

DOLBEAULT COHOMOLOGY OF COMPACT NILMANIFOLDS

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Abstract. Let $M = G/\Gamma$ be a compact nilmanifold endowed with an invariant complex structure. We prove that on an open set of any connected component of the moduli space $\mathcal{C}(\mathfrak{g})$ of invariant complex structures on M , the Dolbeault cohomology of M is isomorphic to the cohomology of the differential bigraded algebra associated to the complexification $\mathfrak{g}^{\mathbb{C}}$ of the Lie algebra of G . To obtain this result, we first prove the above isomorphism for compact nilmanifolds endowed with a rational invariant complex structure. This is done using a descending series associated to the complex structure and the Borel spectral sequences for the corresponding set of holomorphic fibrations. Then we apply the theory of Kodaira–Spencer for deformations of complex structures.

Introduction

Let M be a compact nilmanifold of real dimension $2n$, i.e., the quotient of a connected simply connected $(s+1)$ -step nilpotent real Lie group G by a uniform discrete subgroup Γ . It follows from a result of Mal'čev [Ma] that, for any simply connected real nilpotent Lie group G , for which the coefficients in the structure equations are rational numbers, there is a lattice Γ in G of maximal rank (i.e., a discrete uniform subgroup, cf. [Ra]). We will let Γ act on G on the left. It is well known that such a lattice Γ exists in G if and only if the Lie algebra \mathfrak{g} of G has a rational structure, i.e., if there exists a rational Lie subalgebra $\mathfrak{g}_{\mathbb{Q}}$ such that $\mathfrak{g} \cong \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$.

The de Rham cohomology of a compact nilmanifold can be computed by means of the cohomology of the Lie algebra of the corresponding nilpotent Lie group (Nomizu's Theorem [No]).

We assume that M has an invariant complex structure J , that is that J comes from a (left invariant) complex structure J on \mathfrak{g} . Our aim is to relate the Dolbeault cohomology of M with the cohomology ring $H_{\bar{\partial}}^{*,*}(\mathfrak{g}^{\mathbb{C}})$ of the differential bigraded algebra $\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^*$, associated to $\mathfrak{g}^{\mathbb{C}}$ with respect to the operator $\bar{\partial}$ in the canonical decomposition $d = \partial + \bar{\partial}$ on $\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^*$.

The study of the Dolbeault cohomology of nilmanifolds with an invariant complex structure is motivated by the fact that the latter provided the first known examples of

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compact symplectic manifolds which do not admit any Kähler structure [Ab, CFG, Th].

Since there exists a natural map $i : H_{\bar{\partial}}^{*,*}(\mathfrak{g}^{\mathbb{C}}) \rightarrow H_{\bar{\partial}}^{*,*}(M)$ that is always injective (cf. Lemma 9), the problem we will study is to see for which complex structure J on M the above map gives an isomorphism

$$H_{\bar{\partial}}^{p,q}(M) \cong H_{\bar{\partial}}^{p,q}(\mathfrak{g}^{\mathbb{C}}). \quad (1)$$

Note that $H_{\bar{\partial}}^{*,*}(\mathfrak{g}^{\mathbb{C}})$ can be identified with the cohomology of the Dolbeault complex of the forms on G that are invariant by the left action of G (we shall call them briefly G -invariant forms) and $H_{\bar{\partial}}^{*,*}(M)$ with the cohomology of the Dolbeault complex of Γ -invariant forms on G . We shall use these identifications throughout.

Our main result is the following theorem.

Theorem 1. *The isomorphism (1) holds on an open set of any connected component of the moduli space $\mathcal{C}(\mathfrak{g})$ of invariant complex structures on M .*

To obtain Theorem 1 we first consider the case of complex structures J that are rational, i.e., they are compatible with the rational structure of G ($J(\mathfrak{g}_{\mathbb{Q}}) \subseteq \mathfrak{g}_{\mathbb{Q}}$).

Theorem 2. *For any rational complex structure J , the isomorphism (1) holds.*

It is an open problem as to whether the isomorphism (1) holds for any compact nilmanifold endowed with an arbitrary invariant complex structure. We do not know examples for which (1) does not hold.

Theorem 1 will follow from Theorem 2 using the theory of deformations of complex structures [KS, Su]. Indeed by [Sa] the set $\mathcal{C}(\mathfrak{g})$ of complex structures on \mathfrak{g} is at least infinitesimally a complex variety. Using the theory of deformations of complex structures, we are able to prove that for any small deformation of a rational complex structure J , the isomorphism (1) holds (Lemma 10).

If M is a compact complex parallelizable nilmanifold, i.e., G is a nilpotent complex Lie group and J is also right invariant, we have that $\mathfrak{g}_J^i = \mathfrak{g}^i$ and Theorem 2 follows from [Sak, Theorem 1].

An important class of complex structures is given by the abelian ones (i.e., those satisfying the condition $[JX, JY] = [X, Y]$, for any $X, Y \in \mathfrak{g}$ [BD, DF]). The nilmanifolds with an abelian complex structure are to some extent dual to complex parallelizable nilmanifolds: indeed, in the complex parallelizable case $d\lambda^{1,0} \subset \lambda^{2,0}$, and in the abelian case $d\lambda^{i,0} \subset \lambda^{i,1}$, where $\lambda^{p,q}$ denotes the space of (p, q) -forms on \mathfrak{g} . In this last case we will compute the minimal model of the Dolbeault cohomology of M and prove that the isomorphism (1) holds for any abelian complex structure. In [CFGU] a similar result was proved for the Dolbeault cohomology of M endowed with a nilpotent complex structure; this is a slight generalization of the abelian one.

Note that, if M is a complex solvmanifold G/Γ (with G a solvable not nilpotent Lie group), the isomorphism (1) does not hold in general, as shown in [Le]; a discussion of the behavior of the Dolbeault cohomology of homogeneous manifolds under group actions can be found in [Ak]. If G is a compact even dimensional (semisimple) Lie group endowed with a left invariant complex structure, the Dolbeault cohomology of G does not arise from just invariant classes, as the example in [Pi] shows.

This paper is organized as follows: In Section 1, following [Sa], we define a descending series of subalgebras $\{\mathfrak{g}_J^i\}$ (with $\mathfrak{g}_J^0 = \mathfrak{g}$ and $\mathfrak{g}_J^{s+1} = \{0\}$) for the Lie algebra \mathfrak{g} associated

to the complex structure J . In general, the subalgebra \mathfrak{g}_J^i is not a rational subalgebra of \mathfrak{g}_J^{i-1} . If J is rational, any \mathfrak{g}_J^i is rational in \mathfrak{g}_J^{i-1} .

The importance of this series is twofold. First (Section 2), in case the subalgebra \mathfrak{g}_J^i is rational in \mathfrak{g}_J^{i-1} (in particular if J is rational), it allows us to define a set of holomorphic fibrations of nilmanifolds:

$$\begin{aligned} \tilde{p}_0: M = G/\Gamma &\rightarrow G_J^{0,1}/p_0(\Gamma) \quad \text{with standard fibre } G_J^1/\Gamma^1, \\ \tilde{p}_{s-1}: G_J^{s-1}/\Gamma^{s-1} &\rightarrow G_J^{s-1,s}/p_{s-1}(\Gamma^{s-1}) \quad \text{with standard fibre } G_J^s/\Gamma^s. \end{aligned}$$

For the above fibrations we will consider the associated Borel spectral sequence (E_r, d_r) ([Hi, Appendix II by A. Borel, Theorem 2.1], [FW, LeP]), which relates the Dolbeault cohomology of each total space with the Dolbeault cohomology of each base and fibre.

Second (Section 3), following [Sa], we will prove that one can choose a basis of $(1,0)$ -forms (and also $(0,1)$ -forms) on \mathfrak{g} which is compatible with the above descending series. This basis will give a basis of $G_J^{i-1,i}$ -invariant $(1,0)$ -forms on the nilmanifolds $G_J^{i-1,i}/p_{i-1}(\Gamma^{i-1})$ and of G_J^i -invariant $(1,0)$ -forms on the nilmanifolds G_J^i/Γ^i , $i = 0, \dots, s$.

Next, in Section 4, we consider a spectral sequence $(\tilde{E}_r, \tilde{d}_r)$ concerning the G_J^{i-1} -invariant Dolbeault cohomology of each total space G_J^{i-1}/Γ^{i-1} , the $G_J^{i-1,i}$ -invariant Dolbeault cohomology of each base $G_J^{i-1,i}/p_{i-1}(\Gamma^{i-1})$ and the G_J^i -invariant Dolbeault cohomology of each fibre G_J^i/Γ^i . In this way $(\tilde{E}_r, \tilde{d}_r)$ is relative to the Dolbeault cohomologies of the Lie algebras \mathfrak{g}_J^i , \mathfrak{g}_J^{i-1} and $\mathfrak{g}_J^{i-1}/\mathfrak{g}_J^i$. Note that the latter are the underlying Lie algebras of the fibre, the total space and the base, respectively, of the above holomorphic fibrations.

In Section 5 we compare the spectral sequence $(\tilde{E}_r, \tilde{d}_r)$ with the Borel spectral sequence (E_r, d_r) . Inductively (starting with $i = s$) these two spectral sequences allow us to give isomorphisms between the Dolbeault cohomologies of the total spaces and the cohomology of the corresponding Lie algebras. The last step gives Theorem 2.

Note that our construction of this set of holomorphic fibrations is in the same vein as principal holomorphic torus towers, introduced by Barth and Otte [BO]. In some cases, like the case of abelian complex structures, G/Γ is really a principal holomorphic torus tower.

In Section 6 we will give a proof of Theorem 1. In Section 7 we give examples of compact nilmanifolds with non-rational complex structures.

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1. A descending series associated to the complex structure

We recall that, since G is $(s + 1)$ -step nilpotent, one has the descending central series $\{\mathfrak{g}^i\}_{i \geq 0}$, where

$$\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] \supseteq \mathfrak{g}^2 = [\mathfrak{g}^1, \mathfrak{g}] \supseteq \dots \supseteq \mathfrak{g}^s \supseteq \mathfrak{g}^{s+1} = \{0\}. \tag{D}$$

We define the following subspaces of \mathfrak{g}

$$\mathfrak{g}_J^i := \mathfrak{g}^i + J\mathfrak{g}^i.$$

Note that \mathfrak{g}_J^i is J -invariant.

- Lemma 3.** (1) \mathfrak{g}_J^i is an ideal of \mathfrak{g}_J^{i-1} .
 (2) $\mathfrak{g}_J^{i-1}/\mathfrak{g}_J^i$ is an abelian algebra.
 (3) \mathfrak{g}_J^s is an abelian ideal of \mathfrak{g}_J^{s-1} .

Proof. (1) For any $X = X_1 + JX_2 \in \mathfrak{g}_J^{i-1}$ and $Y = Y_1 + JY_2 \in \mathfrak{g}_J^i$ (with $X_l \in \mathfrak{g}^{i-1}$ and $Y_k \in \mathfrak{g}^i$), we have that $[X, Y] = [X_1, Y_1] + [X_1, JY_2] + [JX_2, Y_1] + [JX_2, JY_2]$. We can easily see that $[X_1, Y_1]$, $[X_1, JY_2]$ and $[JX_2, Y_1]$ belong to \mathfrak{g}^i by definition of the descending central series. Moreover $[JX_2, JY_2]$ belongs to \mathfrak{g}_J^i because J satisfies an integrability condition, namely, the Nijenhuis tensor N of J , given by $N(Z, W) = [Z, W] + J[JZ, W] + J[Z, JW] - [JZ, JW]$, $Z, W \in \mathfrak{g}$, must be zero [NN].

(2) For any $X = X_1 + JX_2, Y = Y_1 + JY_2$ elements of \mathfrak{g}_J^{i-1} , we have that $[X + \mathfrak{g}_J^i, Y + \mathfrak{g}_J^i] = [X_1, Y_1] + [X_1, JY_2] + [JX_2, Y_1] + [JX_2, JY_2] + \mathfrak{g}_J^i$. Then using the same argument as in (1) it follows that $[X + \mathfrak{g}_J^i, Y + \mathfrak{g}_J^i] = \mathfrak{g}_J^i$.

(3) Using the fact that \mathfrak{g}^s is central (i.e., $[\mathfrak{g}^s, \mathfrak{g}] = 0$) and that $N = 0$ it is possible to prove that $[X, Y]$ vanishes for any $X, Y \in \mathfrak{g}_J^s$. \square

Observe moreover that any \mathfrak{g}_J^i is nilpotent. Hence we have the descending series

$$\mathfrak{g} = \mathfrak{g}_J^0 \supset \mathfrak{g}_J^1 \supset \mathfrak{g}_J^2 \supset \dots \supset \mathfrak{g}_J^s \supset \mathfrak{g}_J^{s+1} = \{0\}. \tag{DJ}$$

Remark 1. The first inclusion $\mathfrak{g}_J^1 \subset \mathfrak{g}$ is always strict [Sa, Corollary 1.4].

Observe that in case of complex parallelizable nilmanifolds [Sak], the filtration $\{\mathfrak{g}_J^i\}$ coincides with the descending central series $\{\mathfrak{g}^i\}$ and then the $\{\mathfrak{g}_J^i\}$ are rational. In general, given a rational structure $\mathfrak{g}_\mathbb{Q}$ for \mathfrak{g} , we say that a \mathbb{R} -subspace \mathfrak{h} of \mathfrak{g} is rational if \mathfrak{h} is the \mathbb{R} -span of $\mathfrak{h}_\mathbb{Q} = \mathfrak{h} \cap \mathfrak{g}_\mathbb{Q}$. In general \mathfrak{g}_J^1 is not a rational subalgebra of \mathfrak{g} . When J is rational, it is possible to prove that \mathfrak{g}_J^i is rational in \mathfrak{g}_J^{i-1} . Indeed, we have that \mathfrak{g}^i is rational in \mathfrak{g}^{i-1} . Then $\mathfrak{g}^i = \mathbb{R}\text{-span}\{\mathfrak{g}^i \cap \mathfrak{g}_\mathbb{Q}^{i-1}\}$. Since $J\mathfrak{g}_\mathbb{Q}^{i-1} \subseteq \mathfrak{g}_\mathbb{Q}^{i-1}$, it follows that $\mathfrak{g}_J^i = \mathbb{R}\text{-span}\{\mathfrak{g}_J^i \cap \mathfrak{g}_\mathbb{Q}^{i-1}\}$. Moreover when J is abelian, \mathfrak{g}_J^i is an ideal of \mathfrak{g} , for any i and the center

$$\mathfrak{g}_1 = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] = 0\}$$

is a rational J -invariant ideal of \mathfrak{g} .

2. Holomorphic fibrations and Borel spectral sequences

In this section we assume that the complex structure J is rational and we associate a set of holomorphic fibrations to the above descending series. We recall that a *holomorphic fibre bundle* $\pi : T \rightarrow B$ is a holomorphic map between the complex manifolds T and B , which is locally trivial, and whose typical fibre F is a complex manifold such that the transition functions are holomorphic. By definition the structure group (i.e., the group of holomorphic automorphisms of the typical fibre) is a complex Lie group.

To define the above fibrations, we first consider the surjective homomorphism

$$p_{i-1} : \mathfrak{g}_J^{i-1} \rightarrow \mathfrak{g}_J^{i-1}/\mathfrak{g}_J^i,$$

for each $i = 1, \dots, s$. If G_J^{i-1} and $G_J^{i-1,i}$ denote the simply connected nilpotent Lie group corresponding to \mathfrak{g}_J^{i-1} and $\mathfrak{g}_J^{i-1}/\mathfrak{g}_J^i$ respectively, we have the surjective homomorphism

$$p_{i-1} : G_J^{i-1} \rightarrow G_J^{i-1,i}.$$

We define inductively G_J^i to be the fibre of p_{i-1} : Remark that the Lie algebra of G_J^i is \mathfrak{g}_J^i .

Given the uniform discrete subgroup Γ of $G = G^0$, we consider the continuous surjective map

$$\tilde{p}_0 : G/\Gamma \rightarrow G_J^{0,1}/p_0(\Gamma).$$

Since J is rational, \mathfrak{g}_J^1 is a rational subalgebra of \mathfrak{g} , and then $\Gamma^1 := \Gamma \cap G_J^1$ is a uniform discrete subgroup of G_J^1 [CG, Theorem 5.1.11]. Then, by [CG, Lemma 5.1.4 (a)], $p_0(\Gamma)$ is a uniform discrete subgroup of $G_J^{0,1}$ (i.e., $G_J^{0,1}/p_0(\Gamma)$ is compact, cf. [Ra]). Note moreover that G_J^1 is simply connected. This follows from the homotopy exact sequence of the fibering p_0 . Indeed we have $\dots \rightarrow \pi_2(G_J^{0,1}) = (e) \rightarrow \pi_1(G_J^1) \rightarrow \pi_1(G) = (e) \rightarrow \dots$. Finally it is not difficult to see that G_J^1 is connected. Indeed, if C is the connected component of the identity in G_J^1 , $\text{id} : G \rightarrow G$ induces a covering homomorphism $G/C \rightarrow G/G_J^1 \cong G_J^{0,1}$ which must be the identity, since $G_J^{0,1} \cong \mathbb{R}^{N_0}$. Thus $C = G_J^1$.

Now one can repeat the same construction for any i , since \mathfrak{g}_J^i is a rational ideal of \mathfrak{g}_J^{i-1} . So, for any $i = 1, \dots, s$ we have a map

$$\tilde{p}_{i-1} : G_J^{i-1}/\Gamma^{i-1} \rightarrow G_J^{i-1,i}/p_{i-1}(\Gamma^{i-1}).$$

Lemma 4. $\tilde{p}_{i-1} : G_J^{i-1}/\Gamma^{i-1} \rightarrow G_J^{i-1,i}/p_{i-1}(\Gamma^{i-1})$ is a holomorphic fibre bundle.

Proof. Observe first that \tilde{p}_{i-1} is the induced map of p_{i-1} taking quotients of discrete subgroups. The tangent map of \tilde{p}_{i-1}

$$\mathfrak{g}_J^{i-1} \rightarrow \mathfrak{g}_J^{i-1}/\mathfrak{g}_J^i$$

is J -invariant. Thus \tilde{p}_{i-1} is a holomorphic submersion. In particular, it is a holomorphic family of compact complex manifolds in the terminology of [KS] (see also [Sun]). The fibres of \tilde{p}_{i-1} are all holomorphically equivalent to G_J^i/Γ^i (the typical fibre). Thus a theorem of Grauert and Fisher [FG] applies, implying that \tilde{p}_{i-1} is a holomorphic fibre bundle. \square

Note that G_J^{i-1}/Γ^{i-1} , G_J^i/Γ^i , $G_J^{i-1,i}/p_{i-1}(\Gamma^{i-1})$ are compact connected nilmanifolds.

Given a holomorphic fibre bundle it is possible to construct the associated Borel spectral sequence that relates the Dolbeault cohomology of the total space T with that of the basis B and the fibre F . We will need the following Theorem (which follows from [Hi, Appendix II by A. Borel, Theorem 2.1] and [FW]).

Theorem 5. *Let $p : T \rightarrow B$ be a holomorphic fibre bundle with compact connected fibre F and T and B connected. Assume that either (I) F is Kähler or (I') the scalar cohomology bundle $\mathbf{H}^{u,v}(F) = \bigcup_{b \in B} H_{\bar{\partial}}^{u,v}(p^{-1}(b))$ is trivial.*

Then there exists a spectral sequence (E_r, d_r) ($r \geq 0$) with the following properties:

(i) E_r is 4-graded by the fibre degree, the base degree, and the type. Let ${}^{p,q}E_r^{u,v}$ be the subspace of elements of E_r of type (p, q) , fibre degree u , and base degree v . We have ${}^{p,q}E_r^{u,v} = 0$ if $p + q \neq u + v$ or if any p, q, u, v is negative. The differential d_r maps ${}^{p,q}E_r^{u,v}$ into ${}^{p,q+1}E_{r+1}^{u+r, v-r+1}$.

(ii) If $p + q = u + v$, then ${}^{p,q}E_2^{u,v} \cong \sum_k H_{\bar{\partial}}^{k, u-k}(B) \otimes H_{\bar{\partial}}^{p-k, q-u+k}(F)$.

(iii) The spectral sequence converges to $H_{\bar{\partial}}(T)$.

3. An adapted basis of (1, 0)-forms

In this section we prove that one can choose a basis of (1,0)-forms on \mathfrak{g} that is compatible with the descending series (DJ) . We consider, as in [Sa], some subspaces V_i ($i = 0, \dots, s + 1$) of $V := (T_e G)^* \cong \mathfrak{g}^*$, that determine a series, which is related to the descending central series (D) .

Indeed, we define

$$V_0 = \{0\}, V_1 = \{\alpha \in V \mid d\alpha = 0\}, \dots, V_i = \{\alpha \in V \mid d\alpha \in \Lambda^2 V_{i-1}\}, \dots, V_{s+1} = V.$$

Note that V_i is the annihilator $(\mathfrak{g}^i)^\circ$ of the subspace \mathfrak{g}^i and that $\{0\} = V_0 \subseteq V_1 \subseteq \dots \subseteq V_{s+1} = V$ [Sa, Lemma 1.1].

If we now let $(\mathfrak{g}_J^i)^\circ \cap \lambda^{1,0} =: V_i^{1,0}$, by [Sa, Lemma 1.2], we have that there exists a basis of (1,0)-forms $\{\omega_1, \dots, \omega_n\}$ such that if $\omega_l \in V_i^{1,0}$, then $d\omega_l$ belongs to the ideal (in $(\mathfrak{g}^{\mathbb{C}})^*$) generated by $V_{i-1}^{1,0}$ [Sa, Theorem 1.3]. In particular, there exists at least a closed (1, 0)-form (this implies Remark 1).

Moreover we have the following isomorphisms: $(\mathfrak{g}_J^{i-1}/\mathfrak{g}_J^i)^{\mathbb{C}} \cong V_i^{1,0}/V_{i-1}^{1,0} \oplus V_i^{0,1}/V_{i-1}^{0,1}$, $i = 1, \dots, s$, where $V_i^{0,1}$ is the conjugate of $V_i^{1,0}$. With respect to the subspaces $V_i^{1,0}$ the above basis can be ordered as follows (we let $n_i := \dim_{\mathbb{C}} \mathfrak{g}_J^i$): $\omega_1, \dots, \omega_{n-n_1}$ are elements of $V_1^{1,0}$ (such that $d\omega_l = 0$) or $\mathfrak{g}/\mathfrak{g}_J^1$ is the real vector space underlying $V_1^{1,0}$; $\omega_{n-n_1+1}, \dots, \omega_{n-n_2}$ are elements of $V_2^{1,0} \setminus V_1^{1,0}$ or $\mathfrak{g}_J^1/\mathfrak{g}_J^2$ is the real vector space underlying the quotient $V_2^{1,0}/V_1^{1,0}$; \dots ; $\omega_{n-n_{s-1}+1}, \dots, \omega_{n-n_s}$ are elements of $V_s^{1,0} \setminus V_{s-1}^{1,0}$; $\omega_{n-n_s+1}, \dots, \omega_n$ are elements of $\lambda^{1,0} \setminus V_s^{1,0}$. Here $V_i^{1,0} \setminus V_{i-1}^{1,0}$ denotes a complement of $V_{i-1}^{1,0}$ in $V_i^{1,0}$ (which corresponds to the choice of a complement of \mathfrak{g}_J^i in \mathfrak{g}_J^{i-1}). Hence, by definition, the elements of $V_i^{1,0} \setminus V_{i-1}^{1,0}$ and, by identification, the elements of the quotient $V_i^{1,0}/V_{i-1}^{1,0}$, are (1,0)-forms on \mathfrak{g} which vanish on \mathfrak{g}_J^i . So they may be identified with forms on the quotient $\mathfrak{g}_J^{i-1}/\mathfrak{g}_J^i$.

In this way we can consider:

- the elements of $\lambda^{1,0}/V_1^{1,0} = V_2^{1,0}/V_1^{1,0} \oplus V_3^{1,0}/V_2^{1,0} \oplus \dots \oplus \lambda_{1,0}/V_s^{1,0}$ as (1, 0)-forms on \mathfrak{g}_J^1 ,
- the elements of $\lambda^{1,0}/V_{s-1}^{1,0} = V_s^{1,0}/V_{s-1}^{1,0} \oplus \lambda^{1,0}/V_s^{1,0}$ as (1, 0)-forms on \mathfrak{g}_J^{s-1} and
- the elements of $\lambda^{1,0}/V_s^{1,0}$ as (1,0)-forms on \mathfrak{g}_J^s .

Thus we can prove a lemma on the existence of a basis of (1, 0) forms on \mathfrak{g} related to the series (DJ) .

Lemma 6. *It is possible to choose a basis of $(1, 0)$ -forms on \mathfrak{g} such that (with respect to the previous order) $\{\omega_{n-n_{i-1}+1}, \dots, \omega_{n-n_i}, \dots, \omega_n\}$ is a basis of $(1, 0)$ -forms on \mathfrak{g}_J^{i-1} . Moreover we can consider (up to identifications) $\{\omega_{n-n_i+1}, \dots, \omega_n\}$ as forms on \mathfrak{g}_J^i and $\{\omega_{n-n_{i-1}+1}, \dots, \omega_{n-n_i}\}$ as forms on $\mathfrak{g}_J^{i-1}/\mathfrak{g}_J^i$.*

Proof. By the above arguments, for any $i = 1, \dots, s + 1$, it is possible to choose a basis of $(1, 0)$ -forms on \mathfrak{g}_J^{i-1} as elements of $\lambda^{1,0}/V_{i-1}^{1,0} = \lambda^{1,0}/V_i^{1,0} \oplus V_i^{1,0}/V_{i-1}^{1,0}$. With respect to the above decomposition the forms on $\lambda^{1,0}/V_i^{1,0}$ can be identified with forms on \mathfrak{g}_J^i extended by zero on \mathfrak{g}_J^{i-1} and the forms on $V_i^{1,0}/V_{i-1}^{1,0}$ with forms on $\mathfrak{g}_J^{i-1}/\mathfrak{g}_J^i$, because these forms vanish on \mathfrak{g}_J^i . \square

Remark 2. $d\omega_i, i = n - n_{i-1} + 1, \dots, n - n_i$ belongs to the ideal generated by $\{\omega_l, l = 1, \dots, n - n_{i-1}\}$.

Remark 3. If J is abelian, it is possible to choose a basis of $(1, 0)$ -forms $\{\omega_1, \dots, \omega_n\}$ on \mathfrak{g} such that $d\omega_i \in \Lambda^2 \langle \omega_1, \dots, \omega_{i-1}, \bar{\omega}_1, \dots, \bar{\omega}_{i-1} \rangle \cap \lambda^{1,1}$.

4. A spectral sequence for the complex of invariant forms

We construct a spectral sequence ${}^{p,q}\tilde{E}_r^{u,v}$ for the complexes of G_J^{s-1} -invariant forms on G_J^{s-1}/Γ^{s-1} whose Dolbeault cohomology identifies with $H_{\bar{\partial}}((\mathfrak{g}_J^{s-1})^{\mathbb{C}})$. To do this, we give a filtration of the complex $\Lambda_{\mathfrak{t}} = \oplus \Lambda_{\mathfrak{t}}^{p,q}$ of differential forms of type (p, q) on $\mathfrak{t} = \mathfrak{g}_J^{i-1}$.

We know from Lemma 6 that there exists a basis of $(1, 0)$ -forms $\omega_h^{\mathfrak{t}}$ on \mathfrak{t} (and of $(0, 1)$ -forms $\bar{\omega}_h^{\mathfrak{t}}$) such that part of them are $(1, 0)$ -forms $\omega_j^{\mathfrak{b}}$ on $\mathfrak{b} = \mathfrak{g}_J^{i-1}/\mathfrak{g}_J^i$ and part are forms $\omega_k^{\mathfrak{f}}$ on $\mathfrak{f} = \mathfrak{g}_J^i$. We define

$$\tilde{L}_k := \{\omega^{\mathfrak{t}} \in \Lambda_{\mathfrak{t}} \mid \omega^{\mathfrak{t}} \text{ is a sum of monomials } \omega_j^{\mathfrak{b}} \wedge \bar{\omega}_j^{\mathfrak{b}} \wedge \omega_{\mathfrak{f}}^{\mathfrak{f}} \wedge \bar{\omega}_{\mathfrak{f}}^{\mathfrak{f}}, \text{ in which } |I| + |J| \geq k\},$$

where $|A|$ denotes the number of elements of the finite set A . Note that $\tilde{L}_0 = \Lambda_{\mathfrak{t}}$ and that $\tilde{L}_k = 0$ for $k > \dim_{\mathbb{R}} \mathfrak{b}$, $\tilde{L}_k \supset \tilde{L}_{k+1}$, $\bar{\partial}\tilde{L}_k \subseteq \tilde{L}_k, k \geq 0$. The above shows that $\{\tilde{L}_k\}$ defines a bounded decreasing filtration of the differential module $(\Lambda_{\mathfrak{t}}, \bar{\partial})$. Of course $\tilde{L}_k = \sum_{p,q} {}^{p,q}\tilde{L}_k, {}^{p,q}\tilde{L}_k = \tilde{L}_k \cap \Lambda_{\mathfrak{t}}^{p,q}$ and the filtration is compatible with the bigrading provided by the type (and also with the total degree).

Recall that, by definition (see e.g., [GH]) ${}^{p,q}\tilde{E}_r^{u,v} = {}^{p,q}Z_r^{u,v}/({}^{p,q}Z_{r-1}^{u+1,v-1} + {}^{p,q}B_{r-1}^{u,v})$, where

$$\begin{aligned} {}^{p,q}Z_r^{u,v} &= {}^{p,q}\tilde{L}_u(\Lambda_{\mathfrak{t}}^{u+v}) \cap \ker \bar{\partial}({}^{p,q+1}\tilde{L}_{u+r}(\Lambda_{\mathfrak{t}}^{u+v+1})), \\ {}^{p,q}B_r^{u,v} &= {}^{p,q}\tilde{L}_u(\Lambda_{\mathfrak{t}}^{u+v}) \cap \bar{\partial}({}^{p,q-1}\tilde{L}_{u+r}(\Lambda_{\mathfrak{t}}^{u+v-1})). \end{aligned}$$

Moreover (cf. [GH]), ${}^{p,q}\tilde{E}_0^u = {}^{p,q}\tilde{L}_u/{}^{p,q}\tilde{L}_{u+1}$, where we denote by ${}^{p,q}\tilde{E}_r^{u,v}$ and ${}^{p,q}\tilde{E}_r^u$ the spaces of elements of type (p, q) and total degree $u + v$ and degree u , respectively, in the grading defined by the filtration.

Note also that an element of \tilde{L}^k is identified with an element of $\sum_{c+d \geq k} \Lambda_{\mathfrak{f}}^{a,b} \otimes \Lambda_{\mathfrak{b}}^{c,d}$.

Lemma 7. *Given the holomorphic fibration (with standard fibre G_J^i/Γ^i) $G_J^{i-1}/\Gamma^{i-1} \rightarrow G_J^{i-1,i}/p_{i-1}(\Gamma^{i-1})$, the spectral sequence $(\tilde{E}_r, \tilde{d}_r)$ ($r \geq 0$) converges to $H_{\bar{\partial}}((\mathfrak{g}_J^{i-1})^{\mathbb{C}})$ and*

$${}^{p,q}\tilde{E}_2^{u,v} \cong \sum_k H_{\bar{\partial}}^{k,u-k}((\mathfrak{g}_J^{i-1}/\mathfrak{g}_J^i)^{\mathbb{C}}) \otimes H_{\bar{\partial}}^{p-k,q-u+k}((\mathfrak{g}_J^i)^{\mathbb{C}}). \tag{Ii}$$

Proof. The fact that $(\tilde{E}_r, \tilde{d}_r)$ converges to $H_{\bar{\partial}}((\mathfrak{g}_J^{i-1})^{\mathbb{C}})$ is a general property of spectral sequences associated to filtered complexes (cf. [GH]).

Let $[\omega] \in {}^{p,q}\tilde{E}_0^u = \frac{{}^{p,q}\tilde{L}_u}{{}^{p,q}\tilde{L}_{u+1}}$. We compute the differential $\tilde{d}_0 : {}^{p,q}\tilde{E}_0^u \rightarrow {}^{p,q+1}\tilde{E}_0^u$ defined by $\tilde{d}_0[\omega] = [\bar{\partial}\omega]$. We can write (up to the above identifications) $\omega = \sum \omega_J^b \wedge \bar{\omega}_J^b \wedge \omega_{J'}^f \wedge \bar{\omega}_{J'}^f$, where $|I| + |J| = u$ (because we operate mod \tilde{L}_{u+1}), and $|I'| + |J'| = p + q - u$, since $|I'| + |J'| + |I| + |J| = p + q$. Moreover, using the fact that $\bar{\partial}$ sends forms in \mathfrak{b} on forms that either are in \mathfrak{b} or vanish on \mathfrak{b} (cf. Remark 2) and that \mathfrak{b} is abelian, we get $\bar{\partial}\omega = \sum \bar{\partial}_b(\omega_I^b \wedge \bar{\omega}_J^b) \wedge (\omega_{J'}^f \wedge \bar{\omega}_{J'}^f) + (-1)^s(\omega_I^b \wedge \bar{\omega}_J^b) \wedge \bar{\partial}(\omega_{J'}^f \wedge \bar{\omega}_{J'}^f) = (-1)^s(\omega_I^b \wedge \bar{\omega}_J^b) \wedge \bar{\partial}_f(\omega_{J'}^f \wedge \bar{\omega}_{J'}^f) \text{ mod } \tilde{L}_{u+1}$, where $\bar{\partial}_b$ and $\bar{\partial}_f$ denote the differential on the complexes Λ_b and Λ_f , respectively. Thus $\tilde{d}_0[\omega] = [\bar{\partial}_f\omega]$, which implies

$${}^{p,q}\tilde{E}_1^u \cong \sum_k H_{\bar{\partial}}^{p-k, q-u+k}(\mathfrak{f}^{\mathbb{C}}) \otimes \Lambda_b^{k, u-k}. \tag{k1}$$

Performing the same proof as in [Hi, Appendix II by Borel, Section 6] and using the fact that \mathfrak{b} is abelian, it is possible to prove that \tilde{d}_1 identifies with $\bar{\partial}$ via the above isomorphism and that we have

$${}^{p,q}\tilde{E}_2^u \cong \sum_k H_{\bar{\partial}}^{k, u-k}(\mathfrak{b}^{\mathbb{C}}) \otimes H_{\bar{\partial}}^{p-k, q-u+k}(\mathfrak{f}^{\mathbb{C}}) (\cong \sum_k \Lambda_b^{k, u-k} \otimes H_{\bar{\partial}}^{p-k, q-u+k}(\mathfrak{f}^{\mathbb{C}})). \quad \square \tag{k2}$$

5. Proof of Theorem 2

First we note that Theorem 2 is trivially true if the nilmanifold comes from an abelian group, i.e., it is a complex torus. Namely, if A/Γ is a complex torus, we have

$$H_{\bar{\partial}}(A/\Gamma) \cong H_{\bar{\partial}}(\mathfrak{a}^{\mathbb{C}}). \tag{a}$$

We consider the holomorphic fibrations $G_J^s/\Gamma^s \hookrightarrow G_J^{s-1}/\Gamma^{s-1} \rightarrow G_J^{s-1,s}/p_{s-1}(\Gamma^{s-1}), \dots, G_J^1/\Gamma^1 \hookrightarrow G/\Gamma \rightarrow G_J^{0,1}/p_0(\Gamma)$. The aim is to obtain information about the Dolbeault cohomology of G/Γ inductively through the Dolbeault cohomologies of G_J^i/Γ^i (the nilmanifolds G_J^i/Γ^i play alternately the roles of fibres and total spaces of the above fibre bundles). Note that since the bases are complex tori, $H_{\bar{\partial}}(G_J^{i,i-1}/p_{i-1}(\Gamma^{i-1})) \cong H_{\bar{\partial}}((\mathfrak{g}^{i-1}/\mathfrak{g}^i)^{\mathbb{C}})$.

To this end we will associate to these fibrations two spectral sequences. The first is a version of the Borel spectral sequence (considered in Section 2) which relates the Dolbeault cohomologies of the total spaces with those of fibres and bases. The second is the spectral sequence $(\tilde{E}_r, \tilde{d}_r)$ (constructed in the previous section) relative to the Dolbeault cohomologies of the Lie algebras $\mathfrak{g}^i, \mathfrak{g}^{i-1}, \mathfrak{g}^{i-1}/\mathfrak{g}^i$.

We will proceed inductively on the index i in the descending series (DJ) , starting from $i = s$.

First inductive step. Let us use the holomorphic fibre bundle $\tilde{p}_{s-1} : G_J^{s-1}/\Gamma^{s-1} \rightarrow G_J^{s-1,s}/p_{s-1}(\Gamma^{s-1})$ with typical fibre G_J^s/Γ^s . Recall (cf. Lemma 3) that G_J^s/Γ^s and $G_J^{s-1,s}/p_{s-1}(\Gamma^{s-1})$ are complex tori. Thus by (a)

$$H_{\bar{\partial}}(G_J^s/\Gamma^s) \cong H_{\bar{\partial}}((\mathfrak{g}_J^s)^{\mathbb{C}}), \tag{s}$$

$$H_{\bar{\partial}}(G_J^{s-1,s}/p_{s-1}(\Gamma^{s-1})) \cong H_{\bar{\partial}}((\mathfrak{g}_J^{s-1}/\mathfrak{g}_J^s)^{\mathbb{C}}). \tag{s, s-1}$$

Applying Theorem 5 (since the fibre G_J^s/Γ^s is Kähler) and using (s) and (s, s - 1), we get

$${}^{p,q}E_2^{u,v} \cong \sum_k H_{\bar{\partial}}^{k,u-k}((\mathfrak{g}_J^{s-1}/\mathfrak{g}_J^s)^{\mathbb{C}}) \otimes H_{\bar{\partial}}^{p-k,q-u+k}((\mathfrak{g}_J^s)^{\mathbb{C}}). \tag{Is}$$

Next we use the spectral sequence $(\tilde{E}_r, \tilde{d}_r)$. Note that the inclusion between the Dolbeault complex of G_J^{s-1} -invariant forms on G_J^{s-1}/Γ^{s-1} and the forms on G_J^{s-1}/Γ^{s-1} induces an inclusion of each term in the spectral sequences ${}^{p,q}\tilde{E}_r^{u,v} \subseteq {}^{p,q}E_r^{u,v}$ (which is actually a morphism of spectral sequences).

By Lemma 7, for $i = s$, we have that $(\tilde{E}_r, \tilde{d}_r)$ converges to $H_{\bar{\partial}}((\mathfrak{g}_J^{s-1})^{\mathbb{C}})$ and

$${}^{p,q}\tilde{E}_2^{u,v} \cong \sum_k H_{\bar{\partial}}^{k,u-k}((\mathfrak{g}_J^{s-1}/\mathfrak{g}_J^s)^{\mathbb{C}}) \otimes H_{\bar{\partial}}^{p-k,q-u+k}((\mathfrak{g}_J^s)^{\mathbb{C}}). \tag{\tilde{I}s}$$

Comparing (Is) with ($\tilde{I}s$), we get that $E_2 = \tilde{E}_2$, hence the spectral sequences (E_r, d_r) and $(\tilde{E}_r, \tilde{d}_r)$ converge to the same cohomologies. Thus

$$H_{\bar{\partial}}(G_J^{s-1}/\Gamma^{s-1}) \cong H_{\bar{\partial}}((\mathfrak{g}_J^{s-1})^{\mathbb{C}}). \tag{s-1}$$

General inductive step. We use the holomorphic fibre bundle $\tilde{p}_{i-1} : G_J^{i-1}/\Gamma^{i-1} \rightarrow G_J^{i-1,i}/p_{i-1}(\Gamma^{i-1})$ with the typical fibre G_J^i/Γ^i . We assume inductively that

$$H_{\bar{\partial}}(G_J^i/\Gamma^i) \cong H_{\bar{\partial}}((\mathfrak{g}_J^i)^{\mathbb{C}}). \tag{i}$$

Lemma 8. *The scalar cohomology bundle*

$$\mathbf{H}^{u,v}(G_J^i/\Gamma^i) = \bigcup_{b \in G_J^{i-1,i}/p_{i-1}(\Gamma^i)} H_{\bar{\partial}}^{u,v}(\tilde{p}_{i-1}^{-1}(b))$$

is trivial.

Proof. By [KS, Section 5, Formula 5.3] there exists a locally finite covering $\{U_l\}$ of $G_J^{i-1,i}/p_{i-1}(\Gamma^{i-1})$ such that the action of the structure group of the holomorphic fiber bundle on $U_l \cap (G_J^i/\Gamma^i)$ is the differential of the change of complex coordinates on the fibre, so one can restrict oneself to considering the left translation by elements of G_J^{i-1} as change of coordinates. Then the scalar cohomology bundle $\mathbf{H}^{u,v}(G_J^i/\Gamma^i)$ is trivial since any of its fibres is canonically isomorphic to $H_{\bar{\partial}}((\mathfrak{g}_J^i)^{\mathbb{C}})$. More explicitly, a global frame for $\mathbf{H}^{u,v}(G_J^i/\Gamma^i)$ is given as follows: for any cohomology class $\alpha \in H_{\bar{\partial}}^{u,v}(\tilde{p}_{i-1}^{-1}(1)) = H_{\bar{\partial}}^{u,v}(G_J^i/\Gamma^i) \cong H_{\bar{\partial}}^{u,v}((\mathfrak{g}_J^i)^{\mathbb{C}})$, (1: identity element of $G_J^{i-1,i}/p_{i-1}(\Gamma^i)$) one can take the corresponding cohomology class $\omega \in H_{\bar{\partial}}^{u,v}((\mathfrak{g}_J^i)^{\mathbb{C}})$ and regard it as a G_J^{i-1} -invariant differential form on G_J^{i-1}/Γ^{i-1} . Thus $b \mapsto \omega|_{\tilde{p}_{i-1}^{-1}(b)}$ gives a global holomorphic section of $\mathbf{H}^{u,v}(G_J^i/\Gamma^i)$. Taking a basis of $H_{\bar{\partial}}((\mathfrak{g}_J^i)^{\mathbb{C}})$ one gets a global holomorphic frame of $\mathbf{H}^{u,v}(G_J^i/\Gamma^i)$. \square

Thus the assumption (I') in Theorem 5 is fulfilled. Observe moreover that, since $G_J^{i-1, i}/p_{i-1}(\Gamma^{i-1})$ is a complex torus, by (a),

$$H_{\bar{\partial}}(G_J^{i-1, i}/p_{i-1}(\Gamma^{i-1})) \cong H_{\bar{\partial}}((\mathfrak{g}_J^{i-1}/\mathfrak{g}_J^i)^{\mathbb{C}}). \quad (i, i-1)$$

Hence, by Theorem 5, we have

$${}^{p, q}E_2^{u, v} \cong \sum_k H_{\bar{\partial}}^{k, u-k}((\mathfrak{g}_J^{i-1}/\mathfrak{g}_J^i)^{\mathbb{C}}) \otimes H_{\bar{\partial}}^{p-k, q-u+k}((\mathfrak{g}_J^i)^{\mathbb{C}}). \quad (Ii)$$

Then we proceed exactly as in the first inductive step. Namely, we consider the spectral sequence ${}^{p, q}\tilde{E}_r^{u, v}$ for the complexes of G_J^{i-1} -invariant forms on G_J^{i-1}/Γ^{i-1} whose Dolbeault cohomology identifies with $H_{\bar{\partial}}((\mathfrak{g}_J^{i-1})^{\mathbb{C}})$. By Lemma 7 we have that $(\tilde{E}_r, \tilde{d}_r)$ converges to $H_{\bar{\partial}}((\mathfrak{g}_J^{i-1})^{\mathbb{C}})$ and

$${}^{p, q}\tilde{E}_2^{u, v} \cong \sum_k H_{\bar{\partial}}^{k, u-k}((\mathfrak{g}_J^{i-1}/\mathfrak{g}_J^i)^{\mathbb{C}}) \otimes H_{\bar{\partial}}^{p-k, q-u+k}((\mathfrak{g}_J^i)^{\mathbb{C}}). \quad (\tilde{I}i)$$

Using (Ii) and proceeding as in the first inductive step, we get the proof of Theorem 2.

Remark 4. (On abelian complex structures) If the invariant structure J is abelian, the centre \mathfrak{g}_1 is a rational J -invariant ideal of \mathfrak{g} (cf. Remark 1 in Section 1).

We give an alternative (and simpler) proof of that given in [CFGU] in the abelian case. In the same vein of [No] we consider the principal holomorphic fibre bundle $G/\Gamma \rightarrow G/(\Gamma G_1)$, with typical fibre $G_1/(\Gamma \cap G_1) \cong \Gamma G_1/G_1$ (where G_1 is the simply connected Lie group corresponding to \mathfrak{g}_1).

First we can consider the Borel spectral sequence (E_r, d_r) associated to the Dolbeault complex $\Lambda^{*,*}(G/\Gamma)$ (cf. Theorem 5) and a spectral sequence $(\tilde{E}_r, \tilde{d}_r)$ associated to $\Lambda^{*,*}(\mathfrak{g})^{\mathbb{C}}$ and constructed as in Section 4. As to the latter spectral sequence $(\tilde{E}_r, \tilde{d}_r)$, observe that it is possible to prove that there exists a basis $\{\omega_1, \dots, \omega_n\}$ of $(1, 0)$ -forms on \mathfrak{g} such that $\{\omega_1, \dots, \omega_{n-k}\}$ is a basis on $\mathfrak{g}/\mathfrak{g}_1$ (with $\dim \mathfrak{g}_1 = 2k$) and $\{\omega_{n-k+1}, \dots, \omega_n\}$ is a basis on \mathfrak{g}_1 . So one can perform the same construction as in Section 4 (with $\mathfrak{t} = \mathfrak{g}$, $\mathfrak{b} = \mathfrak{g}/\mathfrak{g}_1$ and $\mathfrak{f} = \mathfrak{g}_1$), using the fact that \mathfrak{f} is abelian. Thus one gets ${}^{p, q}\tilde{E}_2^{u, v} \cong \sum_k H_{\bar{\partial}}^{k, u-k}((\mathfrak{g}/\mathfrak{g}_1)^{\mathbb{C}}) \otimes H_{\bar{\partial}}^{p-k, q-u+k}((\mathfrak{g}_1)^{\mathbb{C}})$ and (\tilde{E}_r, d_r) converges to $H_{\bar{\partial}}((\mathfrak{g})^{\mathbb{C}})$. Moreover, since $G_1/\Gamma \cap G_1$ and $G/(\Gamma G_1)$ are complex tori, $H_{\bar{\partial}}(G_1/\Gamma \cap G_1) \cong H_{\bar{\partial}}((\mathfrak{g}_1)^{\mathbb{C}})$ and $H_{\bar{\partial}}(G/(\Gamma G_1)) \cong H_{\bar{\partial}}((\mathfrak{g}/\mathfrak{g}_1)^{\mathbb{C}})$, by Theorem 5 we have ${}^{p, q}E_2^{u, v} \cong \sum_k H_{\bar{\partial}}^{k, u-k}(G/(\Gamma G_1)) \otimes H_{\bar{\partial}}^{p-k, q-u+k}((\mathfrak{g}_1)^{\mathbb{C}})$. This implies that Theorem 2 holds also for complex abelian structures.

Next we construct a minimal model for the Dolbeault cohomology of a nilmanifold endowed with an abelian complex structure.

Recall that a *model* for the Dolbeault cohomology of M is a differential bigraded algebra $(\mathcal{M}^{*,*}, \bar{\partial})$ for which there exists a homomorphism $\rho : \mathcal{M}^{*,*} \rightarrow \Lambda^{*,*}(M)$ of differential bigraded algebras inducing an isomorphism ρ^* on the respective Dolbeault cohomologies [NT].

Suppose \mathcal{M} is free on a vector space V . Then $\bar{\partial}$ is called decomposable if there is an ordered basis of V such that the differential $\bar{\partial}$ of any generator v of V can be expressed

in terms of the elements of the basis preceding v . A model is called *minimal* if \mathcal{M} is free and $\bar{\partial}$ is decomposable [Su].

By [Sa], in the abelian case, there exists a basis $\{\omega_1, \dots, \omega_n\}$ of $(1, 0)$ -forms on \mathfrak{g} such that

$$d\omega_i = \sum_{j < k < i} A_{ijk} \omega_j \wedge \bar{\omega}_k, \quad i = 1, \dots, n. \tag{m}$$

Thus, by Theorem 2 and (m), $(\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}}), \bar{\partial})$ is a minimal model for the Dolbeault cohomology of G/Γ .

6. Proof of Theorem 1

Let $\mathcal{C}(\mathfrak{g}) = \{J \in \text{End}(\mathfrak{g}) \mid J^2 = -id, [JX, JY] = [X, Y] + J[JX, Y] + J[X, JY]\}$ denote the set of complex structures on \mathfrak{g} . We will use the same notation as in [Sa]. If $M = G/\Gamma$ is a nilmanifold associated to \mathfrak{g} , then

$$0 \rightarrow \text{Hom}(\lambda^{1,0}, \lambda^{0,1}) \xrightarrow{\bar{\partial}} \text{Hom}(\lambda^{1,0}, \lambda^{0,2}) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \text{Hom}(\lambda^{1,0}, \lambda^{0,n}) \rightarrow 0$$

is a subcomplex of the Dolbeault complex of M tensored with the holomorphic tangent bundle $T^{1,0}M$. The kernel K of $\bar{\partial}$ acting on $\text{Hom}(\lambda^{1,0}, \lambda^{1,0})$ can be identified with the subspace of invariant classes in the sheaf cohomology space $H^1(M, \mathcal{O}(T))$. By [Sa, Proposition 4.1] if J is a smooth point of $\mathcal{C}(\mathfrak{g})$, then the tangent space $T_J\mathcal{C}(\mathfrak{g})$ to $\mathcal{C}(\mathfrak{g})$ is contained in the complex subspace of $T_J\mathcal{C}$ (where $\mathcal{C} \cong GL(2n, \mathbb{R})/GL(n, \mathbb{C})$ is the set of all almost complex structures on \mathfrak{g}) determined by $K = \ker \bar{\partial}$.

By the above section we know that if J_0 is a rational complex structure, then $H_{\bar{\partial}}^{p,q}(M) \cong H_{\bar{\partial}}^{p,q}(\mathfrak{g}^{\mathbb{C}})$. Given $J_0 \in \mathcal{C}(\mathfrak{g})$, we know by [KS] that there exists a complete complex analytic family $\{M_t = (M, J_t) \mid J_t \in B\}$.

Let $\bar{\partial}_t$ and Δ_t be respectively the $\bar{\partial}$ -operator and the Laplacian determined by the global inner product g_t induced by an invariant Hermitian metric on M compatible with J_t . More precisely, $\Delta_t = \bar{\partial}_t^* \bar{\partial}_t + \bar{\partial}_t \bar{\partial}_t^*$, where $\bar{\partial}_t^*$ is the adjoint of $\bar{\partial}_t$ with respect to g_t .

Lemma 9. (i) Δ_t sends G -invariant forms of type (p, q) to G -invariant forms. Moreover, the orthogonal complement with respect to g_t of the invariant forms on the space $\Lambda_t^{p,q}$ of Γ -invariant forms of type (p, q) on (M, J_t) is preserved by Δ_t .

(ii) $H_{\bar{\partial}}^{p,q}(\mathfrak{g}^{\mathbb{C}})$ is a subspace of $H_{\bar{\partial}}^{p,q}(M)$ for any invariant complex structure J on M .

Proof. (i) follows given that $\bar{\partial}_t$ and $\bar{\partial}_t^*$ preserve G -invariant forms.

(ii) We have to prove that there exists an injective homomorphism $\pi : H_{\bar{\partial}}^{p,q}(\mathfrak{g}^{\mathbb{C}}) \rightarrow H_{\bar{\partial}}^{p,q}(M)$. By the decomposition $\Lambda^{p,q} = \mathcal{H}^{p,q} \oplus \text{Im} \bar{\partial} \oplus \text{Im} \bar{\partial}^*$, where $\mathcal{H}^{p,q}$ denotes the space of Γ -invariant harmonic forms of type (p, q) on M , we have similarly, for the G -invariant forms, the decomposition $\Lambda_{inv}^{p,q} = \mathcal{H}_{inv}^{p,q} \oplus \text{Im} \bar{\partial}|_{inv} \oplus \text{Im} \bar{\partial}^*|_{inv}$, where $H_{\bar{\partial}}^{p,q}(\mathfrak{g}^{\mathbb{C}}) \cong \mathcal{H}_{inv}^{p,q}$.

We can define $\pi(\omega)$ as the orthogonal projection of ω on $\mathcal{H}^{p,q} = (\text{Im} \bar{\partial} \oplus \text{Im} \bar{\partial}^*)^\perp$, for any $[\omega] \in H_{\bar{\partial}}^{p,q}(\mathfrak{g}^{\mathbb{C}})$. To prove that π is injective, suppose that $\pi(\omega) = 0$ on $H_{\bar{\partial}}^{p,q}(M)$. Then $\pi(\omega) = \bar{\partial}\varphi$. We may assume that φ belongs to the orthogonal complement to the G -invariant forms. Since $\bar{\partial}\varphi$ is G -invariant, we have also that $\bar{\partial}^*(\bar{\partial}\varphi)$ is invariant and then the inner product of φ with $\bar{\partial}^*(\bar{\partial}\varphi)$ is zero and thus φ must be G -invariant, i.e., $[\omega] = 0$ in $H_{\bar{\partial}}^{p,q}(\mathfrak{g}^{\mathbb{C}})$. \square

Lemma 10. *For any small deformation of the rational complex structure J , the isomorphism (1) holds.*

Proof. Using the same proof as in [KS, pp.67–68] it is possible to show that for any (p, q) and on any (M, J_t) there exists a complete orthonormal set of forms of type (p, q) orthogonal to those that are G -invariant $\{e_{t1}^{p,q}, \dots; f_{t1}^{p,q}, \dots; g_{t1}^{p,q}, \dots\}$ such that

- (i) $\Delta_t e_{tj}^{p,q} = 0$, for any $j = 1, \dots, d_q$, $d_q = \dim \mathcal{H}_\perp^{p,q}$, where $\mathcal{H}_\perp^{p,q}$ is the space of harmonic forms of type (p, q) orthogonal to those that are G -invariant;
- (ii) $\Delta_t f_{tj}^{p,q} = a_j^{p,q}(t) f_{tj}^{p,q}$;
- (iii) $\Delta_t g_{tj}^{p,q} = a_j^{p,q-1}(t) g_{tj}^{p,q}$; with $a_j^{p,q}(t) > 0$.

Let c be a positive constant and denote by $\nu^{p,q}(t)$ the number of eigenvalues $a_j^{p,q}(t)$ of Δ_t such that $a_j^{p,q}(t) < c$. Let $\Lambda_t^{p,q}(c)_\perp$ be the subspace spanned by the eigenforms of Δ_t (orthogonal to the G -invariant ones) such that the corresponding eigenvalues are less than c . Thus we have $\dim \Lambda_t^{p,q}(c)_\perp = h_{p,q}(t)^\perp + \nu^{p,q}(t) + \nu^{p,q-1}(t)$, where $h_{p,q}(t)^\perp$ is the dimension of the orthogonal complement $H_{\bar{\partial}}^{p,q}(M_t)^\perp$ of the space of G -invariant forms $H_{\bar{\partial}}^{p,q}(\mathfrak{g}^{\mathbb{C}})$ in $H_{\bar{\partial}}^{p,q}(M_t)$. Given a point $J_{t_0} \in \mathcal{C}(\mathfrak{g})$, we can choose c such that $0 < c < a_j^{p,q}(t_0)$ for any $p, q = 0, \dots, n$ and for any j . Moreover let U be a sufficiently small neighbourhood of t_0 in $\mathcal{C}(\mathfrak{g})$. Since any eigenvalue $a_j^{p,q}(t)$ is a continuous function of t [KS, Theorem 2, p.47] we have that the dimension of $\Lambda_t^{p,q}(c)_\perp$ is independent of $t \in U$ and $\Lambda_{t_0}^{p,q}(c)_\perp = H_{\bar{\partial}}^{p,q}(M_{t_0})^\perp$. Thus we have $h_{p,q}(t)^\perp + \nu^{p,q}(t) + \nu^{p,q-1}(t) = h_{p,q}(t_0)^\perp$, for any $t \in U$. This means that the function $h_{p,q}(t)^\perp$ is upper semicontinuous. Since, for any rational complex structure J_0 , we have that $h_{p,q}(t_0)^\perp = 0$, then $h_{p,q}(t)^\perp = 0$ in some neighbourhood of t_0 . Consequently the set $\{t \in \mathcal{C}(\mathfrak{g}) \mid h_{p,q}(t)^\perp = 0\} = \{t \in \mathcal{C}(\mathfrak{g}) \mid H_{\bar{\partial}}^{*,*}(M_t) \cong H_{\bar{\partial}}^{*,*}(\mathfrak{g}^{\mathbb{C}})\}$ is an open set in any connected component of $\mathcal{C}(\mathfrak{g})$ with respect to the induced topology of $\mathfrak{gl}(2n, \mathbb{R})$ on $\mathcal{C}(\mathfrak{g})$. \square

7. Examples of compact nilmanifolds with non-rational complex structures

Let $M = \Gamma \backslash G$ be the Iwasawa manifold. Recall that it can be constructed by taking as a nilpotent Lie group G the complex Heisenberg group

$$G = \left\{ \begin{bmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{bmatrix} : z_i \in \mathbb{C}, i = 1, 2, 3 \right\},$$

and, as a lattice Γ , the subgroup of G consisting of those matrices whose entries are Gaussian integers. It is known that the 1-forms $\omega_1 = dz_1, \omega_2 = dz_2, \omega_3 = dz_3 + z_1 dz_2$, are left invariant on G . G has structure equations $d\omega_1 = d\omega_2 = 0, \quad d\omega_3 = \omega_1 \wedge \omega_2$. If one regards G as a real Lie group and sets

$$\omega_1 =: e^1 + ie^2, \quad \omega_2 =: e^3 + ie^4, \quad \omega_3 =: e^5 + ie^6, \tag{2}$$

then (e^i) is a real basis of \mathfrak{g}^* such that

$$\begin{cases} de^i = 0, & 1 \leq i \leq 4, \\ de^5 = e^1 \wedge e^3 + e^4 \wedge e^2, \\ de^6 = e^1 \wedge e^4 + e^2 \wedge e^3. \end{cases}$$

By [Na], the bi-invariant complex structure J_0 defined by (2) on the complex Heisenberg group has a deformation space of positive dimension. Then the complex structure associated to the subspace $\langle \omega_1, \omega_2, \omega_3 + t\bar{\omega}_1 \rangle$ of $\mathfrak{g}_{\mathbb{C}}^*$ belongs to the orbit of J_0 in $\mathcal{C}(\mathfrak{g})$ with respect to the group of automorphisms of \mathfrak{g} (see [Sa, Section 4]) and it is not rational for appropriate $t \in \mathbb{C}$.

In the same way one can see that every compact nilmanifold $M = G/\Gamma$ of real dimension $2n$, with at least one (rational) complex structure J_0 having a non-trivial deformation, has a non-rational complex structure. Indeed, by [Sa, Theorem 1.3], given the complex structure J_0 it is possible to construct a basis of left invariant $(1, 0)$ -forms $\{\omega_1, \dots, \omega_n\}$ such that $d\omega_{i+1}$ belongs to the ideal generated by the set $\{\omega_1, \dots, \omega_i\}$ ($i = 0, \dots, n-1$) in the complexified exterior algebra. Then the complex structure associated to the subspace $\langle \omega_1, \dots, \omega_n + t\bar{\omega}_1 \rangle$ is a not rational complex structure on M for appropriate $t \in \mathbb{C}$.

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