions as depicted by the parallel connection structure of the block diagram of Figure 1, is the union of the sets of singular values of the elementary transfer functions taken separately, asymptotically as $\epsilon \downarrow 0$.

Proof. Combine the results of Theorems 1 and 2.

It should be stressed that the above is *not* a general property of a transfer function consisting in the summation of many transfer functions of any kind. In our case, it is rather the fact that the elementary transfer functions are a double integrator and oscillators which endows the overall transfer function with a rich structure, responsible for the rather exceptional result of Theorem 3.

3. Approximation

An obvious approximation procedure for the transfer function (3) consists in cutting the expansion at a given order. This stems from the fact that the higher order terms are higher frequency terms which have a minor contribution to the input-output map. This is the so-called *modal truncation*. Although widely used in aerospace engineering [1]-[15], this procedure is, however, still heuristic and in need of justification.

As we shall see, the systematic approximation procedures [17]-[25] developed these last few years provide a justification of modal truncation, at least in an asymptotic sense.

One should distinguish two systematic approximation schemes – the *balanced* and the *optimal* approximations. Both of them can have rather easily been derived from the singular value decomposition of the Hankel operator associated with the impulse response of the system.

Consider the exact singular value decomposition (9). Define

$$\mathcal{O}(t) = [\sqrt{\sigma_1}u_1(t) \ \sqrt{\sigma_2}u_2(t) \ \ldots, \tag{18a}$$

$$\mathcal{C}(\tau) = \begin{bmatrix} \sqrt{\sigma_1} \ v_1(\tau) \\ \sqrt{\sigma_2} \ v_2(\tau) \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}. \tag{18b}$$

The singular value decomposition provides a factorization

$$h^{\epsilon}(t+\tau) = \mathcal{O}(t)\mathcal{C}(\tau). \tag{18c}$$

It is fairly well-known that a factorization of the Hankel operator associated with the impulse response of a sytem readily leads to a state space realization [18], [30]. In addition, in this case, the fact that the factorization (18c) comes from a singular value decomposition leads to a very special state space realization. Indeed, recalling that $h^\epsilon(t)$ is differentiable, one can write $\frac{d}{dt}h^\epsilon(t+\tau) = \frac{1}{d\tau}h^\epsilon(t+\tau)$, which yields $\dot{\mathcal{O}}(t)\mathcal{C}(\tau) = \mathcal{O}(t)\dot{\mathcal{C}}(\tau)$, from which it follows that

$$\dot{\mathcal{O}}(t) = \mathcal{O}(t)\int_0^\infty \dot{\mathcal{C}}(\tau)\mathcal{C}^+(\tau)d\tau, \tag{18d}$$

$$\dot{\mathcal{C}}(\tau) = \int_0^\infty \mathcal{O}^+(t)\dot{\mathcal{O}}(t)dt\mathcal{C}(\tau), \tag{18e}$$

with

$$\mathcal{O}^+(t) = \begin{bmatrix} \frac{1}{\sqrt{\sigma_1}} u_1(t) \\ \frac{1}{\sqrt{\sigma_2}} u_2(t) \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \tag{18f}$$

$$\mathcal{C}^+(\tau) = \left[\frac{1}{\sqrt{\sigma_1}} v_1(\tau) \quad \frac{1}{\sqrt{\sigma_2}} v_2(\tau) \ \ldots \right. \tag{18g}$$

(18d) further yields

$$\int_0^\infty \mathcal{O}^+(t)\dot{\mathcal{O}}(t)dt = \int_0^\infty \dot{\mathcal{C}}(\tau)\mathcal{C}^+(\tau)d\tau \triangleq \tilde{A}. \tag{18h}$$

With this notation, (18d,e) yield

$$\dot{\mathcal{O}}(t) = \mathcal{O}(t)\tilde{A}, \tag{18i}$$

$$\dot{\mathcal{C}}(\tau) = \tilde{A}\mathcal{C}(\tau). \tag{18j}$$

Define

$$\tilde{B} \triangleq \mathcal{C}(0), \tag{18k}$$

$$\tilde{C} \triangleq \mathcal{O}(0). \tag{18\ell}$$

Hence, from (18i-ℓ), one deduces

$$\mathcal{O}(t) = \tilde{C}e^{\tilde{A}t}, \tag{18m}$$

$$\mathcal{C}(\tau) = e^{\tilde{A}\tau}\tilde{B}. \tag{18n}$$

It follows that

$$h^{\epsilon}(t+\tau) = \mathcal{O}(t)\mathcal{C}(\tau) = \tilde{C}e^{\tilde{A}(t+\tau)}\tilde{B}; \quad \textbf{(18o)}$$

hence $(\tilde{A},\tilde{B},\tilde{C})$ is a realization of $h^{\epsilon}(t)$. This realization is quite particular; indeed, the controllability and observabiltiy grammians are, respectively,

$$\int_0^{\infty} e^{\tilde{A}\tau}\tilde{B}\tilde{B}^T e^{\tilde{A}^T\tau}d\tau = \int_0^{\infty} \mathcal{C}(\tau)\mathcal{C}^T(\tau)d\tau = \Sigma, \quad \textbf{(19a)}$$

$$\int_0^{\infty} e^{\tilde{A}^T t}\tilde{C}^T\tilde{C}e^{\tilde{A}t}dt = \int_0^{\infty} \mathcal{O}^T(t)\mathcal{O}(t)dt = \Sigma\,, \quad \textbf{(19b)}$$

with

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix}. \quad \textbf{(19c)}$$

Hence, in the state space realization $(\tilde{A},\tilde{B},\tilde{C})$, the controllability and observability grammians are equal, diagonal, and equal to Σ. $(\tilde{A},\tilde{B},\tilde{C})$ is the so-called *balanced* realization of $h^{\epsilon}(t)$ [17], [25].

Now, let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > \sigma_{r+1} \geq \ldots$ From the singular value decompositon (9), the best rank r approximation [25, Section III] of the Hankel operator $\mathcal{H}^{\epsilon}$ is the operator with kernel $\mathcal{O}_r(t)\mathcal{C}_r(\tau)$, where

$$\mathcal{O}_r(t) = [\sqrt{\sigma_1}\; u_1(t) \ldots \sqrt{\sigma_r}\; u_r(t)], \quad \textbf{(20a)}$$

$$\mathcal{C}_r(\tau) = \begin{bmatrix} \sqrt{\sigma_1}\; v_1(\tau) \\ \cdot \\ \cdot \\ \cdot \\ \sqrt{\sigma_r}\; v_r(\tau) \end{bmatrix}, \quad \textbf{(20b)}$$

since

$$\| h^{\epsilon}(t+\tau) - \mathcal{O}_r(t)\mathcal{C}_r(\tau) \| = \sigma_{r+1},$$

where $\|\bullet\|$ denotes the spectral norm, i.e., the square root of the supremum of the spectrum of the operator premultiplied by its adjoint. The big problem is that $\mathcal{O}_r(t)\mathcal{C}_r(\tau)$ does not, in general, depend only on the sum $t+\tau$ of the two arguments [25]; in other words, the operator of kernel $\mathcal{O}_r(t)\mathcal{C}_r(\tau)$

does not have Hankel structure. Nevertheless, a decent approximation [18, Section III] of this operator is the Hankel operator associated with the impulse response

$$\tilde{h}^{\epsilon}(t) = \tilde{C}_r e^{\tilde{A}_r t}\tilde{B}_r, \tag{20c}$$

where

$$\tilde{A}_r = \int_0^{\infty} \dot{\mathcal{O}}_r^{+}(t)\mathcal{O}_r(t)dt$$

$$= \int_0^{\infty} \dot{\mathcal{C}}_r(\tau)\mathcal{C}_r^{+}(\tau)d\tau, \tag{20d}$$

$$\tilde{B}_r = \mathcal{C}_r(o), \tag{20e}$$

$$\tilde{C}_r = \mathcal{O}_r(o). \tag{20f}$$

(The equality between the two integrals in (20d) stems from the fact that they are both the same top left-hand corner submatrix of $\tilde{A}$; see (18h).) $(\tilde{A}_r, \tilde{B}_r, \tilde{C}_r)$ is the so-called *balanced* approximation of $(\tilde{A}, \tilde{B}, \tilde{C})$ [17], [25]; observe that $\tilde{A}_r$ is the top left-hand corner submatrix of dimension r of $\tilde{A}$, and that $\tilde{B}_r$ and $\tilde{C}_r$ are the corresponding submatrices of $\tilde{B}$ and $\tilde{C}$, respectively.

Amazingly, Krein [19] has proved that there exists a *Hankel* operator $\hat{\mathcal{H}}_r^{\epsilon}$ of rank r such that

$$\|\mathcal{H}^{\epsilon} - \hat{\mathcal{H}}_r^{\epsilon}\| = \sigma_{r+1}.$$

The impulse response $\hat{h}_r^{\epsilon}(t)$ corresponding to $\hat{\mathcal{H}}_r^{\epsilon}$ is the so-called *optimal* approximation of $h^{\epsilon}(t)$.

Note that, if the operator of kernel $\mathcal{O}_r(t)\mathcal{C}_r(\tau)$ has Hankel structure, then $\tilde{h}_r^{\epsilon}(t) = \hat{h}_r^{\epsilon}(t)$; in other words, balanced and optimal approximations are equivalent.

Applied to the transfer function (3), the above yields the following rather surprising result:

Theorem 4. Consider the transfer function (3) with the sequence (10h) monotone decreasing. For any *even* order $r = 2p$ of approximation, modal truncation, balanced approximation, and optimal approximation are equivalent, asymptotically as $\epsilon \downarrow 0$. (Two operators are "asymptotically equivalent as $\epsilon \downarrow 0$" iff the spectral norm of their difference goes to zero as $\epsilon \downarrow 0$.)

Proof. First, it is easily seen that cutting down the asymptotic singular value decomposition (10) is equivalent to modal truncation; indeed, direct calculation shows that

$$\sum_{n=1}^{2p} \sigma_{a,n} u_{a,n}(t) v_{a,n}(\tau) = (t+\tau)e^{-\epsilon(t+\tau)} + \sum_{h=1}^{p-1} \frac{c_k}{w_k} e^{-\epsilon(t+\tau)} \sin w_k(t+\tau).$$

Consider now approximation in the balanced and optimal senses. The central issue is $\mathcal{O}_r(t)\mathcal{C}_r(\tau)$, as defined above. Combining (10), Theorem 1, and (20) yields

$$\|\mathcal{O}_r(t)\mathcal{C}_r(\tau) - (t+\tau)e^{-\epsilon(t+\tau)} - \sum_{k=1}^{p-1} \frac{c_k}{w_k} e^{-\epsilon(t+\tau)} \sin w_k(t+\tau)\| \to 0, \text{ as } \epsilon \downarrow 0.$$

It follows that $\mathcal{O}_r(t)\mathcal{C}_r(\tau)$ has Hankel structure, asymptotically as $\epsilon\downarrow 0$. Thus, as said above, balanced and optimal approximations are equivalent, and $\tilde{h}^\epsilon(t) \cong \hat{h}^\epsilon(t) \cong (t+\tau)e^{-\epsilon(t+\tau)} + \sum_{k=1}^{p-1} \frac{c_k}{w_k} e^{-\epsilon(t+\tau)} \sin w_k(t+\tau)$, where $\bullet \cong \bullet$ means that the norm of the difference between the two memebers goes to zero as $\epsilon\downarrow 0$. Hence, modal truncation, balanced, and optimal approximations are equivalent, asymptotically as $\epsilon\downarrow 0$.

4. Parametric study of six-dimensional case

This section offers a parametric study of singular value analysis, balancing, and approximations of the transfer function

$$h^\epsilon(s) = \frac{1}{(s+\epsilon)^2} + \sum_{k=1}^{2} \frac{c_k}{s^2 + 2\epsilon s + w_k^2 + \epsilon^2}. \tag{21}$$

A minimal state space realization is provided by

$$\dot{x} = Ax + bu, \tag{22a}$$

$$y = cx, \tag{22b}$$

with

$$A = \left[\begin{array}{cc|cc|cc} -\epsilon & 1 & 0 & 0 & 0 & 0 \\ 0 & -\epsilon & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\epsilon & w_1 & 0 & 0 \\ 0 & 0 & -w_1 & -\epsilon & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\epsilon & w_2 \\ 0 & 0 & 0 & 0 & -w_2 & -\epsilon \end{array}\right], \quad b = \left[\begin{array}{c} 0 \\ 1 \\ \hline -1 \\ 1 \\ \hline -1 \\ 1 \end{array}\right], \tag{22c}$$

$$c = \left[\begin{array}{cc|cc|cc} 1 & 0 & \dfrac{c_1}{2w_1} & \dfrac{c_1}{2w_1} & \dfrac{c_2}{2w_2} & \dfrac{c_2}{2w_2} \end{array}\right]. \tag{22d}$$

The controllability and observability grammians are defined by

$$W_c = \int_0^\infty e^{At} bb^T e^{A^T t} dt, \tag{23a}$$

$$W_o = \int_0^\infty e^{A^T t} c^T c e^{At} dt; \tag{23b}$$

obviously, $W_c = W_c^T > 0$ and $W_0 = W_0^T > 0$. Let T be a similarity transformation of the state, i.e., $(A,b,c) \xrightarrow{T} (TAT^{-1}, Tb, cT^{-1})$. It is readily verified that

$$W_0 \xrightarrow{T} T^{-T} W_0 T^{-1},$$

$$W_c \xrightarrow{T} T W_c T^T,$$

$$W_0 W_c \xrightarrow{T} T^{-T} W_0 W_c T^T.$$

Consider the eigenvalue decomposition of the controllability grammian

$$W_c = U_c \Sigma_c U_c^T, \tag{24}$$

where U_c is the (orthogonal) matrix of eigenvectors of W_c arranged columwise and Σ_c the diagonal matrix of eigenvalues of W_c. It is easily checked that

$$W_c \xrightarrow{\Sigma_c^{-1/2} U_c^T} I,$$

$$W_0 \xrightarrow{\Sigma_c^{-1/2} U_c^T} \Sigma_c^{1/2} U_c^T W_0 U_c \Sigma_c^{1/2} \triangleq W_{0n}.$$

Consider now the eigenvalue decomposition

$$W_{0n} = U_{on} \Sigma^2 U_{0n}^T, \tag{25}$$

where U_{0n} is the (orthogonal) matrix of eigenvectors of W_{0n} arranged columnwise and $\Sigma^2 = \text{diag}\{\sigma_1^2, \ldots, \sigma_6^2\}$ the diagonal matrix of eigenvalues of W_{0n}; the singular value notation is purposely used, and it will be justified very soon. Now, it is easily seen that

$$I \xrightarrow{\Sigma^{1/2} U_{0n}^T} \Sigma,$$

$$\Sigma_c^{1/2} U_C^T W_0 U_c \Sigma_c^{1/2} \xrightarrow{\Sigma^{1/2} U_{0n}^T} \Sigma.$$

In other words, the similarity transformation

$$T = \Sigma^{1/2} U_{0n}^T \Sigma_c^{-1/2} U_c^T \tag{26}$$

defines a state space realization

$$(A,b,c) \xrightarrow{T} (TAT^{-1},Tb,cT^{-1}) = (\tilde{A},\tilde{b},\tilde{c}), \qquad (27a)$$

in which the controllability and observability grammians are equal and diagonal, i.e,

$$W_0 \xrightarrow{T} \Sigma, \qquad (27b)$$

$$W_c \xrightarrow{T} \Sigma. \qquad (27c)$$

It follows that $(\tilde{A},\tilde{b},\tilde{c})$ is the *balanced* state space realization of $c(sI-A)^{-1}b$, and that the σ_i's are the *singular values*. This is another approach for deriving balanced state space realization and singular values, computationally more tractable [31] than the singular value decomposition avenue of approach.

Numerical exploration (in single precision on a DEC 10 computer) of singular value analysis, balancing, and balanced approximation has been performed, as well as comparison between pole/zero configurations of the high order model and the reduced order models.

Computation of Poles/Zeros of High Order Model

The poles of the original model are easily computed from (21). The zeros were computed as the Smith zeros of the system matrix *associated with the state space realization* (22), which were themselves computed as solutions of a generalized eigenvalue problem [38], by making use of the subroutines QZHES, QZIT, and QZVAL of EISPAK [35], [36].

Computation of the Grammians

In place of computing the grammians either as the numerical integrals (23) or as numerical solutions to associated Lyapunov equations [33], [37], it turns out to be more reliable to compute them via their analytic values derived from (22-23). The derivation of the analytical formulas of W_c and W_0 is easy and left to the reader.

Singular Value Analysis, Balancing, and Approximation

This was done by roughly following the procedure outlined in [31]. The eigenanalyses (24) and (25) of the controllability grammian W_c and the new observability grammian W_{0n}, respectively, were performed in single precision using the routines TRED2 and TQL2 of EISPAK [35]-[37]. Although these eigenanalyses are badly conditioned because of the rather large spectra of the eigenvalues of W_c and W_{0n} (typically, $\frac{\lambda_{max}(W_c)}{\lambda_{min}(W_c)} \cong 10^3$ and $\frac{\lambda_{max}(W_{0n})}{\lambda_{min}(W_{0n})} \cong 10^7$, for $\epsilon = 0.05$) due to the underlying physics of the problem, the TRED2 and TQL2 subroutines performed more than satisfactorily, even in single precision. The eigenanalysis of W_c could have been replaced by its Cholesky factorization [31], with equal numerical reliability. The singular values of the transfer function (21) were computed as the square roots of the eigenvalues of W_{0n}; see (25). The "balancing" transformation T was computed as given by (26). The balanced state space realization $(\tilde{A},\tilde{b},\tilde{c})$ was computed according to (27a). The coefficient matrices $\tilde{A}_r$, $\tilde{b}_r$, and $\tilde{c}_r$ of the balanced approximation were simply derived as submatrices of $\tilde{A}$, $\tilde{b}$, and $\tilde{c}$, respectively, as outlined in Section 3.

Computation of Poles/Zeros of Balanced Approximation

Poles of the balanced approximation were computed as the eignevalues of $\tilde{A}_r$, using the routines BALANC, ELMHES, and HQR of EISPAK [35], [36]. These routines performed more than satisfactorily. However, the computation of the zeros of the balanced approximation $(\tilde{A}_r,\tilde{b}_r,\tilde{c}_r)$ is no easy task. It appears, indeed, that these zeros are extremely sensitive to the data $(\tilde{A}_r,\tilde{b}_r,\tilde{c}_r)$. This great sensitivity can be justified by the fact proved in Section 3 that the balanced aproximate system is asymptotically the modally truncated system and hence has its zeros close to the flexible poles; this sensitivity issue can also be more precisely justified by a condition number analysis [39]. Further, the data $(\tilde{A}_r,\tilde{b}_r,\tilde{c}_r)$ is uncertain, since it comes from the balanced realization $(\tilde{A},\tilde{b},\tilde{c})$ which is itself uncertain, because the whole balancing procedure is badly conditioned due to the large spread of singular values typical for this class of systems. It is hence not surprising that the zeros of $(\tilde{A}_r,\tilde{b}_r,\tilde{c}_r)$ are hard to compute. These zeros were computed as the Smith zeros of the associated system matrix, using the routines QZHES, QZIT, and QZVAL of EISPAK [35]-[38]. These zeros were also computed as the poles of the inverse system, that is, as the eigenvalues of the matrix $\left(A_r - \dfrac{b_r c_r A_r^2}{c_r A_r b_r}\right)$, using BALANC, ELMHES, and HQR of EISPAK. Both procedures gave pretty close results. They both provided realistic numerical zeros in the case of a moderate spread of singular values, with a slight preference for the inverse system procedure; in the case of a wide spread of singular values, the numerical zeros were completely irrelevant.

Numerical Results and Interpretation

The numerical results are summarized in Tables 1, 2, and 3. Table 1 provides the singular values and the asymptotic singular values as functions of ϵ. Observe that the singular values fit well with the asymptotic predictions, except that σ_5 and $\sigma_{a,5}$ are quite apart for $\epsilon = 0.001$, but this is a numerical pitfall due to the (too) large spread of singular values for $\epsilon = 0.001$. Also observe that the spread of singular values increases as $\epsilon \downarrow 0$. Table 2 provides the singular values and the asymptotic singular values as functions of the modal constant c_1. Observe that the singular values σ_3, σ_4 and the asymptotic singular values $\sigma_{a,3}$, $\sigma_{a,4}$ corresponding to the first flexible mode increase as c_1 increases, whereas the other singular values are roughly left unchanged, which is not surprising in view of (21). Finally, Table 3 provides the singular values, the asymptotic singular values, and the fourth and second order balanced approximations for some sample values of the parameters. As already said, the computation of the zeros of the balanced approximations was too unreliable for the small values 0.05 and 0.1 of ϵ, because the spreads of singular values were too large, so that these zeros are not listed in Table 3. However, for $\epsilon = 0.5$, the spread was moderate enough to allow a fairly reliable computation of the zeros, and these are given in Table 3. The important fact is to observe that the pole/zero configuration of the balanced approximations is close to the pole/zero configuration of the original transfer function (21) after modal truncation, thereby confirming Theorem 4.

From all the numerical results, it follows that for realistic values of the physical parameters the rigid mode has large singular values and the first and second flexible modes have low singular values. Since singular values provide a measure of controllability/observability, it follows that the flexible modes, and especially the second one, are "weakly controllable/observable," thereby recovering a result already obtained via the Linear-Quadratic-Gaussian avenue of approach [12], [13].

5. Conclusions

We have attempted in this paper to perform a first evaluation of the recently emerged tools centered upon singular value decomposition as applied to physical systems. We have chosen a class of flexible systems, because of their importance with the advent of Large Space Structures, and also because for these systems measures of controllability/observability and systematic approximation procedures like those provided by singular value decomposition are badly needed. In a flexible system consisting of the interconnection of a double integrator and oscillators, singular values have been shown to provide a measure of the contribution of each subsystem to the overall interconnected system and has allowed a fairly rigorous justification of the deletion of the low singular value subsystems in reduction procedures.

Although from a very specific sensitivity standpoint balanced realization is desirable [40], one should, however, keep in mind that balancing so far requires a lot of eigenanalyses which are more and more ill-conditioned as the spread of singular values increases. Hence the computed balanced realization may be far enough from the true value as to invalidate conclusions drawn from this computed balanced realization, as shown by the difficulties encountered in computing the zeros of the balanced approximation.

More reliable numerical procedures for balancing seem to be called for. Specifically, a procedure not involving the unessential multiplications bb^T and c^Tc, typical in the grammians and making the spectral spreads twice as big, would be welcome. In other words, it would be desirable to compute the σ_i's, rather than the σ_i^2's as is usually done with the currently available

ϵ	w_1	w_2	c_1	c_2	σ_1 / $\sigma_{a,1}$	σ_2 / $\sigma_{a,2}$	σ_3 / $\sigma_{a,3}$	σ_4 / $\sigma_{a,4}$	σ_5 / $\sigma_{a,5}$	σ_6 / $\sigma_{a,6}$
0.20	2.7	10.0	0.4	0.05	15.08909	2.583373	0.1881321	0.1665404	0.006326852	0.006155446
					15.08884	2.588835	0.1851852	0.1851852	0.00625	0.00625
0.15	2.7	10.0	0.4	0.05	26.82473	4.599166	0.252944	0.2289220	0.00840336	0.008242955
					26.82459	4.602373	0.2469136	0.2469136	0.0083333	0.0083333
0.10	2.7	10.0	0.4	0.05	60.35540	10.35387	0.3792661	0.3534001	0.01255293	0.01239213
					60.35534	10.35534	0.3703704	0.3703704	0.0125	0.0125
0.05	2.7	10.0	0.4	0.05	241.4214	41.42099	0.7523660	0.7251955	0.02609735	0.02497378
					241.4214	41.42136	0.7407407	0.7407407	0.025	0.025
0.03	2.7	10.0	0.4	0.05	670.6149	115.0592	1.246698	1.219699	0.04162987	0.03814798
					670.6149	115.0593	1.234568	1.234568	0.04166667	0.04166667
0.02	2.7	10.0	0.4	0.05	1508.883	258.8833	1.864733	1.837369	0.06313205	0.06239883
					1508.883	258.8835	1.851852	1.851852	0.0625	0.0625
0.01	2.7	10.0	0.4	0.05	6035.529	1035.529	3.721410	3.689781	0.1706231	0.1249703
					6035.534	1035.534	3.703704	3.703704	0.125	0.125
0.005	2.7	10.0	0.4	0.05	24142.11	4142.115	7.425319	7.393704	0.2896103	0.2499762
					24142.14	4142.135	7.407407	7.407407	0.25	0.25
0.001	2.7	10.0	0.4	0.05	603544.3	103545.3	40.25545	36.99051	19.45976	1.251383
					603553.4	103553.4	37.03704	37.03704	1.25	1.25

Table 1. Singular values and asymptotic singular values of $h^{\epsilon}(s)$, as a function of ϵ.

ϵ	w_1	w_2	c_1	c_2	σ_1 / $\sigma_{a,1}$	σ_2 / $\sigma_{a,2}$	σ_3 / $\sigma_{a,3}$	σ_4 / $\sigma_{a,4}$	σ_5 / $\sigma_{a,5}$	σ_6 / $\sigma_{a,6}$
0.05	1.5	6.0	0.5	0.2	241.4216	41.4167	1.705022	1.598849	0.16779	0.1657916
					241.4214	41.42136	1.666667	1.666667	0.1666667	0.1666667
0.1	1.5	6.0	0.5	0.2	60.35619	10.33783	0.8508420	0.7584192	0.084271	0.0823207
					60.35534	10.35534	0.83333	0.83333	0.083333	0.083333
0.5	1.5	6.0	0.5	0.2	2.446824	0.3173643	0.1004465	0.07340009	0.015308	0.01460638
					2.414214	0.41421	0.1666667	0.1666667	0.01666667	0.01666667
0.5	1.5	3.0	0.7	0.5	2.464407	0.2679445	0.1231249	0.1112925	0.0622465	0.04704958
					2.414214	0.4142135	0.23333	0.233333	0.083333	0.083333

Table 3. Singular values, asymptotic singular values, fourth order and second

softwares. A possibility of avoiding the unessential "squaring up" might be provided by [21], where it is shown that the inherent structure of the controllable canonic form allows the computation of the σ_i's as the solutions to a generalized eigenvalue problem. Another such possibility is given in a recent report [41] on the solution of Lyapunov equations in a "square-root" fashion.

ε	w_1	w_2	c_1	c_2	σ_1 / $\sigma_{a,1}$	σ_2 / $\sigma_{a,2}$	σ_3 / $\sigma_{a,3}$	σ_4 / $\sigma_{a,4}$	σ_5 / $\sigma_{a,5}$	σ_6 / $\sigma_{a,6}$
0.05	2.7	10.0	0.6	0.05	241.4214	41.42083	1.128332	1.087769	0.02497815	0.02452533
					241.4214	41.42136	1.111111	1.111111	0.025	0.025
0.05	2.7	10.0	0.5	0.05	241.4214	41.42091	0.9403207	0.9064842	0.02524023	0.02496893
					241.4214	41.42136	0.9259259	0.9259259	0.025	0.025
0.05	2.7	10.0	0.4	0.05	241.4214	41.42099	0.7523660	0.7251955	0.02609735	0.02497378
					241.4214	41.42136	0.7407407	0.7407407	0.025	0.025
0.05	2.7	10.0	0.3	0.05	241.4213	41.42106	0.5641830	0.5438994	0.0251145	0.02492053
					241.4214	41.42136	0.5555556	0.5555556	0.025	0.025
0.05	2.7	10.0	0.2	0.05	241.4213	41.42115	0.3761815	0.3625994	0.02547584	0.02496273
					241.4214	41.42136	0.3703704	0.3703704	0.025	0.025

Table 2. Singular values and asymptotic values of $h^{\epsilon}(s)$ as function of c_1.

	fourth order balanced approximation		second order balanced approximation	
zeros of original	poles	zeros	poles	zeros
-0.05± j 5.644228 j 1.222964	-0.049± j 1.499732 -0.049± j 0.0007557145	computation too unreliable	-0.0495± j 0.0046498	computation too unreliable
-0.1± j 5.644227 j 1.222964	-0.0990± j 1.499144 -0.0999± j 0.003074864	computation too unreliable	-0.09687± j 0.017242	computation too unreliable
-0.5 ±j 5.644228 ±j 1.222964	-0.47867± j 0.0877055 -0.5064326± j 1.550953	-0.468507 ±j 1.392281	-0.3329831± j 0 -0.9285787± j 0	∞
-0.5± j 2.685597 j 1.129693	-0.405 -0.7686 -0.65389± j 1.143566	-0.6209861 ±j 0.934361	-0.3297905 -0.93333	∞

order balanced approximations of $h^{\epsilon}(s)$.

References

1. W. B. Gevarter, "Basic relations for the control of flexible vehicles," *AIAA 7th Aerospace Sciences Meeting,* New York City, New York, January 20-22, 1969, AIAA Paper No. 69-115.
2. G. Porcelli, "Attitude control of flexible space vehicles," *AIAA Journal,* vol. 10, pp. 807-812, June 1972.

3. V. J. Modi, "Attitude dynamics of satellites with flexible appendages – a brief review," *J. Spacecraft,* vol. 11, pp. 743-751, November 1974.
4. M. Healey et al., "Design of a digital-stability augmentation and control system with gust alleviations and modal suppression," *Proc. IEE,* vol. 112, pp. 437-440, April 1975.
5. J. E. Cochran and D. E. Christensen, "Free-flight rocket attitude motion due to transverse vibration," *J. Spacecraft,* vol. 17, pp. 425-431, Sept.-Oct. 1980.
6. G. D. Martin and A. E. Bryson, "Attitude control of a flexible spacecraft," *J. Guidance and Control,* vol. 3, pp. 37-41, Jan.-Feb. 1980.
7. D. H. L. Poelaert, "Exact modal analysis for spinning flexible spacecraft," *Proc. Symposium on Dynamics and Control of Non-Rigid Spacecraft,* Frascati, Italy, May 24-26, 1976, European Space Agency Special Publication ESA SP 117, pp. 31-38.
8. R. J. Benhabib et al., "Stability of distributed control for large flexible structures using positivity concepts," *AIAA Guidance and Control Conference,* Boulder, Colorado, August 6-8, 1979, pp. 540-548.
9. R. P. Iwens et al., "A unified approach to the design of large space structure control systems," *Joint Automatic Control Conference,* 1980.
10. M. K. Mackay, "Synthesis of low order observers to reconstruct a desired subset of states or modes of a high order linear system," *Joint Automatic Control Conference,* 1980.
11. Y. Ohkami and P. Likins, "The influence of spacecraft flexibility on system controllability and observability," *IV IFAC Symposium on Automatic Control in Space,* August 1974, Papers Session XIII, pp. 51-64.
12. E. Jonckheere, "On the observability of the deformable modes in a class of nonrigid satellites," *Proc. Symposium on Dynamics and Control of Non-Rigid Spacecraft,* Frascati, Italy, May 24-26, 1976, European Space Agency Special Publication ESA SP 117, pp. 251-262.
13. E. Jonckheere, "Robustness of observers for estimating the state of a deformable satellite," *European Space Agency Conference on Attitude and Orbit Control Systems,* Noordwijk, The Netherlands, October 3-6, 1977, Preprint Book, pp. 191-202.
14. J. N. Juang, "Controllability and observability of large flexible spacecraft in noncircular orbits," *J. Guidance and Control,* vol. 3, pp. 186-188, March-April 1980.
15. P. C. Hughes and R. E. Skelton, "Controllability and observability for flexible spacecraft," *J. Guidance and Control,* vol. 3, pp. 452-459, Sept.-Oct. 1980.
16. E. J. Davison, "A method for simplifying linear dynamic systems," *IEEE Trans. Automat. Contr.,* vol. AC-11, pp. 93-101, January 1966.
17. B. C. Moore, "Singular value analysis of linear systems, part I: external variables," Systems Control Report No. 7801, University of Toronto, Canada, July 1978.
18. S. Y. Kung, "A new identification and model reduction algorithm via singular value decompositions," *Proc. 12th Asilomar Conf. Circuits, Syst., Comput.,* Nov. 1978, pp. 705-714.
19. V. M. Adamjan, D. Z. Arov, and M. G. Krein, "Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem," *Math. USSR Sbornik,* vol. 15, pp. 31-73, 1971.

20. V. M. Adamjan, D. Z. Arov, and M. G. Krein, "Infinite Hankel block matrices and related extension problems," *Amer. Math. Society Translations,* vol. 111, pp. 133-156, 1978.
21. L. M. Silverman and M. Bettayeb, "Optimal approximation of linear systems," *IEEE Trans. Automat. Contr.,* submitted for publication.
22. M. Bettayeb, L. M. Silverman, and M. G. Safonov, "Optimal approximation of continuous-time systems," *Proc. IEEE Conference on Decision and Control,* Albuquerque, New Mexico, December 1980.
23. S. Y. Kung and D. W. Lin, "Optimal Hankel-norm model reductions: multivariable systems," *IEEE Trans. Automat. Contr.,* submitted for publication.
24. S. Y. Kung and D. W. Lin, "Optimal multivariable model reductions: State-space approach," *IEEE Trans. Automat. Contr.,* submitted for publication.
25. L. M. Silverman, "Approximation of linear systems," Technical Report, Department of Electrical Engineering, University of Southern California, Los Angeles, California, 1980.
26. I. C. Gohberg and M. G. Krein, *Introduction to the Theory of Linear Non-Self Adjoint Operators.* Providence, RI: Amer. Math. Soc., 1969.
27. V. C. Klema and A. J. Laub, "The singular value decompositions: its computation and some applications," *IEEE Trans. Automat. Contr.,* vol. AC-25, pp. 164-176, April 1980.
28. F. Riesz et B. Sz.-Nagy, *Leçons d'Analyse Fonctionnelle.* Paris: Gauthier-Villars, 1968.
29. T. Kato, *Perturbation Theory for Linear Operators.* New York: Springer-Verlag, 1966.
30. L. M. Silverman, "Realization of linear dynamical systems," *IEEE Trans. Automat. Contr.,* vol. AC-16, pp. 554-567, December 1971.
31. A. J. Laub, "On computing "balancing" transformation," *Joint Automatic Control Conference,* San Francisco, California, Aug. 1980, pp. FA8-E.
32. C. Van Loan, "Computing integrals involving the matrix exponential," *IEEE Trans. Automat. Contr.,* AC-23, pp. 395-404, 1978.
33. R. H. Bartels and G. W. Stewart, "Solution of the matrix equation $AX+XB=C$," *Comm. ACM,* vol. 15, pp. 820-826, 1972.
34. R. S. Martin, G. Peters, and J. H. Wilkinson, "Symmetric decomposition of a positive definite matrix," *Numer. Math.,* vol. 7, pp. 362-383, 1965.
35. B. T. Smith et al., *Matrix Eigensystem Routines – EISPAK Guide.* Second Edition, Lecture Notes in Computer Sciences, vol. 6, Springer-Verlag, New York, 1976.
36. B. S. Garbow et al., *Matrix Eigensystem Routines – EISPAK Guide Extension.* Lecture Notes in Computer Sciences, vol. 51, Springer-Verlag, New York, 1977.
37. A. J. Laub, "Survey of computational methods in control theory," in Erisman, et al. (Eds.) *Electric Power Problems: The Mathematical Challenge,* SIAM, Philadelphia, 1980, pp. 231-260.
38. A. J. Laub and B. C. Moore, "Calculation of transmission zeros using QZ techniques," *Automatica,* vol. 14, pp. 557-556, 1978,
39. G. W. Stewart, "Gershgorin theory for the generalized eigenvalue problem $Ax = \lambda Bx$," *Mathematics of Computation,* vol. 29, pp. 600-606, April 1975.

40. C. T. Mullis and R. A. Roberts, "Synthesis of minimum roundoff noise fixed point digital filters," *IEEE Trans. Circuits and Systems,* vol. CAS-23, pp. 551-562, September 1976.

41. S. J. Hammarling, "The numerical solution of the Lyapunov equation," NPL Report DNACS 49/81, National Physical Laboratory, Teddington, Middlesex, United Kingdom, Aug. 1981.