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Disordered System with n Orbitals per Site: Lagrange Formulation, Hyperbolic Symmetry, and Goldstone Modes

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We give a Lagrange formulation of the gauge invariant n-orbital model for disordered electronic systems recently introduced by Wegner. The derivation proceeds analytically without use of diagrams, and it identifies the previously discussed $n \to \infty$ limit as the saddle-point approximation of the Lagrangian formulation. We discover that the Lagrangian model crucially depends on the position with respect to the real axis of the energies involved. If the energies occur on both sides of the real axis as is the case in the calculation of the conductivity, then the order parameter field takes values in a set of complex non-hermitean matrices. If all energies are on the same side of the real axis then a hermitean matrix model emerges. This difference reflects a difference in the symmetries. Whereas in the latter case normal unitary symmetry holds, the symmetry in the former case is of hyperbolic nature. The corresponding symmetry group is not compact and this might be a source of singularities also in the region of localized states.
Eliminating massive modes in tree approximation we derive an effective Lagrangian for the Goldstone modes. The structure of this Lagrangian resembles the non-linear σ-model and is a very general consequence of broken isotropic symmetry. We also consider the first correction to the tree approximation which is related to the invariant measure of the generalized non-linear σ-model.

1. Introduction

In two recent papers Oppermann and Wegner [1, 2] have considered a model for a disordered electronic system with n orbitals per site. In [1] the $n=\infty$ limit for the averaged one- and two-particle Green's functions has been obtained by summing all tree graphs similarily to the procedure by Wigner [3] and Arnold [4]. In [2] a diagrammatic perturbation expansion in powers of $1/n$ was given. It became apparent that the system shows features characteristic for a broken isotropic symmetry. It was suggested that $d_c = 2$ is the lower critical dimensionality at and below which no metallic behaviour exists, and critical exponents were given for the mobility edge behaviour in the limit $d \to 2_+$. The identification of $d_c = 2$ as lower critical dimensionality is in agreement with the observation by Thouless and Licciardello [5] and with the results of Götze, Wölfle, and Prelovsek [6], and of Abraham, Anderson, Licciardello, and Ramakrishnan [7]. The β-function in [7] allows the prediction of critical exponents in the limit $d \to 2_+$ in agreement with those of [2] for the real matrix ensemble.
The nature of the underlying symmetry was discovered by Wegner [8]. He showed that the system has an internal broken isotropic symmetry with the frequency being the symmetry breaking source and the density of states being the order parameter. The property that the Green's functions decay rapidly unless the points coincide pairwise led Aharony and Imry [9] to introduce a field theory with composite variables $Q(r)$ to describe the mobility edge behaviour. A model of interacting Q-fields obeying the

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isotropic symmetry yields [8] exponents and the location of the mobility edge in agreement with the n-orbital model to the extent calculated, and it recently has been used to analyse the behaviour near the mobility edge of the participation ratio [10]. Using a Lagrangian formulation of this model Harris and Lubensky [11] have suggested that eight is the upper critical dimensionality.

In Refs. 1, 2 a diagrammatic perturbation theory was used. It is the aim of this paper to discuss the symmetries of the Lagrangian $L[Q]$ underlying this perturbation theory. A peculiar structure emerges: In representing an average over a product of several Green's functions $\mathcal{G}(r_p, r'_p; z_p)$ the adequate model crucially depends on the position of the energies z_p. If all z_p are on the same side of the real axis then a model defined on the set of hermitean (real symmetric) matrices can be used. The set depends on the ensemble considered. If the energies are on different sides of the real axis then we have to integrate over a set of complex non-hermitean matrices. This difference reflects a difference in the symmetries of these two cases. In the former case the system is invariant under unitary (orthogonal) transformations and in mean-field theory it is governed by a single invariant saddle point. In the latter case the symmetry is of hyperbolic nature. This symmetry is spontaneously broken and we find a non-compact surface of saddle points.

We stress that our derivation of $L[Q]$ is analytically rigorous - except for possible problems connected to the $m \to 0$ trick which we use without further examination. It is only by insisting on mathematical rigour that one can detect the important difference between the hyperbolic and the unitary case. We prove that on the level of perturbation theory the two models are equivalent.

The hyperbolic symmetry is spontaneously broken, and we can identify the massive modes and the Goldstone modes of broken symmetry. Eliminating the massive modes in saddle-point approximation we derive a generalized non-linear σ-model for the Goldstone modes. This model has been suggested before [8]. It is defined on a set of matrices with two m-fold degenerate fixed eigenvalues. Models of this type have recently been discussed by Brézin, Hikami, and Zinn-Justin [12] from the point of view of renormalization. Evaluating the gradient-independent contribution due to Gaussian fluctuation of the massive modes, we recover the invariant measure of the set of matrices with the above mentioned eigenvalue constraints. This supports the suggestion that the critical behaviour of the generalized non-linear σ-model and the original matrix model should be the same.

The organization of this article is as follows: In Sect. 2 we use the $m=0$ trick to construct a Lagrangian for vector fields starting from the gauge invariant models of Refs. 1, 2. Here we also discuss the symmetries. Section 3 is devoted to the derivation of the matrix Lagrangian. We establish the connection to the work of Refs. 1, 2 by showing that the limit of large n is identical to the saddle-point approximation and the $1/n$-expansion results from expanding around the saddle point (loop expansion). In Sect. 4 we derive the generalized non-linear σ-model. Section 5 contains our conclusions. Some properties of the hyperbolic symmetry operations are discussed in Appendix A, and Appendix B contains a very general derivation of non-linear σ-models which sheds some light on the special case considered here.

2. The Model and its Invariance Properties

2.1. The Model in Vector Representation

We consider the gauge invariant model of Refs. 1, 2. At each lattice site r of a regular lattice there are n orbitals $|r, \alpha\rangle$ $\alpha = 1, \ldots, n$. In the following we will often abbreviate the pair of indices r, α by x. The system is governed by the one-particle Hamiltonian

$$H = \frac{1}{\sqrt{n}} \sum_{x,x'} |x\rangle f_{xx'} \langle x'| \tag{2.1}$$

where either the $f_{xx'}$ are the elements of a real symmetric matrix (real matrix ensemble, RME), or they form a complex hermitean matrix (phase invariant ensemble, PIE). In both cases the matrix elements $f_{xx'}$ are random variables distributed according to the normalized Gaussian weight

$$P[f] = \mathcal{N}^{-1} \exp\left\{ -\frac{\beta}{2} \sum_{xx'} (M_{rr'})^{-1} |f_{xx'}|^2 \right\} \tag{2.2}$$

with $\beta = \frac{1}{2}$ (RME) and $\beta = 1$ (PIE), respectively. First and second moments of this distributions are given by

$$\begin{aligned} &\bar{f}_{xx'} = 0 \\ &\overline{f_{x_1 x'_1} f_{x_2 x'_2}} = (\delta_{x_1 x'_2} \delta_{x_2 x'_1} + \kappa \delta_{x_1 x_2} \delta_{x'_1 x'_2}) M_{r_1 r'_1} \end{aligned} \tag{2.3}$$

where $\kappa = 1$ (RME) and $\kappa = 0$ (PIE). The bar denotes the ensemble average

$$\bar{A} = \int \mathcal{D}[f]\, P[f]\, A[f]. \tag{2.4}$$

Here $\int \mathcal{D}[f]$ stands for the integrals over the independent matrix elements $f_{xx'}$ for the real matrix ensemble or $\mathrm{Re} f_{xx'}$ and $\mathrm{Im} f_{xx'}$ for the phase invariant ensemble. We note that both $(M_{rr'})^{-1}$ and $(M^{-1})_{rr'}$ $\pm (M_{rr'})^{-1}$ will occur in our derivation.

The averaged Green's functions can be expressed as expectation values of products of vectors **S** [13]

$$\overline{\langle x|(z_p-H)^{-1}|x'\rangle}=s_p\langle S_a^p(x)\,S_a^{p*}(x')\rangle \tag{2.5}$$

$$\overline{\langle x_1|(z_1-H)^{-1}|x_1'\rangle\langle x_2|(z_2-H)^{-1}|x_2'\rangle}$$
$$=s_1 s_2\langle S_a^1(x_1)\,S_a^{1*}(x_1')\,S_b^2(x_2)\,S_b^{2*}(x_2')\rangle \tag{2.6}$$

where the system is governed by the new Hamiltonian

$$\mathscr{H}=\sum_{pp',aa',x}(s_p s_{p'})^{\frac{1}{2}}S_a^{p*}(x)\,\hbar_{aa'}^{pp'}(r)\,S_{a'}^{p'}(x)+\mathscr{H}_i \tag{2.7}$$

$$\exp(-\mathscr{H}_i)=\overline{\exp\{\beta n^{-\frac{1}{2}}\sum_{p,xx',a}s_p S_a^{p*}(x)f_{xx'}S_a^p(x')\}} \tag{2.8}$$

with

$$\hbar_{aa'}^{pp'}(r)=\beta z_p\delta_{pp'}\delta_{aa'} \tag{2.9}$$

$$s_p=-i\,\mathrm{sign}\,\mathrm{Im}\,z_p. \tag{2.10}$$

The vector field **S** is real for the real matrix ensemble and complex for the phase invariant ensemble. The number of the components S_a^p, $a=1,\ldots,m$; $p=1,2$ vanishes due to the replica trick: $m=0$. We note that the variables used in [8] correspond to $s_p^{1/2}S_a^{p*}(r)$ in the present notation. The expectation value is defined by

$$\langle A\rangle=\int\mathscr{D}[\mathbf{S}]\,A(\mathbf{S})\exp(-\mathscr{H})\hat{\mathscr{Z}}^{-1} \tag{2.11}$$

$$\hat{\mathscr{Z}}=\int\mathscr{D}[\mathbf{S}]\exp(-\mathscr{H}). \tag{2.12}$$

The probability distribution $P[f]$ is invariant under orthogonal (RME) and unitary (PIE) transformations in the space of the orbitals at any given site. Due to this local gauge invariance the averaged Green's functions vanish unless the points coincide pairwise. Moreover they do not depend on the labels α and one obtains

$$\overline{\langle x|(z_p-H)^{-1}|x'\rangle}=\delta_{xx'}G(r;z_p) \tag{2.13}$$

$$\overline{\langle x_1|(z_1-H)^{-1}|x_1'\rangle\langle x_2|(z_2-H)^{-1}|x_2'\rangle}$$
$$=(\delta_{x_1x_2'}\delta_{x_1'x_2}+\kappa\delta_{x_1x_2}\delta_{x_1'x_2'})\,C(r_1,r_1';z_1,z_2)$$
$$+\delta_{x_1x_1'}\delta_{x_2x_2'}(G(r_1;z_1)\,G(r_2;z_2)+C''(r_1,r_2;z_1,z_2)). \tag{2.14}$$

If in (2.7) we allow for general $\hbar_{aa'}^{pp'}(r)$ then these functions can be obtained from

$$F=\ln\hat{\mathscr{Z}}[\hbar] \tag{2.15}$$

by functional differentiation with respect to $\hbar$ evaluated for $\hbar$ given by (2.9):

$$nG(r;z_p)=-\frac{\delta F}{\delta\hbar_{aa}^{pp}(r)} \tag{2.16}$$

$$n\overline{\sum_\alpha\langle x_1|(z_1-H)^{-1}|r_2\alpha\rangle\langle r_2\alpha|(z_2-H)^{-1}|x_1\rangle}$$
$$=\frac{\delta^2F}{\delta\hbar_{ba}^{21}(r_1)\,\delta\hbar_{ab}^{12}(r_2)}=nK(r_1,r_2;z_1,z_2) \tag{2.17}$$

$$n\overline{\sum_\alpha\langle x_1|(z_1-H)^{-1}|x_1\rangle\langle r_2\alpha|(z_2-H)^{-1}|r_2\alpha\rangle}$$
$$-n^2G(r_1;z_1)\,G(r_2;z_2)=\frac{\delta^2F}{\delta\hbar_{aa}^{11}(r_1)\,\delta\hbar_{bb}^{22}(r_2)}$$
$$=nK''(r_1,r_2;z_1,z_2) \tag{2.18}$$

$$K(r_1,r_2;z_1,z_2)=n\,C(r_1,r_2;z_1,z_2)$$
$$+\delta_{r_1r_2}[\kappa\,C(r_1,r_1;z_1,z_2)+G(r_1;z_1)\,G(r_2;z_2)$$
$$+C''(r_1,r_1;z_1,z_2)] \tag{2.19}$$

$$K''(r_1,r_2;z_1,z_2)=n\,C''(r_1,r_2;z_1,z_2)$$
$$+\delta_{r_1r_2}(1+\kappa)\,C(r_1,r_1;z_1,z_2). \tag{2.20}$$

We stress that as a result of local gauge invariance a source term $\hbar_{aa'}^{pp'}(r)$ coupled to a *product of vectors at the same point* is sufficient to construct any physical observable of the system. This is in contrast to usual vector models where the source term is coupled to a single vector $S_a^p(x)$.

2.2. Symmetries

With the Gaussian distribution (2.2) of $f_{xx'}$ the interaction $\mathscr{H}_i$ is easily evaluated.

$$\mathscr{H}_i=-\frac{\beta}{2n}\sum_{xx',pp',aa'}M_{rr'}s_p s_{p'}$$
$$\cdot S_a^{p*}(x)\,S_{a'}^{p'}(x)\,S_a^p(x')\,S_{a'}^{p'*}(x'). \tag{2.21}$$

The symmetries of this interaction have been discussed before [8], and we here repeat this discussion stressing the difference between the cases $s_1=s_2$ or $s_1=-s_2$, which will play a crucial role in the subsequent development. The symmetries follow from the fact that $\mathscr{H}_i$ depends on the spin fields only via the combination

$$I_{xx'}=\sum_{p,a}s_p S_a^{p*}(x)\,S_a^p(x') \tag{2.22}$$

The same combination can be introduced into the quadratic part of the Hamiltonian (2.7)

$$\sum_{pp',aa'}(s_p s_{p'})^{\frac{1}{2}}S_a^{p*}(x)\,\hbar_{aa'}^{pp'}(r)\,S_{a'}^{p'}(x)$$
$$=\beta E I_{xx}-\sum_{pp',aa'}(s_p s_{p'})^{\frac{1}{2}}S_a^{p*}(x)$$
$$\cdot[(-1)^p\frac{\omega}{2}\beta\delta_{pp'}\delta_{aa'}-\delta\hbar_{aa'}^{pp'}(r)]\,S_{a'}^{p'}(x). \tag{2.23}$$

Here we have used the form

$$z_p = E - (-1)^p \frac{\omega}{2}, \tag{2.24}$$

and we have explicitly added a source term $\delta\mathcal{H}$. In the absence of the symmetry breaking terms $\omega/2$, $\delta\mathcal{H}$ the Hamiltonian is invariant under all transformations which do not affect $I_{xx'}$.

For the case $s_1 = s_2$ these transformations are easily identified as the global, unitary (PIE) or orthogonal (RME), transformations in $(2m)$-dimensional vector space. Thus in this case the Hamiltonian shows the normal rotational invariance. For $s_1 = -s_2$ the situation is different since then $I_{xx'}$ is constructed with an indefinite metric. It may easily be checked that I is invariant under transformations $\mathbf{S}(x) \to \hat{T}\mathbf{S}(x)$ obeying the relation

$$\hat{T}^{\oplus}\hat{T} = \hat{T}\hat{T}^{\oplus} = 1 \tag{2.25}$$

where

$$\hat{T}^{\oplus} = s^{+}\,\hat{T}^{+}\,s \tag{2.26}$$

$$s^{pp'}_{aa'} = s^{p}\,\delta_{pp'}\,\delta_{aa'}. \tag{2.27}$$

Transformations obeying (2.25) will be called 'pseudounitary' (psu). For the real matrix ensemble we have to restrict the group of symmetry operations to psu transformations the elements of which are real in order to guarantee that the vector field is real. We note that for $s_1 = s_2$ the notion of pseudounitary becomes identical to unitarity, and thus the discussion of the symmetries may be summarized in stating that for $\omega = 0$, $\delta\mathcal{H} = 0$ the model is invariant under pseudounitary transformations.

A more detailed understanding of the structure of pseudounitary operators may be helpful. We show in Appendix A that any psu operator can be represented in the form

$$T = U_1\,R\,U_2 \tag{2.28}$$

where the U_i are unitary operators which do not mix the subspaces $p=1$ and $p=2$:

$$U_{i,\,aa'}^{pp'} = \delta_{pp'}\,U_{i,\,aa'}^{p}. \tag{2.29}$$

For $s_1 \neq s_2$ the 'special' psu operator R is parametrized by m real variables φ_a according to

$$R^{pp'}_{aa'} = \delta_{aa'}\,R^{pp'}(\varphi_a) \tag{2.30}$$

$$R(\varphi) = \begin{pmatrix} \cosh\varphi & i\sinh\varphi \\ -i\sinh\varphi & \cosh\varphi \end{pmatrix}. \tag{2.31}$$

This reveals that for $s_1 \neq s_2$ the subspaces $p \neq p'$ are connected by a "hyperbolic" symmetry operation in contrast to the spherical symmetry present for $s_1 = s_2$. This latter case obviously results from the former by choosing φ imaginary.

3. The Lagrangian in Matrix Representation

3.1. Formal Derivation

We assume that the matrix $M_{rr'}$ is translational invariant and positive definite, and we write

$$(M^{-1})_{rr'} = 4\,w_{rr'}/E_0^2 \tag{3.1}$$

where

$$E_0^2 = 4\sum_{r'} M_{rr'} \tag{3.2i}$$

$$1 = \sum_{r'} w_{rr'}. \tag{3.2ii}$$

Moreover we assume that $w_{rr'} \leqq 0$ for $r \neq r'$. This condition and (3.2ii) guarantee that w and consequently M are positive definite.

The quantity E_0 is used as energy scale and z_p is written in the form

$$z_p = E_0(\varepsilon - (-1)^p\,\hat{\omega}) \tag{3.3}$$

where

$$\hat{\omega} = \frac{\omega}{2E_0}. \tag{3.3i}$$

In addition we introduce the notation

$$\widetilde{\delta\mathcal{H}}(r) = \frac{2}{\beta E_0}\,\delta\mathcal{H}(r). \tag{3.4}$$

The interaction $\mathcal{H}_i$ (2.21) is difficult to handle since it couples vectors at different points $x \neq x'$. We can decouple it by a Gaussian transformation introducing a matrix field $Q(r)$ and a Lagrangian $L[Q]$. Formally this Lagrangian is derived via the following steps:

(i) Decoupling of $\mathcal{H}_i$

$$\begin{aligned}\mathcal{Z} &= \int \mathcal{D}[S]\,\exp\Big\{-\sum_{pp',\,aa',\,x}(s_p s_{p'})^{\frac{1}{2}} \\ &\cdot S_a^{p*}(x)\,\mathcal{H}^{pp'}_{aa'}(r)\,S^{p'}_{a'}(x) - \frac{\beta}{2n}\sum_{xx'} M_{rr'}|J_{xx'}|^2\Big\} \\ &= \mathcal{Z}_0^{-1}\int\mathcal{D}[\mathbf{S}]\int\mathcal{D}[Q]\,\exp\Big\{-\beta\frac{E_0}{2}\sum_{x,\,pp',\,aa'}(s_p s_{p'})^{\frac{1}{2}} \\ &S_a^{p*}(x)\,[2\varepsilon\delta_{aa'}\delta_{pp'} - Q^{pp'}_{aa'}(r)]\,S^{p'}_{a'}(x) \\ &-\beta n\sum_{rr'} w_{rr'}\,\mathrm{Tr}\,(\tfrac{1}{2}Q(r)Q(r') + (\tilde{\omega} + \widetilde{\delta\mathcal{H}}(r))\,Q(r'))\Big\}.\end{aligned} \tag{3.5}$$

Note that $J_{x'x} = -J^{*}_{xx'}$.

Here $\mathscr{Z}_0$ is a normalization factor for the Q-integration

$$\mathscr{Z}_0 = \int \mathscr{D}[Q] \exp\{-\beta n \sum_{rr'} w_{rr'} \operatorname{Tr}(\tfrac{1}{2} Q(r) Q(r') + \tilde{\omega}(Q(r'))\} \exp\{n\beta \sum_{rr'} w_{rr'} \operatorname{Tr}(\tfrac{1}{2}\delta\tilde{\hbar}(r)\delta\tilde{\hbar}(r') + \tilde{\omega}\delta\tilde{\hbar}(r'))\} \tag{3.6}$$

and the matrix $\tilde{\omega}$ is defined as

$$\tilde{\omega}_{aa'}^{pp'} = -\delta_{pp'}\delta_{aa'} 2\hat{\omega}(-1)^p \tag{3.7}$$

(ii) Integration over the field **S**

$$\hat{\mathscr{Z}} = \mathscr{Z}_0^{-1} \int \mathscr{D}[Q] e^{-L[Q]} \equiv \mathscr{Z}/\mathscr{Z}_0 \tag{3.8}$$

where

$$\frac{1}{\beta n} L[Q] = \tfrac{1}{2} \sum_{rr'} w_{rr'} \operatorname{Tr}(Q(r')) Q(r')) + \sum_{rr'} w_{rr'} \operatorname{Tr}((\tilde{\omega} + \delta\tilde{\hbar}(r)) Q(r')) + \sum_{r} \operatorname{Tr} \ln(\tilde{\varepsilon} - Q(r)) \tag{3.9}$$

$$\tilde{\varepsilon}_{aa'}^{pp'} = 2\delta_{pp'}\delta_{aa'}\varepsilon. \tag{3.10}$$

The Q-integration is defined in terms of the independent matrix elements of Q in order to allow in a simple way for the shift of Q implicit in the derivation of (3.5). We note that the Lagrangian $L[Q]$ has also been given in Ref. 11.

To make this formal derivation rigorous we have to define the domain of the Q-integration in such a way that the following conditions hold:

(i) Convergence of the Q-integrals in (3.5), (3.6), uniformly in **S**.

(ii) Convergence of the **S**-integrals for fixed Q.

Furthermore the symmetry operation $\mathbf{S} \to \hat{T}\mathbf{S}$ induces the transformation $Q \to TQT^{\oplus}$ where

$$T = s^{\frac{1}{2}} \hat{T} s^{+\frac{1}{2}}. \tag{3.11}$$

For psu $\hat{T}$ also T is pseudounitary. If $\hat{T}$ is real then T^{11} and T^{22} are real whereas T^{12} and T^{21} are imaginary. Such a psu transformation will be called pseudoorthogonal (pso).

For an explicitly symmetric formulation in the absence of symmetry-breaking terms both the Lagrangian $L[Q]$ and the domain of the Q integration should be invariant under global symmetry transformations $Q(r) \to TQ(r)T^{\oplus}$. Since for $\tilde{\omega} = \delta\tilde{\hbar} = 0$ the Lagrangian (3.9) is invariant under *any* nonsingular transformation T the symmetry requirement for $L[Q]$ is trivially fulfilled.

3.2. *The Domain of Integration in Matrix Space*

In examining the convergence conditions we have to distinguish between the cases $s_1 = s_2$ and $s_1 = -s_2$. We simplify the discussion by restricting the energies z_p to the most interesting situations

a) $s_1 = s_2$, $\hat{\omega} = 0$, $\operatorname{Im}\varepsilon \neq 0$ (3.12)

b) $s_1 = -s_2$, $\operatorname{Im}\hat{\omega} > 0$, $\operatorname{Im}\varepsilon = 0$. (3.13)

For $s_1 = s_2$ the set of matrices Q has to be invariant under unitary (PIE) and orthogonal (RME) transformations, respectively. This suggests to choose $Q(r)$ hermitean (PIE) and real symmetric (RME). It is easily checked that with this choice condition (i) is satisfied. Here it is used that $w_{rr'}$ is positive definite. The condition (ii) holds since the energies z_p have non-zero imaginary parts of the same sign. Thus for $s_1 = s_2$ the hermitean (real symmetric) matrix model is rigorously equivalent to the original vector model. (Throughout the paper the reader should replace hermitean matrices Q by real symmetric matrices Q for the RME.) It is tempting to use these sets of matrices also for $s_1 \neq s_2$. However, in that case the integral over S is not convergent. We will construct an adequate set of matrices below. Here we first note that as long as we restrict ourselves to perturbation theory the hermitean matrix model is sufficient. This follows if we expand the r.h.s. of Eq. (3.5) in powers of $(s_p s_{p'})^{\frac{1}{2}} S_a^{p*} Q_{aa'}^{pp'} S_{a'}^{p'}$. For each finite power the S-integral can be carried out, and the resulting perturbation expansion in powers of (M/z_p^2) is identical to the expansion of the original model (2.7), (2.12).

To get some hint on the correct choice of the matrices $Q(r)$ for $s_1 \neq s_2$ we now determine the saddle points of the partition function Z in the invariant situation $\tilde{\omega} = 0 = \delta\tilde{\hbar}$. For r-independent Q the Lagrangian $L[Q]$ then depends only on the eigenvalues λ_j of Q, and the saddle point equation reads

$$0 = \frac{\partial}{\partial\lambda_j} \frac{1}{\beta n N} L[Q] = \lambda_j - (2\varepsilon - \lambda_j)^{-1}; \quad j = 1, \ldots, 2m \tag{3.14}$$

where N denotes the number of lattice sites of the system. This yields

$$\lambda_j = \varepsilon \pm i(1-\varepsilon^2)^{\frac{1}{2}} \equiv \varepsilon \pm i\lambda_0. \tag{3.15}$$

Since for each of the eigenvalues we have two solutions independently we have to choose the physically relevant saddle point. This choice is dictated by the requirement that we have to be able to deform the integration path in Q-space such that it passes through the saddle point. We first consider case a): $s_1 = s_2$. It is obvious that we can shift the integration

paths for the diagonal elements of $Q(r)$ as long as we do not cross the singularities of the logarithm in $L[Q]$. Since all these singularities are on the same side of the real axis only the saddle point matrix

$$Q^p_{\text{s.p.}} = \lambda^{(p)} 1 \tag{3.16}$$

$$\lambda^{(p)} = \varepsilon + s_p \lambda_0 \tag{3.17}$$

can be reached. Clearly this matrix satisfies all the invariance requirements with respect to the transformations T. We note that for real ε we find two equivalent saddle points (3.16). For $\varepsilon = \text{Re}\,\varepsilon \pm i0$ this symmetry is spontaneously broken.

For case b) ($s_1 = -s_2$, Im $\varepsilon = 0$) it seems to be reasonable to consider the generalization of $Q^p_{\text{s.p.}}$:

$$Q_{\text{s.p.}} = \varepsilon 1 + \lambda_0 s. \tag{3.18}$$

This choice guarantees that the single-particle Green's function $G(r; z_p)$ (see Eq. 2.16) is always governed by the same saddle-point matrix element $(Q_{\text{s.p.}})^{pp}_{aa}$ independent of whether $s_1 = s_2$ or $s_1 = -s_2$. However, this choice is not invariant under psu transformations; under these transformations a non-compact manifold of equivalent saddle points emerges. All these saddle points should belong to the set of matrices $Q(r)$, and we therefore are led to define the matrix field $Q(r)$ for $s_1 = -s_2$ as

$$Q(r) = \lambda_0\, T(r)\, s\, T^{\oplus}(r) + T_0\, P(r)\, T_0^{\oplus} \tag{3.19}$$

where

$$P^{pp'}_{aa'}(r) = \delta_{pp'} P^p_{aa'}(r) \tag{3.20}$$

and T_0 denotes a fixed space-independent psu (PIE) and pso (RME) transformation. Equations (2.29) and (3.19) show that $Q(r)$ is independent of the choice of U_2 in the representation (2.28) of T. Thus we can restrict $T(r)$ to the set of all psu (PIE) and pso (RME) transformations of the form

$$T = U_1 R U_1^+ \tag{3.21}$$

which is the set of all hermitean psu and pso transformations. We have absorbed the part $\varepsilon 1$ of $Q_{\text{s.p.}}$ into $P(r)$ which ranges over the set of all hermitean matrices obeying the constraint (3.20) ('block diagonal matrices').

Since (3.19) specifies the set of matrices which we will use for $s_1 = -s_2$, a more explicit analysis of this expression is appropriate. We find that the first contribution on the r.h.s. of equation (3.19) can be written in the form

$$\lambda_0 [T(r)\, s\, T^{\oplus}(r)]^{pp'} = V^{pp'}(\hat{Q}^{12})$$
$$= \begin{cases} \hat{Q}^{pp'}(r), & p \neq p' \\ -i(\lambda_0^2 1^{(1)} + \hat{Q}^{12}(r)\, \hat{Q}^{21}(r))^{\frac{1}{2}}, & p = p' = 1 \\ i(\lambda_0^2 1^{(2)} + \hat{Q}^{21}(r)\, \hat{Q}^{12}(r))^{\frac{1}{2}}, & p = p' = 2 \end{cases} \tag{3.22}$$

where

$$\hat{Q}^{21} = \hat{Q}^{12+}. \tag{3.23}$$

In Eq. (3.22) it is understood that we choose the matrix square root continuing analytically from $s_p \lambda_0 1^{(p)}$. The distinction of the sign again implies that a discrete symmetry is broken. If T ranges over all matrices of the form (3.21) the $\hat{Q}^{12}$ ranges over all real (RME) and complex (PIE) $m \times m$ matrices. To prove these statements we note that by construction the matrix $V(r)$ has m-fold degenerate eigenvalues $-i\lambda_0$ (subspace $p=1$) and $+i\lambda_0$ (subspace $p=2$). Given that the off-diagonal blocks fulfill relation (3.23) the diagonal blocks are uniquely specified in terms of Q^{12} and of the eigenvalues. To derive Eq. (3.22) it therefore is sufficient to show that

$$\lambda_0 [T(r)\, s\, T^{\oplus}(r)]^{21} = \lambda_0 [T(r)\, s\, T^{\oplus}(r)]^{12+} \tag{3.24}$$

and that $[T(r)\, s\, T^{\oplus}(r)]^{12}$ is real for the RME. (Note that λ_0 is real by virtue of Im $\varepsilon = 0$). Both statements are easily checked with the help of (2.26) and of the relations $s_1 = -i = -s_2$ which holds in the present case. This proves that any matrix $\lambda_0\, T s\, T^{\oplus}$ can be written in the form of $V(r)$. On the other hand we can also prove that each matrix $V(r)$ can be written in the form of $\lambda_0\, T s\, T^{\oplus}$. This proof is quite similar to the derivation of relation (2.28) and will be sketched in Appendix A. As a consequence the first contribution to $Q(r)$ (Eq. 3.19) ranges over all hermitean block off-diagonal matrices to which antihermitean blocks in the diagonal are added according to Eq. (3.22). On the other hand the second contribution $T_0\, P(r)\, T_0^{\oplus}$ ranges over all hermitean block diagonal matrices to which blocks outside the diagonal are coupled by the fixed transformation T_0. Thus the set $Q(r)$ is similar to the set of all hermitean matrices, and it has enough freedom to allow for the shifting of variables implicit in the derivation of (3.5).

We now prove that with the set (3.19) the formal derivations of Subsect. 3.1 are justified. The convergence of the S-integrals in Eq. (3.5) (requirement ii) is easily established since the S-dependent exponential in the integrand takes the form

$$\sum_{pp', aa'} (s_p s_{p'})^{\frac{1}{2}} S^{p*}_a(x) (\tilde{\varepsilon} - Q(r))^{pp'}_{aa'} S^{p'}_{a'}(x)$$
$$= \langle \mathbf{S}(x) | s^{\frac{1}{2}} (\tilde{\varepsilon} - Q(r))\, s^{\frac{1}{2}} | \mathbf{S}(x) \rangle \tag{3.25}$$

where the hermitean part of the operator $s^{\frac{1}{2}}(\tilde{\varepsilon} - Q)\, s^{\frac{1}{2}}$ is positive definite. Requirement (i) is fulfilled *provided* Imω *is not zero*. With the choice (3.19) we find for the relevant part of the exponential

$$\sum_{rr'} w_{rr'} \operatorname{Tr}(\tfrac{1}{2} Q(r) Q(r') + \tilde{\omega} Q(r'))$$
$$= \tfrac{1}{2} \sum_{rr'} w_{rr'} \operatorname{Tr}(P(r) P(r'))$$
$$+ \frac{\lambda_0^2}{2} \sum_{rr'} w_{rr'} \operatorname{Tr}(T(r) s T^{\oplus}(r) T(r') s T^{\oplus}(r'))$$
$$+ \lambda_0 \sum_{rr'} w_{rr'} \operatorname{Tr}(T_0 P(r) T_0^{\oplus} T(r') T^{+}(r') s)$$
$$+ 2i\hat{\omega} \sum_{r} \operatorname{Tr}(T_0 P(r) s T_0^{+})$$
$$- 2i\hat{\omega}\lambda_0 \sum_{r} \operatorname{Tr}(T(r) T^{+}(r)). \tag{3.26}$$

The third term on the r.h.s. of (3.26) is purely imaginary. The $Q-\mathbf{SS}$ coupling (3.25) does not affect the convergence and thus the $P(r)$-integrals converge by virtue of the first term on the r.h.s. of (3.26). The second term can be rewritten as

$$\frac{\lambda_0^2}{2} \sum_{rr'} w_{rr'} \operatorname{Tr}(T_{rr'} s T_{rr'}^{-1} s)$$

with

$$T_{rr'} = T^{\oplus}(r) T(r'). \tag{3.27}$$

Since $T_{rr'}$ is pseudounitary we may use the representation (2.28). Since the block-diagonal U's commute with s expression (3.27) reduces to

$$\frac{\lambda_0^2}{2} \sum_{rr'} w_{rr'} \operatorname{Tr}(R_{rr'} s R_{rr'}^{-1} s) \tag{3.28}$$

which with (2.31) yields

$$-\lambda_0^2 \sum_{rr'} w_{rr'} \sum_{a} \cosh 2\varphi_{rr',a}. \tag{3.29}$$

This guarantees the convergence of the integral over the fluctuations of $T(r)$ provided $w_{rr'} \leqq 0$ for $r \neq r'$. For $r = r'$ one has $\varphi \equiv 0$. Similarly one can see that the continuum limit $w_{rr'} \to (1-(2d)^{-1} R^2 \Delta)\delta(r-r')$ yields a convergent integral. Although we need this requirement on w in order to obtain convergence we do not see a physical indication for a different behavior of the system if this requirement is violated. In perturbation theory a necessity for this requirement does not show up.

If the operators $T(r)$ would represent a compact group this discussion would be sufficient to prove the existence of the Q-integrals. In our case the saddle point manifold is not bounded and this forces us to invoke the symmetry-breaking term to guarantee convergence.

$$-i\hat{\omega} \sum_{r} \operatorname{Tr}(T(r) T^{+}(r))$$
$$= -i2\hat{\omega} \sum_{r} \sum_{a} \cosh 2\varphi_a(r). \tag{3.30}$$

(Note that by virtue of (3.13) Im $\hat{\omega}$ is positive.) This term essentially cuts off the contribution of the saddle point surface at $2\varphi_a \sim -\ln \operatorname{Im} \hat{\omega}$. As a result the Q-integrals converge and we have shown that (3.19) to (3.21) define a set of matrices for which the matrix model (3.8) (3.9) is identical to the original model in the case $s_1 = -s_2$.

In the present formulation the set $Q(r)$ is not invariant under global transformations T; rather this transformation yields a set specified by $T_0 T$ in place of T_0. However, the invariance of the theory is recovered since the partition function is independent of T_0. An attempt to introduce a manifestly invariant set of matrices by the choice $Q(r) = T(r)\ (\lambda_0 s + P(r))\, T^{\oplus}(r)$ leads to serious convergence problems. These difficulties are due to a coupling between $P(r)$ and the $T(r)$-fluctuations which for rapid $T(r)$ fluctuations renders the $P(r)$-integration divergent, and for $P(r)$ of order λ_0 yields a negative mass of the $T(r)$ fluctuations. One possibility to circumvent these problems could be to couple in Eq. (3.19) T_0 locally to $T(r)$ in such a way that $T(r)$ fluctuations are strongly damped in $T_0(r)$ in order to suppress the above mentioned effects. Such a formulation would be at least very cumbersome and we therefore prefer to work with the simpler set (3.19).

We have mentioned above that in perturbation theory it is sufficient to work with the hermitean matrix model which rigorously is equivalent to the vector model only for $s_1 = s_2$. It may be worth while to reconsider this question here. Perturbation theory formally results from the expansion of $\ln(\tilde{\varepsilon} - Q(r))$ in $L[Q]$ in powers of $(\tilde{\varepsilon} - Q_{\text{s.p.}})^{-1}(Q(r) - Q_{\text{s.p.}})$. Linear and quadratic terms are kept in the exponential whereas higher-order terms are expanded in a Taylor series. The equivalence in perturbation theory of the choice (3.19) and the hermitean matrix model is guaranteed if in a Gaussian integral we can deform the integration paths from the set (3.19) to the set of all hermitean matrices. That this indeed is possible is most easily seen for the special choice $T_0 = 1$ where the off-diagonal blocks Q^{12}, Q^{21} are those of a hermitean matrix (compare (3.23)). Then it is sufficient to shift $P(r)$ for fixed $Q^{12}(r)$ (i.e. fixed $T(r)$) by a finite amount such that the matrix $Q(r)$ as a whole becomes hermitean. In particular this argument allows us to rewrite $\mathscr{Z}_0$ in the form

$$\mathscr{Z}_0 = \int \mathscr{D}[Q] \exp\left\{-\frac{\beta n}{2} \sum_{rr'} w_{rr'} \operatorname{Tr}(Q(r) Q(r'))\right\}$$
$$\cdot \exp\left\{\frac{\beta n}{2} \sum_{rr'} w_{rr'} \operatorname{Tr}[(\tilde{\omega} + \delta\widetilde{\hbar}(r))(\tilde{\omega} + \delta\widetilde{\hbar}(r'))]\right\} \tag{3.31}$$

where the integral ranges over all hermitean matrices $Q(r)$. If combined with equation (3.9) this shows that

$\mathscr{Z}$ does depend only on the combination $(\tilde{\omega}+\delta\tilde{h}(r))$ apart from the ε-dependence.

3.3. Relation to Earlier Work

We end this section by presenting the connection of the present formulation with the work of Refs. 1, 2. Obviously the physical correlation functions can be expressed in terms of correlation functions of the matrix field $Q(r)$. Equations (2.15) to (2.18) together with the results of Sect. 3.1 yield

$$G(r; z_p)=\frac{2}{E_0}(\langle Q_{aa}^{pp}(r')\rangle-(-1)^p 2\hat{\omega}) \tag{3.32i}$$

$$K(r_1,r_2; z_1,z_2)=\frac{4}{E_0^2}\left(n\sum_{rr'} w_{r_1 r} w_{r_2 r'}\right.$$
$$\left.\cdot\langle Q_{ab}^{12}(r') Q_{ba}^{21}(r)\rangle^c-\frac{1}{\beta} w_{r_1 r_2}\right) \tag{3.32ii}$$

$$K''(r_1,r_2; z_1,z_2)=\frac{4n}{E_0^2}\sum_{rr'} w_{r_1 r} w_{r_2 r'}$$
$$\langle Q_{aa}^{11}(r) Q_{bb}^{22}(r')\rangle^c \tag{3.32iii}$$

where the superscript c indicates the cumulant average. The results of Ref. 1 for the gauge invariant models in the limit $n\to\infty$ follow immediately from the saddle-point approximation applied to these correlation functions. For example (3.15), (3.32i) yield

$$G_{\text{s.p.}}(r; E)=\frac{2}{E_0}\lambda^{(p)} \tag{3.33}$$

which for $\operatorname{Im} E\to 0$ immediately implies the well known semicircle law [3, 4] for the density of states $\rho(E)$:

$$\rho(E)=\frac{1}{\pi}\operatorname{Im} G(r; E-i0)=\frac{2}{\pi E_0}\cos\varphi \tag{3.34}$$

$$\varphi=\arcsin E/E_0. \tag{3.35}$$

The results of [1] for K, K'' here follow in the standard way by solving the saddle-point equations for r-dependent $Q(r)$ in the presence of an infinitesimal source $\delta h(r)$. To construct the Lagrangian underlying the $1/n$-expansion of [2] we use the hermitean matrix model and we shift $Q(r)$ according to

$$Q(r)=\tilde{Q}(r)+\tilde{\xi}-\tilde{\omega} \tag{3.36}$$

where

$$\xi_{aa'}^{pp'}=\delta_{pp'}\delta_{aa'}\xi^{(p)}$$
$$=\delta_{pp'}\delta_{aa'}\left[\frac{z_p}{E_0}+s_p\left(1-\left(\frac{z_p}{E_0}\right)^2\right)^{\frac{1}{2}}\right]. \tag{3.37}$$

Up to constant terms which do not contribute in the limit $m\to 0$ the Lagrangian (3.9) is transformed to

$$\frac{1}{\beta n}\tilde{L}[\tilde{Q}]=\tfrac{1}{2}\sum_{rr'} w_{rr'}\operatorname{Tr}(\tilde{Q}(r)\tilde{Q}(r'))$$
$$+\sum_{rr'} w_{rr'}\operatorname{Tr}[(\tilde{\xi}+\delta\tilde{h}(r))\tilde{Q}(r')]$$
$$+\sum_{r}\operatorname{Tr}\ln(1-\tilde{\xi}^{\frac{1}{2}}\tilde{Q}(r)\tilde{\xi}^{\frac{1}{2}})$$
$$-\sum_{r}\operatorname{Tr}(\delta\tilde{h}(r)(\tilde{\xi}-\tilde{\omega})). \tag{3.38}$$

If we take into account the normalization (3.31) the term proportional to $\operatorname{Tr}(\delta\tilde{h}\cdot\tilde{\omega})$ is cancelled and a term $\sim\operatorname{Tr}(\delta\tilde{h}(r)\,\delta\tilde{h}(r'))$ is added. The remaining part of $\mathscr{Z}_0$ just provides the normalization. Rewriting the result in terms of a field $\hat{Q}=n^{\frac{1}{2}}\tilde{\xi}^{\frac{1}{2}}\tilde{Q}\tilde{\xi}^{\frac{1}{2}}$ we obtain the Lagrangian underlying the work of Oppermann and Wegner [2]:

$$\frac{1}{\beta}\tilde{L}=\frac{1}{2}\sum_{rr'}\sum_{pp',aa'}\hat{Q}_{aa'}^{pp'}(r)$$
$$[(\tilde{\xi}^{(p)}\tilde{\xi}^{(p')})^{-1} w_{rr'}-\delta_{rr'}]\,\hat{Q}_{a'a}^{p'p}(r')$$
$$-\sum_{k=3}^{\infty}\frac{n^{1-k/2}}{k}\operatorname{Tr}(\hat{Q}^k)+\frac{n}{2}\sum_{rr'}\operatorname{Tr}(\delta\tilde{h}(r)\,\delta\tilde{h}(r'))$$
$$+\sum_{rr'}\sum_{pp',aa'}(n\,\tilde{\xi}^{(p)}\tilde{\xi}^{(p')})^{\frac{1}{2}}\,\delta\tilde{h}_{aa'}^{pp'}(r)$$
$$(\tilde{\xi}^{(p)}\tilde{\xi}^{(p')})^{-1} w_{rr'}\,\hat{Q}_{aa'}^{pp'}(r). \tag{3.39}$$

This Lagrangian defines a diagrammatic expansion with vertices of order $k\geqq 3$ and coupling $n^{1-k/2}$ and with propagators

$$g_{pp'}(q)=[(\tilde{\xi}^{(p)}\tilde{\xi}^{(p')})^{-1}\sum_{r} w_{rr'} e^{iq(r-r')}-1]^{-1}. \tag{3.40}$$

These vertices and propagators are identical to those introduced in [2], and a combinatorial analysis shows that the diagrams of [2] are those of the Lagrangian (3.39) in the limit $m\to 0$.

4. The Effective Lagrangian for Goldstone Modes

4.1. Analogies to the Ferromagnet

The model constructed here shows close formal analogies to the familiar spin model of a Heisenberg ferromagnet. These analogies result from the fact that we are concerned with a continuous symmetry which is broken by a term coupled linearly to the fluctuating field. Thus the symmetry breaking field $\tilde{\omega}$ is the analogon of the magnetic field, and the field conjugate to $\tilde{\omega}$, i.e. the analogon of the magnetization, is constructed from the expectation value of Q:

$$\frac{1}{m}\frac{\partial}{\partial\tilde{\omega}}\ln\hat{\mathscr{Z}}=-2\beta n\sum_r[\langle Q^{11}_{aa}(r)-Q^{22}_{bb}(r)\rangle+4\hat{\omega}] \quad (4.1)$$

i.e.

$$\frac{2}{mN}\frac{\partial}{\partial\omega}\ln\hat{\mathscr{Z}}=\beta n(G(r;z_2)-G(r;z_1)). \quad (4.2)$$

Equation (4.2) shows that for $z_p=E_0(\varepsilon-(-1)^p\hat{\omega})$ and vanishing symmetry breaking field $\hat{\omega}$ the symmetry is *spontaneously* broken by the nonvanishing density of states in the energy band. The density of states therefore is analogous to the magnetization and the energy E plays the rôle of temperature. However, at this point a difference to the magnetic case is to be stressed. The transition from localized to extended states will occur for some energy E_c in the band, and spontaneous symmetry breaking occurs on *both* sides of the transition. Indeed, no pecularities are expected for the density of states at $E=E_c$.

Under global symmetry transformations the partition function transforms according to the law

$$\hat{\mathscr{Z}}[\hat{\omega}+\widetilde{\delta h}]=\hat{\mathscr{Z}}[T^{\oplus}(\tilde{\omega}+\widetilde{\delta h})\,T]. \quad (4.3)$$

For an infinitesimal transformation $T=1+\delta T$, $\delta T^{\oplus}=-\delta T$, this implies the standard Ward identity

$$\sum_r\sum_{pp',aa'}[\tilde{\omega}+\widetilde{\delta h}(r),\delta T]^{pp'}_{aa'}\frac{\delta}{\delta\widetilde{h}^{pp'}_{aa'}(r)}\ln\hat{\mathscr{Z}}=0 \quad (4.4)$$

where $[,]$ denotes the commutator. Out of the variety of results following from this relation we especially note the identity [8, 14]

$$(G(r;z_1)-G(r;z_2))/(z_2-z_1)=\sum_{r'}K(r,r';z_1,z_2) \quad (4.5)$$

which obviously is the analogon of the relation between M/H and the transverse Green's function well known for the isotropic magnet. Equations (4.1) and (4.5) suggest that the diagonal blocks Q^{pp} of Q play the role of 'longitudinal' components whereas the off diagonal blocks Q^{12}, Q^{21} are the analogon of transverse fields. This interpretation is supported by an examination of the form of the mass matrix in the perturbation theory of Ref. 2. Equation (3.40) yields

$$(m^2)^{pp'}_{aa'}=(\tilde{\xi}^{(p)}\,\tilde{\xi}^{(p')})^{-1}-1. \quad (4.6)$$

For $z_p=E_0(\varepsilon-(-1)^p\hat{\omega})$ we find from Eq. (3.37)

$$(m^2)^{pp'}_{aa'}=\delta_{pp'}(e^{2s_p\varphi}-1)+O(\hat{\omega}) \quad (4.7)$$

where the angle φ has been defined in Eq. (3.35). For $\hat{\omega}=0$ the transverse mass ($p\neq p'$) vanishes whereas the longitudinal mass ($p=p'$) stays finite within the band.

4.2. Definition of the Effective Lagrangian

The close analogy between our problem and the ferromagnet suggests that we should be able to construct a generalization of the non-linear σ-model, i.e. an effective Lagrangian for the Goldstone modes of broken isotropic symmetry which correctly treats the limit of small momentum transfer. In this limit a continuum approximation to the d-dimensional lattice is adequate:

$$\sum_r\to a^{-d}\int d^dr \quad (4.8\,\mathrm{i})$$

$$w_{rr'}\to a^d\left(1-\frac{R^2}{2d}\Delta_r\right)\delta^d(r-r'). \quad (4.8\,\mathrm{ii})$$

In this approximation the Lagrangian (3.9) reads

$$\frac{1}{\beta n}L_c[Q]=a^{-d}\int d^dr\left\{\frac{1}{2}\mathrm{Tr}\left[Q(r)\left(1-\frac{R^2}{2d}\Delta\right)Q(r)\right]\right.$$
$$\left.+\mathrm{Tr}\ln(\tilde{\varepsilon}-Q(r))+\mathrm{Tr}\left[(\tilde{\omega}+\widetilde{\delta h}(r))\left(1-\frac{R^2}{2d}\Delta\right)Q(r)\right]\right\}. \quad (4.9)$$

In the following we restrict ourselves to the case of broken symmetry, $-s_1=s_2=i$, and in Eq. (3.19) which defines the set of matrices $Q(r)$ we take for simplicity $T_0=1$.

The effective Lagrangian is defined by integrating over all longitudinal fields $Q^{11}(r)$, $Q^{22}(r)$. With the choice $T_0=1$ this amounts to integrating over $P(r)$:

$$\exp\{-\mathscr{L}[Q^{12}]\}=\int\mathscr{D}[P]\exp\{-L_c[Q]\}\cdot e^{\mathrm{const.}}. \quad (4.10)$$

To evaluate this expression we determine the saddle point and the corresponding saddle point Lagrangian for fixed $Q^{12}(r)$, $Q^{21}(r)$ and vanishing symmetry breaking field. Expansion around this saddle point in principle allows for the evaluation of $\mathscr{L}$ to arbitrary order in $1/n$. In a diagrammatic formulation the saddle point sums the contributions of all tree-like structures of longitudinal lines and the $1/n$ expansion proceeds according to the number of longitudinal loops.

4.3. $\mathscr{L}$ in Saddle Point Approximation

For the normal $(\mathbf{S}^2)^2$-theory it is known [15] that elimination of the longitudinal field component in tree-approximation below T_c and in zero external field yields the Lagrangian of the non-linear σ-model. In our case this procedure yields a Lagrangian $\mathscr{L}^0$ which is a generalization of the non-linear σ-model:

$$\mathscr{L}^0[Q^{12}]=\frac{\beta}{2}\hat{K}\int d^dr\,\mathrm{Tr}(\nabla V_\varepsilon(r))^2+\beta n a^{-d}$$
$$\cdot\int d^dr\,\mathrm{Tr}\,[(\tilde{\omega}+\widetilde{\delta h}(r))\,V_\varepsilon(r)]$$
$$+O(\nabla^4,\tilde{\omega}\nabla^2,\widetilde{\delta h}\nabla^2). \tag{4.11}$$

The constant $\hat{K}$ and the field V_ε are determined as

$$\hat{K}=\frac{nR^2}{2d}a^{-d} \tag{4.12}$$

$$V_\varepsilon(r)=\varepsilon 1+V(r) \tag{4.13}$$

where $V(r)$ is defined in (3.22). Such an effective Lagrangian with the coupling constant (4.12) has been reported in [8]. We note that the matrix $V_\varepsilon(r)$ is $Q_{\mathrm{s.p.}}$ transformed by $T(r)$ (Eq. 3.18):

$$V_\varepsilon(r)=T(r)(\varepsilon 1+\lambda_0 s)\,T^{\oplus}(r). \tag{4.14}$$

According to the discussion in connection with (3.22) $V_\varepsilon(r)$ ranges over all matrices with hermitean conjugate off diagonal blocks $Q^{12}(r)=Q^{21}(r)^+$ and with m-fold degenerate eigenvalues $\lambda^{(p)}=\varepsilon+\lambda_0 s_p$.

To derive these results we define Q^{pp} as function of $Q^{12}=Q^{21+}$ via the saddle point equation

$$0=\frac{\delta L_i}{\delta Q^{pp}(r)}-\hat{K}\,\Delta Q^{pp}(r)|_{Q^{pp}=Q^{pp}[Q^{12}]} \tag{4.15}$$

where

$$\frac{1}{\beta n}L_i=a^{-d}\int d^dr\{\tfrac{1}{2}\mathrm{Tr}(Q(r)\,Q(r))+\mathrm{Tr}\ln(\tilde{\varepsilon}-Q(r))\} \tag{4.16}$$

We decompose $Q^{pp}[Q^{12}]$ according to

$$Q^{pp}[Q^{12}(r)]=V_\varepsilon^{pp}(r)+\Gamma^p(r) \tag{4.17}$$

where V_ε^{pp} is defined by

$$0=\frac{\delta L_i}{\delta Q^{pp}(r)}\bigg|_{Q^{pp}=V_\varepsilon^{pp}}. \tag{4.18}$$

Thus Γ^p is of order ∇^2 and substituting (4.17) into the Lagrangian (4.9) we find

$$\mathscr{L}^0=\frac{\beta}{2}\hat{K}\int d^dr\,\mathrm{Tr}(\nabla V_\varepsilon(r))^2+L_i[V_\varepsilon]+\beta n a^{-d}$$
$$\cdot\int d^dr\,\mathrm{Tr}[(\tilde{\omega}+\widetilde{\delta h}(r))\,V_\varepsilon(r)]+O(\nabla^4,\tilde{\omega}\nabla^2,\widetilde{\delta h}\nabla^2) \tag{4.19}$$

where we have used (4.18) to eliminate a term proportional to Γ^p.

We still have to prove that $L_i[V_\varepsilon]$ is constant and that V_ε indeed has the form given by (4.13). This immediately follows from the way we have constructed the set of matrices $Q(r)$ (Eq. (3.19)). For fixed Q^{12} and $T_0=1$ the contribution $\lambda_0 T(r)\,s\,T^{\oplus}(r)$ is fixed, and (4.18) is fulfilled if the $P(r)$-integration hits the saddle point surface, i.e. for $P(r)=\varepsilon 1$. This immediately yields the result (4.14) which then leads to

$$L_i[V_\varepsilon]=\beta n m N(\tfrac{1}{2}(\lambda^{(1)2}+\lambda^{(2)2})$$
$$+\ln((\varepsilon-\lambda^{(1)})(\varepsilon-\lambda^{(2)}))). \tag{4.20}$$

The results (4.11) to (4.13) are a special case of a very general statement which we quote here since it allows to distinguish general aspects of the broken symmetry from properties specific for the Lagrangian L_c (Eq. (4.9)). Consider a Lagrangian

$$L=K\int d^dr\,\mathrm{Tr}(\nabla Q)^2+L_i[Q] \tag{4.21}$$

where the otherwise arbitrary local interaction is invariant under nonsingular transformations $T^{-1}QT\to Q$. Assume that the symmetry is broken in such a way that the eigenvalues of the saddle point matrix fall into l groups with $\lambda_j=\lambda^{(p)}$ for j in the p'th group and $\lambda^{(p)}\neq\lambda^{(p')}$ for $p\neq p'$. Writing the matrix as a block matrix $Q^{pp'}$ we take the diagonal blocks as longitudinal, the others as transverse. Eliminating the longitudinal blocks in tree-approximation we derive a generalized non-linear σ-model of the form (4.11) where for fixed transverse blocks the longitudinal blocks of $V_\varepsilon(r)$ are completely determined by the eigenvalue constraints $\lambda_j=\lambda^{(p)}$. We will prove this statement in Appendix B. It shows that it is not the form of $\mathscr{L}^0$ but the form of the matrices V_ε which is specific for the problem under consideration. For the disordered electron problem the V_ε have only two different eigenvalues. This and the hyperbolic symmetry characterize the physics of the disordered system.

4.4. *The Fluctuation Contribution and the Invariant Measure*

We now analyse the correction of order $1/n$ and we show that the gradient-independent part of this correction is closely connected to the invariant measure on the set of matrices $V_\varepsilon(r)$. Expanding L_c around the saddle point $Q^{pp}[Q^{12}]$ and keeping the tranverse components Q^{12},Q^{21} fixed we find

$$L_c=\mathrm{const}+\mathscr{L}^0+\frac{\beta}{2}a^{-d}\int d^dr\,\mathrm{Tr}\,\{\delta Q(r)\,\delta Q(r)$$
$$-(\tilde{\varepsilon}-V_\varepsilon(r))^{-1}\,\delta Q(r)\,(\tilde{\varepsilon}-V_\varepsilon(r))^{-1}\,\delta Q(r)\}$$
$$+O(\nabla^2,\tilde{\omega}+\widetilde{\delta h},n^{-\frac{1}{2}}) \tag{4.22}$$

where

$$n^{-\frac{1}{2}}\delta Q^{pp'}=\delta_{pp'}[Q^{pp}-Q^{pp}[Q^{12}]]. \tag{4.23}$$

The gradient-independent part of the fluctuation integral is found as

$$\mathscr{F}=\int \mathscr{D}[\delta Q]\exp\left\{-\frac{\beta}{2}a^{-d}\int d^dr\,\mathrm{Tr}\,\{\delta Q(r)\,\delta Q(r)\right.$$
$$\left.-(\tilde{\varepsilon}-V_\varepsilon(r))^{-1}\,\delta Q(r)\,(\tilde{\varepsilon}-V_\varepsilon(r))^{-1}\,\delta Q(r)\}\right\}. \tag{4.24}$$

Equations (4.14) and (3.21) yield

$$(\tilde{\varepsilon}-V_\varepsilon(r))^{-1}=U_1(r)\,R(r)\left(\frac{\tilde{\varepsilon}}{2}-\lambda_0 s\right)^{-1}R^{\oplus}(r)\,U_1^{+}(r). \tag{4.25}$$

The unitary transformation $U_1(r)$ can be absorbed into $\delta Q(r)$, and taking into account the structure (2.30) of the special psu transformation R we find for the fluctuation integral

$$\mathscr{F}=\int \mathscr{D}[\delta Q]\exp\left\{-\frac{\beta}{2}a^{-d}\int d^dr\sum_{pp',aa'}\delta Q^{pp}_{aa'}(r)\,[\delta_{pp'}\right.$$
$$\left.-X^{pp'}_{a}(r)\,X^{p'p}_{a}(r)]\,\delta Q^{pp'}_{a'a}(r)\right\} \tag{4.26}$$

where

$$X^{pp'}_{a}(r)=\left[R(r)\left(\frac{\tilde{\varepsilon}}{2}-\lambda_0 s\right)^{-1}R^{\oplus}(r)\right]^{pp'}_{aa}. \tag{4.27}$$

This integral is easily calculated since it separates into a product of integrals with fixed pairs (a,a'). Going back to the discretized form of the r-integral and using the expression

$$X_a=\begin{pmatrix}\varepsilon-i\lambda_0\cosh 2\varphi_a & -\lambda_0\sinh 2\varphi_a\\ -\lambda_0\sinh 2\varphi_a & \varepsilon+i\lambda_0\cosh 2\varphi_a\end{pmatrix} \tag{4.28}$$

which follows from (4.27), (3.10), and (2.31) we find

$$\mathscr{F}=\begin{cases}\displaystyle\prod_r\left[\left(\frac{4}{\lambda_0}\right)^{m\frac{m+1}{2}}\prod_{a\leqq a'}(\cosh 2\varphi_a+\cosh 2\varphi_{a'})^{-1}\right]\\ (\mathrm{RME})\\ \displaystyle\prod_r\left[\left(\frac{2}{\lambda_0}\right)^{m^2}\prod_{a,a'}(\cosh 2\varphi_a+\cosh 2\varphi_{a'})^{-1}\right].\\ (\mathrm{PIE})\end{cases} \tag{4.29}$$

Next we wish to introduce a measure $I(V_\varepsilon)\mathscr{D}[V_\varepsilon^{12}]$ on the saddle point surface which is invariant under psu (pso) transformations. This measure can be obtained from the mapping

$$V_\varepsilon+\delta V=T[Q_{\mathrm{s.p.}}+\delta Q]\,T^{\oplus} \tag{4.30}$$

with $Q_{\mathrm{s.p.}}+\delta Q$ (as well as $V_\varepsilon+\delta V$) on the saddle point surface. Since I times the volume spanned by the variations δV^{12} equals the volume spanned by δQ^{12} one obtains I as the Jacobian

$$I^{-1}=\delta V^{12}/\delta Q^{12}. \tag{4.31}$$

The unitary transformation U_1 in the representation (3.21) of T contributes a factor of 1 to the Jacobian and we therefore can restrict T to the set of special psu transformations R. Equation (4.30) yields the relation

$$\delta V^{12}_{aa'}=\cosh\varphi_a\,\delta Q^{12}_{aa'}\cosh\varphi_{a'}$$
$$+\sinh\varphi_a(\delta Q^{12}_{a'a})^{*}\sinh\varphi_{a'} \tag{4.32}$$

and its complex conjugate. These relations couple the (real and complex, respectively) independent differentials $\delta V^{12}_{aa'}$, $\delta V^{12}_{a'a}$ to the differentials $\delta Q^{12}_{aa'}$, $\delta Q^{12}_{a'a}$. The Jacobians are easily calculated and yield

$$I=\begin{cases}\displaystyle\prod_r\{2^{m\frac{m+1}{2}}\prod_{a\leqq a'}(\cosh 2\varphi_a+\cosh 2\varphi_{a'})^{-1}\}\\ (\mathrm{RME})\\ \displaystyle\prod_r\{2^{m^2}\prod_{a,a'}(\cosh 2\varphi_a+\cosh 2\varphi_{a'})^{-1}\}.\\ (\mathrm{PIE})\end{cases} \tag{4.33}$$

Up to a constant factor $\mathscr{F}$ and I coincide. This result shows that also in order $1/n$ the infrared singularities of the generalized non-linear σ-model and the original matrix model are identical.

5. Conclusion

We have established a matrix model of a disordered electronic system. In our derivation we have taken care that the Gaussian transformations involved are not only formally but also analytically correct. As a result we find two different models, depending on the position of the energies involved. If the energies are on the same side of the cut representing the energy band a hermitean (real symmetric, resp.) matrix model is adequate which has an internal unitary (orthogonal) symmetry, and this symmetry is not broken spontaneously. However, in the limit of vanishing imaginary part of z the Green's functions depend crucially on the half plane from which the real axis is approached. This can be interpreted as a spontaneous breaking of a discrete symmetry.

On the other hand, if the energies are on different sides of the cut, the symmetry becomes of hyperbolic type. Spontaneous symmetry breaking occurs and forces us to define the model on a more complicated set of complex non-hermitean matrices. The fact that from this latter model we also can determine the one-particle Green's function does not contradict these statements: Due to the $m=0$ property in a calcu-

lation of G_p only subspace p is relevant, and if restricted to such a subspace both models become identical. Also in this case the discrete symmetry is broken. Thus if $\tilde{\varepsilon}+V$ is a saddle point matrix contained in the domain of integration, then $\tilde{\varepsilon}-V$ does not belong to the matrices admitted for the integration even though they have the same eigenvalues.

The hyperbolic symmetry has quite unusual features. It does not correspond to a compact group and therefore a symmetry-breaking term in the Lagrangian is necessary to make the functional integrals well defined. We believe that this could be the source of the difference between the disordered electron problem and the normal second-order phase transitions. For normal phase transitions the correlation functions are completely regular above T_c. For the disordered electron problem, however, two-particle correlation functions are singular on *both* sides of the transition as ω goes to zero. Clearly this could be an effect of the infinite volume of the saddle point manifold. As another important feature we note that spontaneous symmetry breaking exists on both sides of the transition and vanishes only at the band edge.

Eliminating the longitudinal modes in tree-approximation we have derived a generalized non-linear σ-model which should govern the Goldstone modes of the broken continuous symmetry. This model is defined on the set of all matrices with hermitean conjugate off diagonal blocks and fixed $(m=0)$-fold degenerate eigenvalues $\lambda^{(p)}$. We have also been able to prove that the Gaussian fluctuations of the longitudinal degrees of freedom reproduce the invariant measure which is connected with this set of matrices. Thus it is suggested that the critical behaviour of the original model and of the generalized non-linear σ-model is identical - a suggestion which is supported also by the results of Oppermann and Wegner [2] who have shown that in perturbation theory based on the Lagrangian (3.39) the terms combine to yield a vanishing interaction as the momenta go to zero. Non-linear σ-models of the type described above (but with real eigenvalues and unitary symmetry) recently have been discussed in Ref. 12. The results for the critical behaviour coincide with the results of [2] to the extent calculated. This is not surprising since we have shown that in perturbation theory there is no difference between the normal unitary symmetry and the hyperbolic symmetry discussed here. What is more important is that these results are in agreement with the scaling laws [16] expected for the disordered electron system. This suggests that indeed the expansion on the generalized non-linear σ-model can be used in in the region of extended states. For the region of localized states, on the other hand, we do not know of any appropriate perturbation method and we believe that in this region a proper treatment of the hyperbolic symmetry is essential.

Appendices

A. Properties of Pseudounitary Operators and Related Problems

In this appendix we explicitly will consider psu operators. The result for pso operators follow from obvious modifications of the arguments.

Psu operators as introduced in (2.25) to (2.27) form a group with respect to matrix multiplication. The special psu transformations represent a subgroup: $R(\varphi)R(\varphi')=R(\varphi+\varphi')$

To derive the representation (2.28)

$$T=U_1 R U_2 \tag{A.1}$$

we decompose T into its four blocks $T^{pp'}$ and we introduce unitary operators $U_i^{(p)}$ which diagonalize T^{pp}

$$T^{pp}=U_1^{(p)}A^{(p)}U_2^{(p)}, \quad p=1,2. \tag{A.2}$$

We note that such a representation with positive diagonal matrix $A^{(p)}$ is possible for any $m\times m$ matrix. We now argue that due to pseudo-unitary also the off-diagonal blocks can be diagonalized by this transformation:

$$T^{pp'}=U_1^{(p)}A^{pp'}U_2^{(p')}, \quad p\neq p' \tag{A.3}$$

$$A^{pp'}_{aa'}=\delta_{aa'}A^{pp'}_a. \tag{A.4}$$

Pseudounitary yields

$$s_p^+ T^{+pp}s_p T^{pp}+s_p^+ T^{+pp'}s_{p'}T^{p'p}=1; \quad p'\neq p. \tag{A.5}$$

Using Eq. (A.2) and the expressions for $s_p, s_{p'}$ we find

$$(A^{(p)})^2-\mathcal{O}^+\mathcal{O}=1 \tag{A.6}$$

with

$$\mathcal{O}=U_1^{(p')+}T^{p'p}U_2^{(p)+}, \quad p'\neq p. \tag{A.7}$$

The same calculation starting from $TT^{\oplus}$ instead of $T^{\oplus}T$ yields

$$(A^{(p')})^2-\mathcal{O}\mathcal{O}^+=1. \tag{A.8}$$

Since $\mathcal{O}\mathcal{O}^+$ and $\mathcal{O}^+\mathcal{O}$ have the same eigenvalues, the diagonal matrix elements of $A^{(1)}$ and $A^{(2)}$ coincide (we use that they are positive). Multiplication of the U's by appropriate permutation operators (which themselves are orthogonal) yields

$$A^{(1)}=A^{(2)}=A. \tag{A.9}$$

Thus $\mathcal{O}$ and $\mathcal{O}^+$ commute which allows to diagonalize $\mathcal{O}$

$$\mathcal{O}=BDB^+ \tag{A.10}$$

with unitary B and diagonal D. From this one obtains

$$B^+A^2B=DD^++1. \tag{A.11}$$

Since A^2 and B^+A^2B are both diagonal and have the same eigenvalues they are identical apart from permutations which again can be absorbed into B. Then, however, it follows that $B^+A^2B=A^2$ and since A is positive $B^+AB=A$. Finally choosing $\hat{U}_1=U_1B$, $\hat{U}_2=B^+U_2$ we conclude that

$$T=\hat{U}_1A\hat{U}_2 \tag{A.12}$$

where

$$A_{aa'}^{pp'}=\delta_{aa'}A_a^{pp'}. \tag{A.13}$$

It is easily checked that A_a is psu by virtue of the block structure of $\hat{U}_i$. The most general 2×2 psu matrix with real diagonal elements has the form

$$A_a=\begin{pmatrix}\cosh\varphi_a & -ie^{i\chi}\sinh\varphi_a\\ ie^{-i\chi_a}\sinh\varphi_a & \cosh\varphi_a\end{pmatrix}. \tag{A.14}$$

The phase χ_a can be absorbed into the unitary transformations $\hat{U}_i$ and the form (A.1) with R given by (2.30), (2.31) results.

We next prove that any matrix V of the form (3.22) can be diagonalized by a psu transformation. We first note that the diagonal blocks are antihermitean and can therefore be diagonalized by a unitary block matrix

$$U^{(p)}V^{pp}U^{(p)+}=iB^{(p)} \tag{A.15}$$

where $B^{(p)}$ is real diagonal. From the equation

$$(V-\lambda^{(1)})(V-\lambda^{(2)})=0 \tag{A.16}$$

which expresses the fact that V has only two different eigenvalues $\lambda^{(p)}$ it follows that U diagonalizes also the off-diagonal blocks $V^{pp'}$, in complete analogy to the argument given above. We thus find

$$V=U^+BU \tag{A.17}$$

where

$$B_{aa'}^{pp'}=\delta_{aa'}B_a^{pp'} \tag{A.18}$$

The matrix B_a is of the structure given by (3.22):

$$B_a=\begin{pmatrix}-i(\lambda_0^2+|B_a^{12}|^2)^{\frac{1}{2}} & B_a^{12}\\ B_a^{12*} & i(\lambda_0^2+|B_a^{12}|^2)^{\frac{1}{2}}\end{pmatrix}. \tag{A.19}$$

It is easily checked that this matrix is diagonalized by a transformation of the form (A.14) with

$$\sinh 2\varphi_a=|B_a^{12}|/\lambda_0,\qquad e^{i\chi_a}=B_a^{12}/|B_a^{12}|.$$

B. Generalized σ-Model for Arbitrary Interaction

Let $Q=Q(r)$ be a hermitean matrix field. We consider a Lagrangian of the type

$$L=K\int d^dr\,\mathrm{Tr}(\nabla Q(r))^2+L_i[Q] \tag{B.1}$$

where L_i is local and is invariant under arbitrary non-singular transformations. We eliminate a subset of the matrix elements of Q in tree approximation. We want to show that as a result we find a Lagrangian

$$\mathcal{L}^0=K\int d^dr\,\mathrm{Tr}(\nabla V(r))^2+O(\nabla^4) \tag{B.2}$$

provided the set of longitudinal elements was chosen correctly. We first note that for any choice of longitudinal elements we can go through the steps given in equations (4.15) to (4.19). As a result we only have to prove

$$L_i[V]=\text{const} \tag{B.3}$$

where the longitudinal elements $V^l(r)$ of V are defined as function of the transverse elements $V^t(r)\equiv Q^t(r)$ by the saddle point condition

$$\frac{\delta L_i}{\delta V^l(r)}=0. \tag{B.4}$$

By virtue of the symmetry $L_i[Q]$ depends only on the eigenvalues of the $N\times N$ matrix Q and thus equation (B.4) can be written as

$$0=\sum_{\nu=1}^{N}\frac{\delta\lambda_\nu}{\delta V^l(r)}\frac{\partial L_i}{\partial\lambda_\nu}\bigg|_{\lambda_\nu=\lambda_\nu[V]}. \tag{B.5}$$

We want to prove

$$0=\frac{\partial L_i}{\partial\lambda_\nu}\bigg|_{\lambda_\nu=\lambda_\nu[V]} \tag{B.6}$$

for all ν and all $V=V[V^t]$. It will be sufficient to prove this relation for a neighbourhood of the mean field solution

$$V_{\mathrm{s.p.},cc'}=\delta_{cc'}M_c \tag{B.7}$$

since then it follows for all V by analytic continuation. Our proof consists in showing that with the proper choice of V^l (B.5) yields a set of N homogeneous equations for the N derivatives $\partial L_i/\partial\lambda_\nu$, and the determinant $\mathrm{Det}(\delta\lambda_\nu/\delta V^l)$ of this set is different from zero in a neighbourhood of $V_{\mathrm{s.p.}}$. To calculate λ_ν we use the exact reformulation of the eigenvalue problem

$$P_\nu[Q-\lambda_\nu-Q\bar{P}_\nu[\bar{P}_\nu Q\ \bar{P}_\nu-\lambda_\nu]^{-1}\bar{P}_\nu Q]\bar{P}_\nu|x_\nu\rangle=0 \tag{B.8}$$

where $|x_\nu\rangle$ is the eigenvector with eigenvalue λ_ν and P_ν is the orthogonal projection on the subspace of components c with $M_c = M_\nu = \lambda_\nu[V(V^t=0)]$. $\bar{P}_\nu$ denotes the orthogonal complement to P_ν: $\bar{P}_\nu + P_\nu = 1$.

Let us first assume that the symmetry is completely broken, i.e. $M_{c'} \neq M_c$ if $c \neq c'$. Then we identify the set of longitudinal components with the set of all diagonal elements of Q. Differentiating (B.8) with respect to Q_{cc} we find

$$\frac{\partial \lambda_\nu}{\partial Q_{cc}} = \delta_{c\nu} + O(Q^t)^2 \tag{B.9}$$

and therefore

$$\mathrm{Det}\left|\frac{\partial \lambda_\nu}{\partial Q_{cc}}\right| = 1 + O(Q^t)^2. \tag{B.10}$$

Equation (B.6) follows. If the eigenvalue λ_ν is degenerate on the mean field level, i.e. $M_c = M_\nu$ for $c = c_1, \ldots, c_k$; $c_1 \equiv \nu$, then we first diagonalize Q in the k-dimensional subspace $P_\nu|x\rangle = |x\rangle$

$$\bar{Q}_{c_i c_j} = \delta_{ij} \sum_{l,m=1}^{k} U_{il} Q_{c_l c_m} (U^+)_{mj}. \tag{B.11}$$

Equation (B.9) then holds for the derivative $\partial/\partial \bar{Q}_{c_i c_i}$, and since this derivative is a linear combination of the derivatives $\partial/\partial Q_{c_i c_j}$ Eq. (B.6) holds, provided we include in the set of longitudinal components all $Q_{cc'}$ with $M_c = M_{c'}$. In this way the block structure emerges.

In general any solution $(\lambda_1, \ldots, \lambda_N)$ of (B.6) represents a point in eigenvalue space well separated from the other solutions and therefore the elimination of the longitudinal components leads to a model with fixed eigenvalues $(\lambda_1, \ldots, \lambda_N)$, and $L_i[V]$ is constant. There may be exceptional cases in which the solutions of Eq. (B.6) form a continuous set and then the eigenvalues of V can vary over this set. However, $L_i[V]$ has still to be constant, and in this exceptional case L_i essentially possesses another symmetry not connected to the transformation T, which should be properly taken into account. We also note that the choice of a hermitean matrix model is not essential. The same arguments apply e.g. to the set defined in (3.19).

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Note Added in Proof

The non-compactness of the symmetry group has also been observed by G. Parisi (preprint obtained after submission of the present paper).