On the Indeterminancy of the Problem of Stratified Fluid Flow Over a Barrier and Related Problems

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I. Introduction

It has long been recognized that the stationary problem associated with the flow of a stratified fluid over a barrier, is indeterminate when the Froude number is less than some critical value. The same is true for the problem associated with the formation of standing waves on the surface of water running over a corrugated bed when $U^2 < gh$, and for other related problems, see Lamb [1]. Here U and h are the velocity and the depth respectively of the water, and g is the gravitational acceleration. This difficulty has in the past been tackled in two ways: 1) by introducing a small friction force (a small fictitious viscosity) which is due to Rayleigh [2], and is adopted by Long [3] and Crapper [4] in the stratified fluid problem; 2) by solving the problem from an initial state and letting the time tend to infinity, (Stoker [5], Palm [6], and Engevik [7]). These two approaches seem quite different. However, it is the purpose of this note to show the connection between them, and that they in fact are quite equivalent. This has not yet been done as far as we know. It follows easily by using the theorem from the theory of the Laplace transform which reads: $\lim_{t\to\infty} F(t) = \lim_{p\to 0} p\bar{F}(p)$, where $\bar{F}(p) = \int_0^\infty F(t)e^{-pt}dt$ is the Laplace transform of $F(t)$. The fictitious viscosity is shown to be proportional to p. We demonstrate the equivalence of the two approaches for the stratified fluid flow problem only; that the result holds for the other related problems, follows easily.

II. Formulation of the Problem

For simplicity let us assume that the problem is two-dimensional and is considered in an x-z-plane. The fluid, which is stratified and incompressible, is confined between two rigid boundaries; the bottom with the shape $z = \eta(x)$, and the horizontal plane at $z = h$. The fluid is of infinite extension in the x-direction. The basic motion of the fluid is given by: $\underline{v}_b = U\underline{i}$, $\rho_b = \rho_0(z)$, where $\underline{i}$ is the unit-vector in the x-direction, U is a constant, $\rho_0(z)$ is a function of z only, and $\rho_0' < 0$. If we start from the stationary inviscid equations, linearize about the basic state, introduce a stream-function $\psi(x, z)$, we find that the stream-function, when applying the Boussinesq approximation, must satisfy the equation:

$$\frac{\partial}{\partial x}[\nabla^2\psi + F^{-2}\psi] = 0, \tag{2.1}$$

where $F = (-\rho_0 U^2/\rho_0' g)^{1/2}$ is the Froude number. The stream function is defined by:

$$\underline{v}_1 = -\nabla \times \psi(x, z)\mathbf{j}, \tag{2.2}$$

where $\underline{v}_1$ is the perturbation in the velocity field and $\mathbf{j}$ is the unit-vector in the y-direction.

It is assumed that $F \neq h/n\pi$ where n is an integer. (The stationary solution tends to infinity when $F \to h/n\pi$. We will return to this point at the end of the paper.)

The problem is indeterminate when $F < h/\pi$, but this difficulty can be avoided either by introducing a fictitious viscosity or by solving the problem from an initial state as mentioned above. In sections III and IV we give the equations to be solved in these two cases, and show that these two approaches which seem quite different, are in fact equivalent.

III. The 'Small Friction Force' Approach

Rayleigh's idea [2] (which is adopted by Crapper [4]) is to introduce a small friction force $-\rho\mu\underline{v}$ ($\mu > 0$) into the momentum equation. Then the stationary equations which govern the above-mentioned problem are:

$$\left.\begin{aligned} \rho\underline{v}\cdot\nabla\underline{v} &= -\nabla p - \rho g\underline{k} - \rho\mu\underline{v} \\ \nabla\cdot\underline{v} &= 0 \\ \underline{v}\cdot\nabla\rho &= 0, \end{aligned}\right\} \tag{3.1}$$

where $\mathbf{k}$ is the unit-vector in the z-direction.

Crapper [4] pointed out that μ can also be considered as the operator $\partial/\partial t$, and that $\mu \to 0$ corresponds to $t \to \infty$. The boundary conditions to be satisfied are that the normal velocity at the boundaries must be equal to zero.

Applying the Boussinesq approximation and linearizing as above, we find the equation for the stream-function to be:

$$\frac{\partial}{\partial x}[\nabla^2\psi + F^{-2}\psi] = -\frac{\mu}{U}\nabla^2\psi. \tag{3.2}$$

The linearized boundary conditions become:

$$\left.\begin{aligned} \frac{\partial\psi}{\partial x} &= -U\frac{\partial\eta}{\partial x} && \text{at } z = 0, \\ \frac{\partial\psi}{\partial x} &= 0 && \text{at } z = h. \end{aligned}\right\} \tag{3.3}$$

Eq. (3.2) is solved together with the boundary conditions (3.3), and by letting $\mu \to 0$ in the solution, the stationary solution is obtained.

Long [3] proposes to solve another equation, namely:

$$\frac{\partial}{\partial x}[\nabla^2\psi + F^{-2}\psi] = \mu\psi \tag{3.4}$$

together with the boundary conditions (3.3). (Long gives the equation for δ which represents the variation in height of the streamline about its equilibrium height, but δ is proportional to ψ, so we have here given the equation with ψ). Eq. (3.2) and eq. (3.4) differ by the term on the right hand side.

IV. The 'Initial-value Problem' Approach

By this approach the non-stationary *inviscid* equations are used. The linearized equation for the stream-function is found to be:

$$\left\{U^{-1}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right\}^2 \nabla^2\psi + F^{-2}\frac{\partial^2\psi}{\partial x^2} = 0. \tag{4.1}$$

The boundary conditions are given by (3.3), and the initial values are (see [7]):

$$\left.\begin{aligned} \nabla^2\psi &= 0 \\ \left(U^{-1}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\nabla^2\psi &= 0 \end{aligned}\right\} \text{ at } t = 0. \tag{4.2}$$

We are interested in the stationary solution, i.e. $\lim_{t\to\infty}\psi(x, z, t)$, and to show that this is equivalent to the solution of eq. (3.2) and eq. (3.4) when $\mu \to 0$ in these solutions. Let $\bar{\psi}(x, z, p) = \int_0^\infty \psi(x, z, t)e^{-pt}dt$ be the Laplace transform of $\psi(x, z, t)$. We know that $\lim_{t\to\infty}\psi(x, z, t) = \lim_{p\to 0} p\bar{\psi}(x, z, p)$, showing that the stationary solution of $\psi(x, z, t)$ can be found when its Laplace transform is known. Let us define $\varphi(x, z, p) = p\bar{\psi}(x, z, p)$. Applying the Laplace transformation in eq. (4.1) and eq. (3.3) and introducing φ, we find that φ must satisfy the following equation and boundary conditions:

$$\left\{\frac{p}{U} + \frac{\partial}{\partial x}\right\}^2 \nabla^2\varphi + F^{-2}\frac{\partial^2\varphi}{\partial x^2} = 0, \tag{4.3}$$

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial x} &= -U\frac{\partial\eta}{\partial x} && \text{at } z = 0, \\ \frac{\partial\varphi}{\partial x} &= 0 && \text{at } z = h. \end{aligned}\right\} \tag{4.4}$$

We observe that the boundary conditions (4.4) are the same as (3.3).

We are interested in the equation for φ for small values of p, since $\lim_{p\to 0}\varphi$ gives the stationary solution. Neglecting higher order terms in p in eq. (4.3), and integrating once with respect to x, yields:

$$\frac{\partial}{\partial x}[\nabla^2\varphi + F^{-2}\varphi] = -\frac{2p}{U}\nabla^2\varphi. \tag{4.5}$$

The boundary conditions (4.4) are unchanged. Eq. (3.2) is equal to eq. (4.5) if μ is replaced by $2p$, and the boundary conditions (3.3) and (4.4) are also the same. This clearly shows that the two approaches are equivalent.

Let us also show that we can arrive at Long's equation, eq. (3.4). Applying the Fourier transformation in eq. (4.3), dividing the transformed equation by $(ik)(1 + p/ikU)^2$ and neglecting higher order terms in p, yields:

$$(ik)\left[\left(\frac{d^2}{dz^2} - k^2\right)\hat{\varphi} + F^{-2}\hat{\varphi}\right] = \frac{2p}{F^2U}\hat{\varphi}, \quad \text{where} \quad \hat{\varphi} = \int_{-\infty}^{\infty}\varphi e^{ikx}dx.$$

This is the Fourier transformation of the equation:

$$\frac{\partial}{\partial x}[\nabla^2\varphi + F^{-2}\varphi] = \frac{2p}{F^2U}\varphi, \tag{4.6}$$

which is equal to eq. (3.4) if we put $\mu = 2p/F^2U$.

Let us in conclusion consider the case when $F = h/n\pi$. The stationary solution ($\lim_{p\to 0}\varphi$) tends to infinity when $F \to h/n\pi$, (see for instance [3] or [7]). When $F = h/n\pi$ the solution of eq. (4.5) and eq. (4.6) does not exist when $p \to 0$, because $\lim_{p\to 0}\varphi = \infty$. But since $\lim_{p\to 0}\varphi = \lim_{t\to\infty}\psi$ this means that our initial-value problem does not have any stationary solution. This agrees with what Stoker [5] finds in his problem. $F = h/n\pi$ in our problem corresponds to $U^2/gh = 1$ in Stoker's problem, and he finds that the time dependent solution tends to infinity when $t \to \infty$ in this case. It is also known that the stationary solution (corresponding to $\lim_{p \to 0} \varphi$ above) tends to infinity when $U^2/gh \to 1$. This means that when the stationary solution becomes infinite for some values of the parameter, the solution of the initial-value problem will tend to infinity when $t \to \infty$ for these values of the parameter.

Conclusion

The indeterminancy of the stationary problem associated with the stratified fluid flow over a barrier and related problems, has long been recognized, and has been solved either by introducing a small fictitious viscosity or by solving the problem from an initial state and letting $t \to \infty$. In this note the two approaches are shown to be equivalent by using a theorem from the theory of the Laplace transform. It is our opinion that this theorem could be useful in other fluid flow problems where a stationary solution is required.

References

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Abstract

The indeterminancy of the problem of stratified fluid flow over a barrier and related problems has in the past been solved either by introducing small friction forces or by solving the problem from an initial state and letting the time tend to infinity. Here the connection between these two approaches are considered, and they are shown to be equivalent.

Zusammenfassung

Das Problem der Unbestimmtheit der Strömung eines geschichteten Fluids über ein Hindernis und verwandte Fragen sind bisher entweder durch die Einführung von kleinen Reibungskräften gelöst worden, oder durch Lösung des zeitabhängigen Problems von einem Anfangszustand bis zum stationären Endzustand. Hier wird der Zusammenhang zwischen den beiden Methoden betrachtet, und es wird gezeigt, dass sie äquivalent sind.

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