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Boundary Layer Analysis of Oscillating Cylinder Flows in a Viscoelastic liquid

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I. Introduction

It is reported earlier [1], that patterns of the secondary flows induced by an oscillating cylinder have been drastically altered when a small amount of polymers was added to the Newtonian liquid. We speculate that the change in secondary flow patterns probably results from the elasticity of the dilute polymer solution. In this analysis, an attempt is made to explain the novel phenomenon qualitatively by using a boundary layer approach. For a more complete and quantitative analysis, the treatise is given elsewhere [2], and will be presented later [15].

Boundary layer analysis has been successfully applied to various two-dimensional flows for viscoelastic fluids: for example, wedge flows [3, 4], stagnation flow [3, 5, 6, 7], converging channel and flat plate flows [3, 11], etc. Most of these works are under steady state flow condition. For this particular oscillating cylinder flow problem, Frater [8] apparently was the first and only one who made an analysis theoretically by using the Oldroyd convected model. Employing a boundary layer analysis, he showed how viscoelastic effect can change the flow pattern of the secondary streaming. However, some of the arguments used in that paper have been questioned [9].

In order to compare the theoretical prediction with the experimental results in [1], Walters' Liquid B′ [10] will be used. It can be expressed as

$$p_{ik} = -p g_{ik} + p'_{ik} \tag{1}$$

$$p'^{ik}(\mathbf{x}, t) = 2\int_{\infty}^{t} \Phi(t - t')\,\frac{\partial x^i}{\partial x'^m}\,\frac{\partial x^k}{\partial x'^r}\,e^{(1)mr}(\mathbf{x}', t')\,dt' \tag{2}$$

where

$$\Phi(t - t') = \int_0^\infty \frac{N(\tau)}{\tau}\,e^{-(t-t')/\tau}\,d\tau \tag{3}$$

and p_{ik} being the stress tensor, g_{ik} the metric tensor of a coordinate system x^i, $N(\tau)$ the distribution function of relaxation times, and x'^i being the position at time t' of the fluid element which is instantaneously at the point x^i at time $t \cdot$ $e^{(1)}_{mr}$ is the rate of strain tensor, which in terms of the velocity vector v_m is given by

$$e^{(1)}_{mr} = \tfrac{1}{2}(v_{m,r} + v_{r,m}). \tag{4}$$

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The Oldroyd three-parameter convected model is a special case of Eqns (1) and (2). By assuming $N(\tau) = \eta_0\delta(\tau)$, where $\delta(\tau)$ is a Dirac delta function, the Newtonian fluid model with constant viscosity η_0 is recovered.

In the experiment of [1], the polymer solution used is very dilute. It is a reasonable assumption then that the liquid has a short memory, i.e., it has short relaxation times. Under this assumption, the equations of state (1) and (2) can be expanded in terms of relaxation time. By retaining only the first few terms, p'_{ik} can be expressed as [5]

$$p'^{ik} = 2\eta_0\, e^{(1)ik} - 2k_0^* \frac{\delta e^{(1)ik}}{\delta t} \tag{5}$$

where

$$\eta_0 = \int_0^\infty N(\tau)\, d\tau; \qquad k_0^* = \int_0^\infty \tau N(\tau)\, d\tau$$

$$\frac{\delta b^{ik}}{\delta t} = \frac{\partial b^{ik}}{\partial t} + v^m \frac{\partial b^{ik}}{\partial x^m} - \frac{\partial v^k}{\partial x^m} b^{im} - \frac{\partial v^i}{\partial x^m} b^{mk}.$$

η_0 is the zero shear rate viscosity, and k_0^* characterizes the elasticity effect. Terms involving $\int_0^\infty \tau^n N(\tau)\, d\tau (n \geq 2)$ are neglected.

Equation (5) is essentially a type of second order fluid, representing a second order analysis [10]. This approximation is valid for not highly elastic liquid, which is the case for the particular experiment reported in [1]. For the relationship between different equations of state including Eqn. (5), one can refer to Walters [13], and Schowalter [14].

II. Boundary Layer Equations and Solutions

The boundary layer equations can be obtained by substituting Eqns (1) and (5) into the equations of motion with order of magnitude comparison, utilizing boundary layer approximation [12]. (For example, u and x are one order of magnitude higher than v and y respectively.) The resulting approximate equation becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\eta_0}{\rho}\frac{\partial^2 u}{\partial y^2} - \frac{k_0^*}{\rho}\left[u\frac{\partial^3 u}{\partial x\,\partial y^2} + v\frac{\partial^3 u}{\partial y^3} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\,\partial y}\right]. \tag{6}$$

Equation (6) with the absence of the first term, $\partial u/\partial t$, is exactly the same viscoelastic boundary layer equation as obtained by previous workers [5, 6].

We utilized the method of successive approximation for non-steady boundary layer analysis [12] with the assumption that the velocity component was in the form of

$$u(x, y, t) = u_0(x, y, t) + u_1(x, y, t). \tag{7}$$

Let

$$\nu = \frac{\eta_0}{\rho}; \qquad k_0 = \frac{k_0^*}{\rho},$$

then from Eqns (6) and (7), we have

$$\frac{\partial u_0}{\partial t} - \nu \frac{\partial^2 u_0}{\partial y^2} = \frac{\partial U}{\partial t} \tag{8}$$

$$\begin{aligned}\frac{\partial u_1}{\partial t} - \nu \frac{\partial^2 u_1}{\partial y^2} = U \frac{\partial U}{\partial x} - v_0 \frac{\partial u_0}{\partial y} - u_0 \frac{\partial u_0}{\partial x} - k_0 \Big[u_0 \frac{\partial^3 u_0}{\partial x \, \partial y^2} + \\ v_0 \frac{\partial^3 u_0}{\partial y^3} + \frac{\partial u_0}{\partial x} \frac{\partial^2 u_0}{\partial y^2} - \frac{\partial u_0}{\partial y} \frac{\partial^2 u_0}{\partial x \, \partial y} \Big]\end{aligned} \tag{9}$$

where $U = U(x)\, e^{i\omega t}$ is the potential flow just outside the boundary layer. ω is the frequency of oscillation. The boundary conditions are

$$\begin{aligned} &u_0 = u_1 = 0, \quad \text{at } y = 0 \\ &u_1 = 0, \qquad u_0 = U_0(x)\, e^{i\omega t} \quad \text{at } y = \infty. \end{aligned} \tag{10}$$

In the above approximation, the curvature effects are neglected and we assume that $|\partial U/\partial t| \gg |U\, \partial U/\partial x|$, i.e., the amplitude of oscillation is much smaller than the diameter of the oscillating cylinder.

Assume that the stream function can be expressed as

$$\psi_0(x, y, t) = \sqrt{\nu/\omega}\; U_0(x) \zeta_0(\eta)\, e^{i\omega t} \tag{11}$$

where $\eta = y\sqrt{\omega/\nu}$, then velocities u_0 and v_0 are

$$\begin{aligned} u_0 &= U_0(x) \zeta_0'\, e^{i\omega t} \\ v_0 &= -\frac{dU_0}{dx} \sqrt{\nu/\omega}\; \zeta_0(\eta)\, e^{i\omega t} \end{aligned} \tag{12}$$

Combine Eqns (7) and (11), we have the first approximate equation,

$$\zeta_0''' - i\zeta_0' = -i$$

in which the prime denotes a differentiation with respects to η. The boundary conditions become $\zeta_0 - \zeta_0' = 0$ at $\eta = 0$ and $\zeta_0' = 1$ at $\eta = \infty$. The solution for ζ_0' is

$$\zeta_0' = 1 - e^{-(1+i)\eta/\sqrt{2}} \tag{13}$$

Therefore, the velocities of the first approximation are

$$u_0 = U_0(x)[1 - e^{-(1+i)\eta/\sqrt{2}}]\, e^{i\omega t} \tag{14}$$

$$v_0 = -\frac{dU_0}{dx} \sqrt{\nu/\omega}\; e^{i\omega t} \left[\eta - \frac{\sqrt{2}}{1+i} + \frac{\sqrt{2}}{1+i} e^{-(1+i)\eta/\sqrt{2}} \right] \tag{15}$$

For the second approximation, the stream function ψ_1 has the form

$$\psi_1 = \sqrt{\nu/\omega}\; [\zeta_{1a}(\eta)\, e^{2i\omega t} + \zeta_{1b}(\eta)] \frac{U_0(x)}{\omega} \frac{dU_0}{dx} \tag{16}$$

where ζ_{1a} is the unsteady part and ζ_{1b} is the steady part of the second approximation, respectively. The velocity component u_1 becomes

$$u_1 = U_0(x)\frac{dU_0}{dx}\frac{1}{\omega}\left[\zeta'_{0a}(\eta)\, e^{2i\omega t} + \zeta'_{1b}(\eta)\right] \tag{17}$$

Substituting Eqns (14), (15) and (17) into Eqn. (9), we obtain the differential equations for ζ'_{1a} and ζ'_{1b}:

$$2i\zeta'_{1a} - \zeta'''_{1a} = \tfrac{1}{2}[(1 - \zeta'_0\zeta'_0 + \zeta_0\zeta''_0) - K(2\zeta'_0\zeta'''_0 - \zeta_0\zeta''''_0 - \zeta''_0\zeta''_0)] \tag{18}$$

$$\begin{aligned} -\zeta'''_{1b} = \tfrac{1}{2}\{&1 - \zeta'_0\tilde{\zeta}'_0 + \tfrac{1}{2}(\zeta_0\tilde{\zeta}''_0 + \tilde{\zeta}_0\zeta''_0) \\ &- K[\zeta'_0\tilde{\zeta}'''_0 + \tilde{\zeta}'_0\zeta'''_0 - \tfrac{1}{2}(\zeta_0\tilde{\zeta}''''_0 + \tilde{\zeta}_0\zeta''''_0) - \zeta''_0\tilde{\zeta}''_0]\} \end{aligned} \tag{19}$$

where $\tilde{a}$ is a complex conjugate of a, and

$$K = \frac{k_0\omega}{\nu} \tag{20}$$

The boundary conditions are that the normal and tangential velocities vanish at wall, whereas at a large distance from the wall, only the tangential part vanishes. The solution for ζ'_{1a} is

$$\zeta'_{1a} = \frac{K - i}{2}\, e^{-\sqrt{2}(1+i)\eta'} + \left[\frac{i - K}{2} + \frac{(1 + K) - i(1 - K)}{2}\,\eta'\right] e^{-(1+i)\eta'} \tag{21}$$

where $\eta' = \eta/\sqrt{2}$.

For the steady part of the stream function, the solution for ζ'_{1b} was obtained under the condition that one of the boundary conditions has to be relaxed, i.e., the tangential velocity is not zero, but a finite value at a large distance from the wall. The solution is

$$\begin{aligned} \zeta'_{1b} = C_1 &+ \left[\tfrac{1}{4} - K + \left(\frac{K}{4} + 1\right) i\right] e^{-(1+i)\eta'} + \left[\tfrac{1}{4} - K - \left(\frac{K}{4} + 1\right) i\right] e^{-(1-i)\eta'} \\ &+ \tfrac{1}{4}\, e^{-2\eta'} - \frac{(1 + K) + (1 - K)i}{4}\,\eta'\, e^{-(1-i)\eta'} \\ &+ \frac{-(1 + K) + (1 - K)i}{4}\,\eta'\, e^{-(1+i)\eta'} \end{aligned} \tag{22}$$

Where C_1 is to be determined from the boundary condition $\zeta'_{1b} = 0$ at $\eta' = 0$. Since only the real part of ζ'_{1b} has physical meaning, we have

$$\begin{aligned} \zeta'_{1b} = -\tfrac{3}{4} + 2K &+ \left[(\tfrac{1}{2} - 2K)\cos\eta' + \left(\frac{K}{2} + 2\right)\sin\eta'\right] e^{-\eta'} \\ &+ \tfrac{1}{4}\, e^{-2\eta'} + \tfrac{1}{2}[(1 + K)\cos\eta' - (1 - K)\sin\eta']\eta'\, e^{-\eta'} \end{aligned} \tag{23}$$

or by rearranging,

$$\begin{aligned} \zeta'_{1b} = &\left[-\tfrac{3}{4} + \tfrac{1}{4}\, e^{-2\eta'} + (2\sin\eta' + \tfrac{1}{2}\cos\eta')\, e^{-\eta'} - \frac{\eta'}{2}(\cos\eta' - \sin\eta')\, e^{-\eta'}\right] \\ &+ K\left[2 + (\tfrac{1}{2}\sin\eta' - 2\cos\eta')\, e^{-\eta'} - \frac{\eta'}{2}(\cos\eta' + \sin\eta')\, e^{-\eta'}\right] \end{aligned} \tag{24}$$

As we notice that in Eqn. (24), terms in the first bracket are exactly the same as that of Newtonian fluids [12], and the terms in the second bracket are due to the elasticity effects characterized by the parameter K. The asymptotic value for ζ'_{1b} at η' approaches infinity is

$$\zeta'_{1b}(\infty) = -\tfrac{3}{4} + 2K \tag{25}$$

as compared to the Newtonian case, in which ζ'_{1b} has a value of $-\frac{3}{4}$.

III. Discussion and Conclusion

The solution thus obtained is not a uniformly valid solution except for certain value of elasticity parameter K, i.e., $K = 3/8$, because the solution does not satisfy the boundary conditions at infinity due to the boundary layer approximation. Asymptotic expansion techniques have to be applied if a complete solution is required. This attempt is not intended here, since an exact solution without using boundary layer approximation can be obtained [15]. However, it is very instructive at least qualitatively to see what is the elasticity effects on the inner solution of the boundary layer equations. It is believed that the region near the solid boundary has the greatest effect on the whole flow patterns of the steady secondary flow induced by an oscillating cylinder in a viscoelastic liquid.

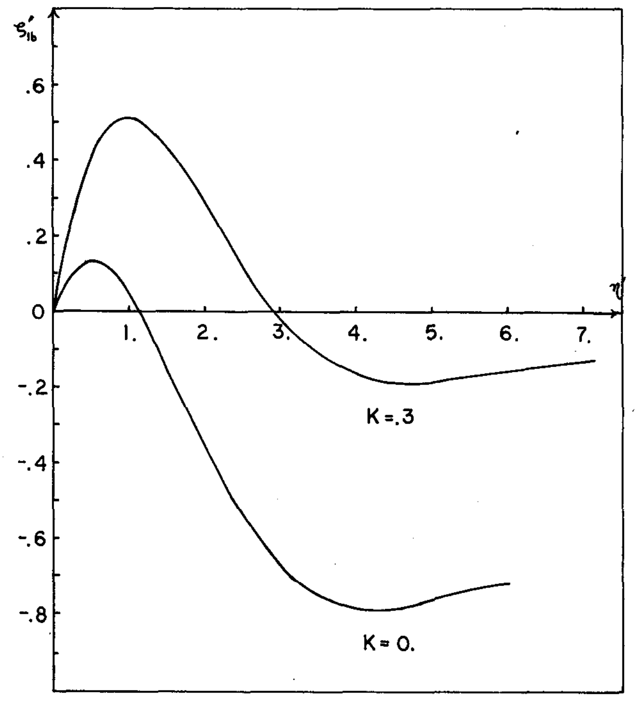

Figure 1
Effect of elasticity parameter *K* on the steady streaming velocity patterns.

The effect of the inherent elasticity of the liquid is presented in Figure 1, as ζ'_{1b} vs η' with different values of K. The figure clearly shows that the effect of elasticity of polymer solution is to increase the thickness of the inner vortex system, and to increase the intensity of the secondary flows near the solid boundary. The increase in the thickness of the inner vortex system indicates the right trend of the elasticity effect by comparing the observed experimental results [1], in which the inner vortex is expanded to occupy the whole flow field, thus the flow direction of the steady streaming is completely reversed. As mentioned before, however, the unsatisfied boundary condition at the infinity leads one to regard the boundary layer analysis as only qualitatively correct, and the solution is valid only at those regions near the solid boundary.

The predicted increase in the intensity of the secondary flows near the solid boundary has certain practical applications. These stronger secondary flows may enhance the heat and mass transfer rates between the solid boundary and the surrounding fluids in the industrial processes.

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Abstract

A boundary layer analysis is applied to the oscillating cylinder system in a viscoelastic liquid. The effect of the inherent elasticity of the liquid is to increase the thickness of the inner vortex system of the steady streaming secondary flows, which is consistent with the experimental observation reported earlier.

Résumé

L'analyse d'un système composé d'un cylindre oscillant dans un liquide visco-élastique est approchée tel un problème de couche limite. L'effet de l'élasticité inhérente du liquide est d'augmenter l'épaisseur du vortex interne des écoulements secondaires en régime permanent, effet en accord avec l'observation expérimentale décrite dans une publication antérieure.

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