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Thermally Driven Acoustic Oscillations, Part III: Second-Order Heat Flux

By Nikolaus Rott, Swiss Federal Institute of Technology, Zurich

1. Introduction

In a recent paper, Merkli and Thomann [1] have given a theory of the distribution of the second-order heat-flux out of a closed gas-filled tube in which a piston-driven standing acoustic wave is maintained. Their theory is based on the first-order solution given by Iberall [2] and by Bergh and Tijdeman [3], in which no restrictions are imposed on the extent of the dissipative layer. For the case that this region is thin compared to the tube radius, the special results of [1] were derived by Rott [4], based on boundary-layer theory; the connexion between the results of the two papers were also briefly discussed. Actually, the final heat-flux formula of [1] is formally simpler than the more restricted result of [4].

The purpose of this paper is to calculate the second-order heat flux for thermally driven acoustic oscillations, which were treated by the author in two previous papers [5], [6]. The acoustic streaming in this case has been already calculated by the author [7], albeit only for thin boundary layers; an effort to do the same type of calculation for 'thermoacoustic streaming', i.e., for the second-order heat flux, has led to a practically unmanageable number of terms, due to the additional effect of a wall-temperature gradient. On the other hand, a generalization of the theory of [1] to the case of variable wall temperature was successful and is presented here. The situation found for thermoacoustic streaming in this respect is opposed to that for mass acoustic streaming, where no closed-form results could be obtained unless thin boundary layers were assumed. The following discussion will shed some light on the reasons for the simplicity of the second-order energy equation obtained without the restriction to thin boundary layers.

2. The Energy Equation

A derivation of the second order energy equation was given by Merkli and Thomann [1]. Therefore, only an outline of a slightly different approach is given here, which is based on the full energy equation for the gas motion in a tube, for which, however, the 'dynamic' boundary-layer simplifications are applied. Neglected are the radial pressure gradient and all dissipative terms involving a differentiation in axial direction. On the other hand, no restriction is applied to the extent of the dissipative region relative to the tube cross-section. (These are the assumptions made for the linear theories in [1], [2], [3], [5] and [6].) Not surprisingly, the non-linear energy equation under these assumptions turns out to be almost identical to the well-known boundary-layer energy equation (see, e.g., Howarth [8], p. 397), with the only differences due to the fact that now the flow in the tube is described by cylindrical coordinates, x and r (say). A further modification through the continuity equation leads to the following form:

$$\frac{\partial(\rho H)}{\partial t} + \frac{\partial(\rho u H)}{\partial x} + \frac{\partial(r\rho v H)}{r\partial r} - \frac{\partial p}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left[\mu r\frac{\partial}{\partial r}\left(\frac{u^2}{2}\right)\right] \tag{1}$$

where

$$H = c_p T + \tfrac{1}{2}u^2. \tag{2}$$

Integration of the equation from 0 to r_w after multiplication with rdr, and time-averaging gives (with $u = v = 0$ at $r = r_w$):

$$r_w\left(\overline{k\frac{\partial T}{\partial r}}\right)_w = \frac{\partial}{\partial x}\int_0^{r_w}\overline{\rho u H}\,rdr. \tag{3}$$

For the purposes of the present paper, an expansion is made in the amplitude of the motion, and the equation is written down that is obtained in second order. For this equation it suffices to replace H by $c_p T$, and to expand $T = T_m + T_1 + T_2 + \cdots$ as well as all other quantities, as has been described in detail elsewhere ([1], [4] and [7]). The result can be simplified by the use of the second-order continuity equation

$$\int_0^{r_w}\overline{\rho_1 u_1}\,rdr + \int_0^{r_w}\rho_m\overline{u_2}\,rdr = 0 \tag{4}$$

to give the following final result:

$$-\bar{q}_2 = \left(k_m\frac{\partial\overline{T_2}}{\partial r}\right)_w + \left(\overline{k_1\frac{\partial T_1}{\partial r}}\right)_w = \frac{1}{2\pi r_w}\frac{\partial\overline{Q_2}}{\partial x} \tag{5}$$

$$\overline{Q}_2 = 2\pi\int_0^{r_w}\rho_m c_p\overline{T_1 u_1}\,rdr. \tag{6}$$

Here, $\overline{q_2}$ is the mean second-order heat flux to the tube wall, per unit area. $\overline{Q_2}$, the total axial second-order heat flux in the gas (over the whole tube cross-section) is the quantity of main interest for the applications here. Thus, the differentiation in (5) does not have to be carried out. This is particularly advantageous in the case of variable wall temperature, when many quantities do depend on x.

The quantity $\overline{Q_2}$ does in general not vanish anywhere except at a closed fixed end. (Any possible axial heat flow through conduction at a closed end is neglected, consistently with simplifications stated above). The determination of the second order mean temperature distribution $\overline{T_2}$ would require one more integration, just as many as the calculation of $\overline{u_2}$. By restricting our attention to $\overline{Q_2}$, the problem is strongly simplified, in particular as $\overline{u_2}$ is not needed for the evaluation of (6).

The relation expressed by (5) and (6) was given by Merkli and Thomann [1]. To obtain the special results in the case of thin viscous and thermal layers, it is not permissible to restrict the evaluation of (6) to the boundary layer. The 'core' has an important contribution to the integral. Indeed, the only function of state whose second-order flux is negligible in the core is the entropy. Based on this fact, it is possible to derive the equivalent of (5) for thin dissipative layers directly, as was shown in [4]. The effect of the core-contribution to (6) has given rise to an additional term in the equation equivalent to (5), which was interpreted as the exchange of mechanical energy between the core and the boundary layer. This term does not appear in a form which is readily integrable with respect to x, and therefore its use in the case of a variable wall temperature is impractical, as was noted above. The subsequent calculations are restricted to the evaluation of (6), based on the results of [5]. Naturally, it is possible to find the special results for thin dissipative layers *a posteriori*.

3. The Evaluation of $\overline{Q_2}$

It remains to insert into (6) the quantities u_1 and T_1 calculated in [5]. The following results are needed:

$$u_1 = \frac{i}{\omega\rho_m}\frac{dp_1}{dx}\left\{1 - \frac{J_0(i\eta)}{J_0(i\eta_w)}\right\}e^{i\omega t} \tag{7}$$

and

$$\begin{aligned}\rho_m c_p T_1 = &\left[p_1 - \frac{\theta}{(\gamma-1)(1-\sigma)}\frac{a^2}{\omega^2}\frac{dp_1}{dx}\right]\left\{1 - \frac{J_0(i\eta\sqrt{\sigma})}{J_0(i\eta_w\sqrt{\sigma})}\right\}e^{i\omega t}\\ &+ \frac{\sigma\theta}{(\gamma-1)(1-\sigma)}\frac{a^2}{\omega^2}\frac{dp_1}{dx}\left\{1 - \frac{J_0(i\eta)}{J_0(i\eta_w)}\right\}e^{i\omega t}\end{aligned} \tag{8}$$

where

$$\eta = \left(\frac{i\omega}{\nu}\right)^{1/2} r, \qquad \eta_w = \left(\frac{i\omega}{\nu}\right)^{1/2} r_w \tag{9}$$

and $\theta = d\log T_m/dx$. It is noted that the second term of (8) has exactly the same radial distribution as (7), and furthermore is out of phase with u_1 by $\pi/2$. Thus, the corresponding contribution to the integral (6) is zero. The time-average in (6) can be computed in complex form [7], and the result of the integration is given without further details:

$$\overline{Q_2} = \pi r_w^2 \,\mathrm{Re}\left\{\left[\tfrac{1}{2}u_{1_c}\tilde{p}_1 - \frac{i}{2\omega}\frac{\theta\rho_m a^2}{(\gamma-1)(1-\sigma)}u_{1_c}\tilde{u}_{1_c}\right]g\right\} \tag{10}$$

where

$$u_{1_c} = \frac{i}{\omega \rho_m} \frac{dp_1}{dx} \tag{11}$$

and

$$g = 1 - \frac{\sigma}{1+\sigma} f(\eta_w) - \frac{1}{1+\sigma} \tilde{f}^*(\eta_w). \tag{12}$$

Complex conjugates are indicated by a tilde. The quantity f was introduced in [5] as follows:

$$f(\eta_w) = \frac{2J_1(i\eta_w)}{i\eta_w J_0(i\eta_w)}, \qquad f^*(\eta_w) = f(\eta_w \sqrt{\sigma}). \tag{13}$$

The first term of the inner bracket of (10) corresponds to the result given by Merkli and Thomann [1]; differentiating this term with respect to x, their result for $\overline{q_2}$ (given in the notation of [3]) is obtained explicitly. Not to be included in this differentiation is the factor g, which is a constant in [1].

Here $\overline{q_2}$ is of secondary interest; it would be important, however, to find the point where $\overline{q_2} = 0$ and therefore $\overline{Q_2}$ has a maximum. In the case of thermally driven acoustic oscillations, as found for instance in a half-open tube stuck into a dewar containing helium gas over liquid helium, where the transition from hot to cold occurs essentially in a short 'bottleneck', the maximum of $\overline{Q_2}$ represents the heat carried by the oscillations, to second order in amplitude, into the dewar. The full problem is rather complicated: both the amplitude distribution and η_w depend on the wall temperature distribution with x, and numerical methods are needed for the solution. Here, the discussion of (10) is restricted to an order-of-magnitude estimate of the maximum of $\overline{Q_2}$ for a very steep wall temperature gradient.

In this case, the form of (10) suggests that the second term of the inner bracket dominates over the first. A detailed investigation (not reproduced here) confirms this view. Furthermore, the retained term is purely imaginary, so that only the imaginary part of g is needed for the final result, which can be simplified by the relation

$$\operatorname{Im} g = \operatorname{Im} \left(\frac{f^* - \sigma f}{1+\sigma} \right) \tag{14}$$

to give for the remaining part of the heat flux (superscript 2)

$$\overline{Q_2^{(2)}} = \pi r_w^2 \frac{1}{2\omega} u_{1_c} \tilde{u}_{1_c} \operatorname{Im} \left(\frac{f^* - \sigma f}{1 - \sigma^2} \right) \rho_m c_p \frac{dT_m}{dx}. \tag{15}$$

It is convenient to introduce an 'effective' coefficient of heat conduction by the relation

$$\overline{Q_2^{(2)}} = -\pi r_w^2 k_{\text{eff}} \frac{dT_m}{dx} \tag{16}$$

so that

$$\frac{k_{\text{eff}}}{k} = \frac{u_{1_c}\tilde{u}_{1_c}}{2\omega\nu} \operatorname{Im}\left(\frac{\sigma^2 f - \sigma f^*}{1 - \sigma^2}\right). \tag{17}$$

For the discussion, the last term in (17) is expanded under the assumption that the boundary layer is thin. By use of the first approximation for f given in [5], the following result[1]) is obtained:

$$\frac{k_{\text{eff}}}{k} = \frac{s_1\tilde{s}_1}{r_w}\sqrt{\frac{\omega}{2\nu}}\left(1 - \frac{1}{(1+\sigma)(1+\sqrt{\sigma})}\right) \tag{18}$$

where $s_1 = u_{1_c}/\omega$ is the amplitude of the particle displacement in the core. The value of the last bracket in (18) for helium, with a Prandtl number $\sigma = 2/3$, is about 0.67.

The value of s_1 to be introduced in (18) is the one found in the region of the steep temperature gradient. There exists a variation of s_1 in such a region even for a sharp temperature jump, but this is negligible for the purposes of this estimate. If the length $L - l$ of the hot part at the closed end (in the notation of [5]) is not more than about a half of the length L of the half-open tube, the approximations made in [5] lead to the following estimate of s_1 expressed by the (real) pressure amplitude $\hat{p}_1$ at the closed end:

$$s_1 = (L - l)\frac{\hat{p}_1}{\gamma p_m}. \tag{19}$$

Thus, s_1/r_w can easily be of order one or much more.

Next, it is noted that the Stokes layer thickness appears in the denominator in (18); its variation along the tube with the temperature is definitely not negligible. For $\mu \sim k \sim T^\beta$, we have

$$k_{\text{eff}}/k \sim T_m^{-(1+\beta)/2}, \qquad k_{\text{eff}} \sim T_m^{-(1-\beta)/2}.$$

The variation of k_{eff} is comparatively small ($\beta = 0.647$ for He), but its ratio to k can vary strongly for a high temperature ratio between the hot and cold end. According to the theoretical results of [6], for which experimental support was given by von Hoffmann *et al.* [9], thermally driven helium-oscillations can be observed, for a ratio of absolute temperatures up to 70, for values of

$$Y_c = r_w\sqrt{\frac{\omega}{\nu_c}}$$

between 10 and 10^3, where ν_c is the kinematic viscosity at the cold end. The corresponding Y-values at the hot end are reduced, for a temperature ratio of 70, by a factor of about 30. Nevertheless, it is possible to obtain, by proper combination of the

[1]) This result was presented by the author at the XIIIth International Congress of Theoretical and Applied Mechanics, Moscow 1972.

parameters and for observed values of $\hat{p}_1$ of the order of 0.1 bar, ratios of several orders of magnitude predicted by the formula (18).

Observations of Bannister [10] for the heat flux into a dewar were successfully correlated with the product of the amplitude and the frequency. Unfortunately it is not known how the temperature gradient behaved as the parameters were varied. According to (18), the effective coefficient of heat conduction should correlate with the product of the square of the amplitude and the square root of the frequency.

4. Limitations of the Present Theory

The calculation of the thermo-acoustic streaming appears, from the point of view of the amplitude expansion, fully satisfactory if conducted up to second order, considering the range of intensities for which the theory is to be applied. Unfortunately, it is doubtful whether the expansion converges in the limit of very sharp temperature gradients, as a fourth-order thermoacoustic streaming term would contain, in its steady part, the temperature gradient up to the third power. To extend the theory to fourth order appears as a hopeless undertaking, and would also be of doubtful value.

For further progress, it is inevitable to proceed to the investigation of actual temperature distributions with finite gradients. This has to be carried out numerically; some calculations of this kind were reported in [6]. The numerical evaluation of the heat flux for such cases could give, on the theoretical side, an indication of the influence of the temperature gradient, and provide quantitative results for the heat flux which could be compared with experiments.

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Summary

The second-order heat flux in thermally driven oscillations of gas columns is calculated and given in a form suitable for numerical evaluation. Order-of-magnitude estimates are made in the case of steep temperature gradients.

Zusammenfassung

Der Wärmestrom zweiter Ordnung, der in thermisch getriebenen akustischen Schwingungen einer Gassäule entsteht, wurde berechnet und ist in einer für numerische Rechnungen geeigneten Form angegeben worden. Die Grössenordnung des Effektes wurde für steile Temperaturgradienten abgeschätzt.

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