Large Plane Deformations of Thin Elastic Sheets of Neo-Hookean Material

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1. Introduction

A general theory of plane stress for large elastic deformations of isotropic materials has been developed by ADKINS, GREEN and NICHOLAS [1] (see also [2, 3]). The theory applies to a thin plane sheet which is stretched by forces in its plane so that it remains plane after deformation, the major surfaces of the sheet being free from traction. It is assumed that the deformation and stress resultants are determined to sufficient accuracy by the deformation of the middle surface of the sheet, and the theory is reduced to two-dimensional form. The one-dimensional case of circular symmetry was treated earlier by RIVLIN and THOMAS [4] and recently YANG [5] has considered approximate and exact solutions for several axisymmetric problems.

The two-dimensional equations of the general theory are difficult to solve exactly and a method of successive approximations for static problems, with solutions expressed as power series in a real parameter ε, has been used to obtain first and second order solutions for unsymmetric problems [1, 2, 3]. Quantities in the first order or infinitesimal solution are $O(\varepsilon)$ as $\varepsilon \to 0$ and they provide the asymptotic form of the solution for vanishingly small strains. Thus the application of the first and second order solutions is limited to a range of deformations near the undeformed state and is inadequate at large strains. In this work we assume that the elastic material of the plane membrane is incompressible and has the neo-Hookean form for the strain energy function and we develop a method of successive substitutions for the solution of problems involving large strains. The first approximation of the present theory is the asymptotic form of the solution for infinitely large strains in contrast to the method of [1, 2, 3], although the two methods are similar in character.

A summary of the basic formulae and equations of the equilibrium theory of plane elastic membranes is given in Section 2. A method of successive substitutions is outlined in Section 3 for the particular case of a neo-Hookean material, and a brief comment is made on the extension of the method to dynamic problems. Section 4 deals with several (static) problems involving infinite membranes with circular or elliptic inclusions, and the first approximations are obtained. For circularly symmetric problems the second approximation can be obtained without difficulty and exact solutions are available through numerical integration. Good agreement was found

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even for moderate deformations between the approximate analytical solutions and the exact numerical solutions in the problem of a rigid circular inclusion with uniform stretching at infinity.

When the membrane has an edge which is traction free, the method of successive substitutions developed in Section 3 breaks down after the first approximation because the transverse extension ratio or thickness ratio λ calculated from the first approximation becomes infinite as the traction-free edge is approached. Section 5 provides an alternative way to calculate a better first approximation to the thickness ratio which remains finite at the traction-free edges.

Several simple examples involving membranes with traction-free edges are considered in Section 6. In the case of the radially symmetric stretching of an infinite membrane with a circular hole, comparison between approximate and exact numerical solutions is made. Agreement between the solutions is again good for moderate deformations but the agreement is not as good as in the problem of the rigid circular inclusion. Other examples treat a finite membrane under a deformation which is close to a homogeneous state of pure shear, the deformation near a boundary with a corner which is traction-free, and the action of a concentrated load at a boundary point.

The method of successive substitutions used here for neo-Hookean materials may be used with obvious modification for materials which have strain energy functions close to the neo-Hookean form over the range of deformation involved, the modification occurring in the second and higher approximations. The comparisons with exact solutions for symmetric problems indicate that the first approximations can give results accurate in the range of moderate deformations where the neo-Hookean form provides a fair approximation for rubber-like materials. For larger deformations, where, for example, the Mooney form may be more appropriate, the results based on the first approximation for the neo-Hookean material can still be of value in indicating the main features and characteristics to be expected in the actual deformation of real materials.

2. Basic Equations for Finite Plane Stress

We suppose that in its initial state the body is a plane sheet of homogeneous elastic material bounded by the surfaces $x_3 = \pm h_0/2$, where (x_1, x_2, x_3) are the coordinates of a particle of the sheet referred to a rectangular cartesian reference frame. The thickness h_0 may depend on x_1, x_2. The sheet undergoes a finite deformation symmetric about the middle plane $x_3 = 0$ and we denote by (y_1, y_2, y_3) the coordinates after deformation of a particle which was at the point (x_1, x_2, x_3) in the unstrained state. The middle plane in the deformed state is $y_3 = 0$ and the major surfaces of the sheet after deformation are given by $y_3 = \pm h/2$, where h is, in general, a function of y_1, y_2. We shall use indicial notation and the summation convention, with Latin indices taking the values 1, 2, 3 and Greek indices the values 1, 2.

We assume that the displacement gradients $y_{i,k}$ throughout the thickness are determined with sufficient accuracy for our purposes by their middle plane values. Because of the symmetry of the deformation we assume in particular that

$$\frac{\partial y_3}{\partial x_\alpha} = 0 , \quad \frac{\partial y_\alpha}{\partial x_3} = 0 ,$$

and we shall write

$$\frac{\partial y_3}{\partial x_3} = \lambda ,$$

where $\lambda(x_1, x_2)$ is the extension ratio in the direction normal to the sheet.

The condition that the major surfaces of the sheet be free from traction is satisfied approximately by requiring the transverse normal stress t_{33} to be zero, and this condition serves to determine λ in terms of $y_{\alpha,\beta}$ through the stress-strain relation for a compressible material. For an incompressible material, however, λ is determined by

$$\lambda = \frac{1}{J} = \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} . \tag{2.1}$$

In either case the strain energy U per unit area of the undeformed middle surface is expressible as

$$U = h_0\, W(y_{\alpha,\beta}; \lambda) = U(y_{\alpha,\beta}; x_\gamma) ,$$

where W is the strain energy per unit volume of the undeformed material. The strain energy U depends on $y_{\alpha,\beta}$ through $y_{\gamma,\alpha}\, y_{\gamma,\beta}$ only. For an isotropic sheet, U is a symmetric function of the principal extension ratios λ_1, λ_2 in the plane of the sheet, and we can write

$$U = U(K, J; x_\alpha)$$

where

$$K = y_{\alpha,\beta}\, y_{\alpha,\beta} = \lambda_1^2 + \lambda_2^2 , \qquad J = |y_{\alpha,\beta}| = \lambda_1 \lambda_2 .$$

When the material of the sheet is incompressible and isotropic, U depends on K and J only through the invariants

$$I_1 = K + \frac{1}{J^2} , \qquad I_2 = \frac{K}{J^2} + J^2 .$$

The stress resultants $T_{\alpha\beta}$ are defined by

$$T_{\alpha\beta} = \int_{-h/2}^{h/2} t_{\alpha\beta}\, dy_3 ,$$

where t_{ik} are the stress components, and they are given by

$$T_{\alpha\beta} = \frac{1}{J} \frac{\partial U}{\partial y_{\beta,\gamma}} \frac{\partial y_\alpha}{\partial x_\gamma} .$$

With the major surfaces of the sheet traction-free, the equations of equilibrium become

$$\frac{\partial T_{\alpha\beta}}{\partial y_\alpha} = 0 \quad \text{or} \quad \frac{\partial}{\partial x_\gamma}\left(J \frac{\partial x_\gamma}{\partial y_\alpha} T_{\alpha\beta}\right) = 0$$

in the absence of body forces.

For the most part we shall confine our discussion to an incompressible material which has the neo-Hookean form for the strain energy function

$$U = h_0\, C_1 (I_1 - 3) = h_0\, C_1 (K + \lambda^2 - 3) \tag{2.2}$$

where C_1 is a material constant. The stress resultants are then

$$T_{\alpha\beta} = 2\, h_0\, C_1\, \lambda \left(\frac{\partial y_\alpha}{\partial x_\gamma} \frac{\partial y_\beta}{\partial x_\gamma} - \lambda^2\, \delta_{\alpha\beta}\right) \tag{2.3}$$

and the equations of equilibrium reduce to

$$\frac{\partial}{\partial x_\gamma}\left(h_0 \frac{\partial y_\beta}{\partial x_\gamma}\right) - J \frac{\partial}{\partial y_\beta}(h_0 \lambda^3) = 0 .$$

For constant initial thickness h_0 we have

$$\nabla^2 y_\beta - 3\lambda \frac{\partial \lambda}{\partial y_\beta} = 0 \tag{2.4}$$

where ∇^2 is the two-dimensional Laplace operator. Equations (2.4) can also be written in the form

$$\nabla^2 y_1 = \frac{\partial \lambda^3}{\partial x_1}\frac{\partial y_2}{\partial x_2} - \frac{\partial \lambda^3}{\partial x_2}\frac{\partial y_2}{\partial x_1}, \qquad \nabla^2 y_2 = \frac{\partial y_1}{\partial x_1}\frac{\partial \lambda^3}{\partial x_2} - \frac{\partial y_1}{\partial x_2}\frac{\partial \lambda^3}{\partial x_1}. \tag{2.5}$$

The load resultant dL_α on a normal section through a line element ds of a curve drawn in the middle plane is given by

$$dL_\alpha = T_{\alpha\beta}\, n_\beta\, ds$$

where n_α is the unit normal to the curve and by

$$dL_\alpha = J \frac{\partial x_\gamma}{\partial y_\beta} T_{\alpha\beta}\, n_\gamma^0\, ds^0 ,$$

where ds^0, n_α^0 refer to the undeformed state. Substituting for $T_{\alpha\beta}$ from (2.3) we obtain

$$dL_\alpha = 2 h_0 C_1 \left(\frac{\partial y_\alpha}{\partial n^0} - \lambda^2 \frac{\partial x_\varrho}{\partial y_\alpha} n_\varrho^0\right) ds^0 \tag{2.6}$$

for a neo-Hookean material. Equations (2.6) can also be written in the form

$$dL_1 = 2 h_0 C_1 \left(\frac{\partial y_1}{\partial n^0} - \lambda^3 \frac{\partial y_2}{\partial s^0}\right) ds^0 , \qquad dL_2 = 2 h_0 C_1 \left(\frac{\partial y_2}{\partial n^0} + \lambda^3 \frac{\partial y_1}{\partial s^0}\right) ds^0 \tag{2.7}$$

where the s^0-, n^0-directions are taken right-handedly.

3. Successive Substitution

As can be seen from (2.5) and (2.7), both the differential equations and the traction boundary conditions are non-linear so that exact solutions will not always be easy to determine. Since the non-linearity in (2.5) and (2.7) comes solely from terms involving $\boldsymbol{\lambda}$, a natural first approximation when $\lambda \ll 1$ is obtained by neglecting all such terms in the equations. This is equivalent to using for the strain energy the form

$$U^{(1)} = h_0 C_1 (K - 2) ,$$

rather than the exact form (2.2). Superscribed quantities here stand for approximate values, with (1) for the first, (2) for the second and so on. We remark that the first approximation $y_\alpha^{(1)}$ to y_α is exact for a sheet with strain energy $U^{(1)}$. Such a sheet is isotropic but is stressed in all-around tension in its reference state.

If the principal extension ratios λ_1, λ_2 in the plane of the sheet are of the order of μ for large μ throughout the sheet, where μ is a parameter which measures the amount of deformation, then

$$U^{(1)} = O(\mu^2)$$

and
$$U = U^{(1)} + O(\mu^{-4}) \, .$$

If the first approximation provides derivatives $y^{(1)}_{\alpha,\beta}$ which are $O(\mu)$ for large μ, correction terms of order $O(\mu^{-5})$ added to these derivatives will change K and therefore $U^{(1)}$ by terms of order $O(\mu^{-4})$. It is therefore to be expected that, on the average, the first approximation determines $y_{\alpha,\beta}$ to within terms which are $O(\mu^{-5})$ for large μ so that we will have
$$y_{\alpha,\beta} = y^{(1)}_{\alpha,\beta} + O(\mu^{-5}) \quad \text{when} \quad y^{(1)}_{\alpha,\beta} = O(\mu) \, . \tag{3.1}$$

If $\lambda_1 = O(1)$ and $\lambda_2 = O(\mu)$, the approximation will be less good and a similar argument leads to
$$y_{\alpha,\beta} = y^{(1)}_{\alpha,\beta} + O(\mu^{-2}) \, .$$

When $\lambda_1 = O(\mu^{-1/2})$ and $\lambda_2 = O(\mu)$, so that each element of the sheet is strained close to a state of simple extension, we will have
$$y_{\alpha,\beta} = y^{(1)}_{\alpha,\beta} + O(\mu^{-1/2}) \, .$$

In this case some of the derivatives $y^{(1)}_{\alpha,\beta}$ can be of the same order, $O(\mu^{-1/2})$, as the correction terms.

Setting $\lambda = 0$ in (2.5) and (2.7) we find that the first approximation satisfies
$$\nabla^2 y^{(1)}_1 = 0 \, , \qquad \nabla^2 y^{(1)}_2 = 0$$

in A^0, with the boundary conditions
$$\frac{\partial y^{(1)}_1}{\partial n^0} \, ds^0 = \frac{1}{2 \, h_0 \, C_1} \, dL^*_1 \, , \qquad \frac{\partial y^{(1)}_2}{\partial n^0} \, ds^0 = \frac{1}{2 \, h_0 \, C_1} \, dL^*_2 \quad \text{on} \quad C^0_T \, , \tag{3.2}$$

and
$$y^{(1)}_1 = y^*_1 \, , \quad y^{(1)}_2 = y^*_2 \quad \text{on} \quad C^0_D \, .$$

Here A^0 is the middle plane of the unstrained sheet with boundary C^0, C^0_T is that part of C^0 where traction components dL^*_α are prescribed and C^0_D is that part where deformed locations y^*_α are given. When A^0 is infinite, conditions at infinity must also be imposed. For example, if the sheet extends to infinity in all directions and if it is under uniform biaxial extension at infinity with principal extension ratios μ_1 and μ_2 along the x_1- and x_2-axes, respectively, the appropriate conditions on y_α are, for zero rotation at infinity,
$$\left.\begin{aligned} \frac{\partial y_1}{\partial x_1} &= \mu_1 - \frac{L^*_1 \cos\theta}{4 \pi h_0 C_1 r} + o\left(\frac{1}{r}\right), & \frac{\partial y_1}{\partial x_2} &= - \frac{L^*_1 \sin\theta}{4 \pi h_0 C_1 r} + o\left(\frac{1}{r}\right), \\ \frac{\partial y_2}{\partial x_1} &= - \frac{L^*_2 \cos\theta}{4 \pi h_0 C_1 r} + o\left(\frac{1}{r}\right), & \frac{\partial y_2}{\partial x_2} &= \mu_2 - \frac{L^*_2 \sin\theta}{4 \pi h_0 C_1 r} + o\left(\frac{1}{r}\right), \end{aligned}\right\} \tag{3.3}$$

as $r \to \infty$, where L^*_α are the components of the resultant of all external forces acting on the interior boundaries of the membrane and (r, θ) are polar coordinates. The first approximation $y^{(1)}_\alpha$ must satisfy (3.3) at infinity.

Once the first approximation to the solution is known, higher approximations can be obtained by a method of successive substitutions as follows. We use the first approximation $y^{(1)}_\alpha$ to estimate the nonlinear terms in (2.5) and (2.7). With $\lambda^{(1)}$

defined by

$$\lambda^{(1)} = \frac{1}{J^{(1)}} = \left[\frac{\partial(y_1^{(1)}\, y_2^{(1)})}{\partial(x_1, x_2)}\right]^{-1}, \tag{3.4}$$

the second approximation solution $y_\alpha^{(2)}$ satisfies

$$\left.\begin{aligned} \nabla^2 y_1^{(2)} &= \frac{\partial}{\partial x_1}[\lambda^{(1)}]^3 \frac{\partial y_2^{(1)}}{\partial x_2} - \frac{\partial}{\partial x_2}[\lambda^{(1)}]^3 \frac{\partial y_2^{(1)}}{\partial x_1}, \\ \nabla^2 y_2^{(2)} &= \frac{\partial y_1^{(1)}}{\partial x_1}\frac{\partial}{\partial x_2}[\lambda^{(1)}]^3 - \frac{\partial y_1^{(1)}}{\partial x_2}\frac{\partial}{\partial x_1}[\lambda^{(1)}]^3, \end{aligned}\right\} \tag{3.5}$$

in A^0, with the boundary conditions

$$\left.\begin{aligned} \frac{\partial y_1^{(2)}}{\partial n^0} ds^0 &= \frac{1}{2\, h_0\, C_1} dL_1^* + [\lambda^{(1)}]^3 \frac{\partial y_2^{(1)}}{\partial s^0} ds^0, \\ \frac{\partial y_2^{(2)}}{\partial n^0} ds^0 &= \frac{1}{2\, h_0\, C_1} dL_2^* - [\lambda^{(1)}]^3 \frac{\partial y_1^{(1)}}{\partial s^0} ds^0, \end{aligned}\right\} \quad \text{on} \quad C_T^0 \tag{3.6}$$

and

$$y_\alpha^{(2)} = y_\alpha^* \quad \text{on} \quad C_D^0 .$$

The process of successive substitution can be repeated, the approximation $y_\alpha^{(n+1)}$ being determined as the solution to a Poisson equation with inhomogeneous terms in the equation and boundary condition determined by $y_\alpha^{(n)}$ and the boundary data. The solution $y_\alpha^{(n+1)}$, if it exists, will be unique provided C_D^0 is non-zero. If $y_\alpha^{(1)}$ and its derivatives are $O(\mu)$ everywhere for μ large and the Jacobian $J^{(1)}$ is such that $\lambda^{(1)} = O(\mu^{-2})$ everywhere, the difference $y_\alpha^{(2)} - y_\alpha^{(1)}$ satisfies a Poisson boundary value problem with inhomogeneous terms which are $O(\mu^{-5})$, in agreement with the earlier estimate (3.1) on the order of error involved in the first approximation. Assuming that the solution $y_\alpha^{(2)} - y_\alpha^{(1)}$ and its derivatives are $O(\mu^{-5})$, the boundary value problem for the difference $y_\alpha^{(3)} - y_\alpha^{(2)}$ will involve inhomogeneous terms of $O(\mu^{-11})$ for large μ, and so on. Thus a related approach would be to assume that, for large enough μ, the functions y_α/μ can be expanded in power series in μ^{-6}, with coefficients which are twice differentiable functions of (x_1, x_2).

When $\lambda_1 = O(1)$ and $\lambda_2 = O(\mu)$ for large μ, the corresponding estimates for $y_\alpha^{(2)} - y_\alpha^{(1)}$ and $y_\alpha^{(3)} - y_\alpha^{(2)}$ are $O(\mu^{-2})$ and $O(\mu^{-5})$, respectively. For a smooth enough first approximation $y_\alpha^{(1)}$ and a smooth enough region A^0, it is to be expected, in this case and in the previous case, that the process will converge when μ is large enough. However the convergence of the method is not so apparent when a large region of the sheet is in a state close to simple extension so that the first approximation $y_\alpha^{(1)}$ involves principal extension ratios

$$\lambda_1^{(1)} = O(\mu^{-1/2}), \qquad \lambda_2^{(1)} = O(\mu)$$

for large μ. In this case we will have $\lambda^{(1)} = O(\mu^{-1/2})$ and the terms in (3.5), (3.6) involving $y_\alpha^{(1)}$ are $O(\mu^{-1/2})$. Since the difference $y_\alpha^{(2)} - y_\alpha^{(1)}$ will then be $O(\mu^{-1/2})$, the error in $y_\alpha^{(1)}$ can be of the same order and therefore $y_\alpha^{(1)}$ may not determine the non-linear terms in (3.5), (3.6) correct to $O(\mu^{-1/2})$.

A difficulty arises with the method described in this section when a portion of the boundary is traction-free. The first approximation then has a Jacobian $J^{(1)}$ which goes to zero as the unloaded boundary is approached. The terms involving $y_\alpha^{(1)}$ in the equations (3.5), (3.6) for $y_\alpha^{(2)}$ are then singular on the unloaded part of C^0 and in fact the singularity is non-integrable in that the solution $y_\alpha^{(2)}$ cannot remain finite. A modification of the method which avoids this difficulty is given later in Section 5.

We remark that the method of this section for equilibrium problems can also be used for dynamic problems of plane motion with large extensions. For a neo-Hookean material and constant initial thickness, the equations of motion are

$$\nabla^2 y_\alpha - 3\,\lambda\,\frac{\partial\lambda}{\partial y_\alpha} = \frac{\varrho_0}{2\,C_1}\,\frac{\partial^2 y_\alpha}{\partial t^2} \tag{3.7}$$

where ϱ_0 is the density, and a first approximation for $\lambda \ll 1$ leads to the ordinary wave equations

$$\nabla^2 y_\alpha^{(1)} = \frac{\varrho_0}{2\,C_1}\,\frac{\partial^2 y_\alpha^{(1)}}{\partial t^2}\,.$$

When the functions $y_\alpha^{(1)}$ are known, they can be used to estimate the nonlinear terms in (3.7) and inhomogeneous wave equations then govern the second approximation $y_\alpha^{(2)}$. However, since the inhomogeneous terms are derived from solutions to the same wave equation resonance will occur, in general, and this approach will be of limited value. The same difficulty arises in the corresponding approach for the determination of second order effects in elastic wave propagation (see [9], for example); an alternative approach avoids the difficulty (see [10], for example).

4. Some Inclusion Problems

Several basic inclusion problems are considered in this section. For simplicity the inclusion shape is taken to be a circle or an ellipse, but other geometries can be treated in a similar manner when the appropriate conformal mapping is known.

As a first example, we consider an infinite membrane with a circular hole of radius a. The edge of the hole is bonded to a rigid inclusion (or otherwise held fixed) and at infinity the sheet is in a state of biaxial extension with principal extension ratios μ_1 and μ_2 along the x_1- and x_2-axes, respectively, the origin being at the center of the hole.

Applying the approach of Section 3, we find that the harmonic functions $y_\alpha^{(1)}$ of the first approximation are determined by the conditions (3.3) with L_α^* equal to zero in the case when there is no resultant force on the inclusion, and at $r = a$,

$$y_1^{(1)} = a\cos\theta\,, \qquad y_2^{(1)} = a\sin\theta\,.$$

We have therefore

$$y_1^{(1)} = \mu_1\,r\left(1 - \frac{k_1^2}{r^2}\right)\cos\theta\,, \qquad y_2^{(1)} = \mu_2\,r\left(1 - \frac{k_2^2}{r^2}\right)\sin\theta\,, \tag{4.1}$$

where

$$k_\alpha^2 = \left(1 - \frac{1}{\mu_\alpha}\right)a^2\,.$$

Because $y_\alpha^{(1)}$ are harmonic functions, it follows that $|\nabla y_\alpha^{(1)}|^2$ and hence $K^{(1)}$ and $U^{(1)}$ are subharmonic. If we exclude the case of constant strain energy $U^{(1)}$, the sum of the squares of the principal extension ratios Λ_1, Λ_2 obtained from the first approximation must then attain its maximum and minimum at an internal boundary or at infinity. We note also that at the edge of the hole $r = a$ we have

$$\Lambda_1 = 2\mu_1 - 1\,, \quad \Lambda_2 = 1 \quad \text{at} \quad \theta = 0, \pi\,,$$

$$\Lambda_1 = 2\mu_2 - 1\,, \quad \Lambda_2 = 1 \quad \text{at} \quad \theta = \pm\frac{\pi}{2}\,.$$

Thus, for large μ_1, μ_2 principal extension ratios which are close to twice the values at infinity occur near the inclusion.

The first approximation $y_\alpha^{(1)}$ is meaningful only if the Jacobian $J^{(1)} = \Lambda_1 \Lambda_2$ is positive everywhere. It can be shown from (4.1) that this condition requires μ_1 and μ_2 to be greater than one-half. However for a neo-Hookean material the principal force resultants are tensile only if $\Lambda_1 \Lambda_2^2$ and $\Lambda_1^2 \Lambda_2$ are greater than unity. From the values of Λ_1, Λ_2 at the inclusion we see that the first approximation requires μ_1 and μ_2 to be both greater than unity in order to avoid compressive stresses near the inclusion, an indication that wrinkling or folding of the sheet would occur if either of μ_1, μ_2 were less than unity.

If the inclusion is acted upon by a force L^* through the origin at an angle δ to the x_1-axis, the terms

$$-\frac{L^* \cos\delta}{4\pi h_0 C_1} \log\frac{r}{a}\,, \qquad -\frac{L^* \sin\delta}{4\pi h_0 C_1} \log\frac{r}{a}$$

must be added to the expressions for $y_1^{(1)}$ and $y_2^{(1)}$, respectively. The value of L^* is restricted by the conditions that $\Lambda_1 \Lambda_2^2$ and $\Lambda_1^2 \Lambda_2$ be greater than unity everywhere in the membrane.

It is apparent from the nature of the inhomogeneous terms in the differential equations for $y_\alpha^{(2)}$ that the solution for $y_\alpha^{(2)}$ will not be elementary. Considerable simplification results when the deformation has axial symmetry, that is when $\mu_1 = \mu_2 = \mu$ and the sheet is subjected to an all-around tension at infinity. When we substitute the expressions

$$y_1 = \varrho(r)\cos\theta\,, \qquad y_2 = \varrho(r)\sin\theta \tag{4.2}$$

into the exact equilibrium equations, we find that ϱ satisfies, for $r > a$, the ordinary differential equation

$$\frac{d^2\varrho}{dr^2} + \frac{1}{r}\frac{d\varrho}{dr} - \frac{\varrho}{r^2} = \frac{3r}{\varrho^3 (d\varrho/dr)^4}\left[\varrho\frac{d\varrho}{dr} - r\left(\frac{d\varrho}{dr}\right)^2 - r\varrho\frac{d^2\varrho}{dr^2}\right]. \tag{4.3}$$

With the boundary conditions

$$\varrho = a \ \text{at}\ r = a\,, \qquad \frac{d\varrho}{dr} = \mu \ \text{as}\ r \to \infty\,, \tag{4.4}$$

equation (4.3) can be integrated numerically to yield an exact solution. Setting $\mu_1 = \mu_2 = \mu$ in (4.1), the first approximation $\varrho^{(1)}$ to ϱ is given by

$$\varrho^{(1)}(r) = \mu r\left(1 - \frac{k^2}{r^2}\right), \qquad k^2 = a^2\left(1 - \frac{1}{\mu}\right). \tag{4.5}$$

The second approximation $\varrho^{(2)}$ then satisfies the differential equation

$$\frac{d}{dr}\left(\frac{d}{dr}\varrho^{(2)} + \frac{\varrho^{(2)}}{r}\right) = -\frac{12k^4}{\mu^5 r^5}\left[\left(1 + \frac{k^2}{r^2}\right)\left(1 - \frac{k^4}{r^4}\right)^3\right]^{-1},$$

and it is found that

$$\varrho^{(2)}(r) = a_1 r + \frac{b_1}{r} + I(r) \tag{4.6}$$

where

$$I(r) = \frac{3k^2}{32\mu^5}\frac{1}{r}\left[\left(5 - \frac{k^2}{r^2}\right)\log\frac{r^2 - k^2}{r^2 + k^2} + \frac{30(k/r)^6 - 24(k/r)^4 + 26(k/r)^2 + 16}{3(k/r)^2(1 - k^2/r^2)(1 + k^2/r^2)^2}\right],$$

$$a_1 = \mu\left(1 - \frac{1}{2\mu^6}\right), \quad b_1 = a\left[(1 - a_1)\,a - I(a)\right].$$

For a case of moderate deformation, $\mu = 1.24$, it was found that both the first and second approximate solutions given by (4.5) and (4.6) gave values for ϱ/r which were within 0.3% of the values obtained from numerical integration of the exact equation (4.3).

When the circular inclusion is rotated counterclockwise through an angle β about its center, the sheet being uniformly strained at infinity as before, the first approximation $y_\alpha^{(1)}$ can easily be obtained. It is found that as β is increased the Jacobian $J^{(1)}$ remains positive only until the value β_0 is reached, where

$$\beta_0 = \cos^{-1}\left(\frac{1}{\mu_1 + \mu_2} + \frac{|\mu_1 - \mu_2|}{\mu_1 + \mu_2}\right).$$

For a neo-Hookean material the first approximation indicates that the stresses in the sheet at points near the inclusion will cease to be tensile at a value of β somewhat smaller than β_0. For values of β greater than this critical value, folding of the sheet will occur and as β increases the sheet will wrap around the inclusion (assuming it is thicker than the sheet).

The case of an infinite membrane with an elliptic rigid inclusion, semi-axes a, b $(a > b)$, under homogeneous deformation at infinity as before can be treated in a similar fashion. In the case when there is no resultant force on the inclusion, we find that

$$y_1^{(1)} = \frac{c\,\mu_1}{2}\left(e^{\xi} - \varkappa_1 e^{-\xi}\right)\cos\eta, \quad y_2^{(1)} = \frac{c\,\mu_2}{2}\left(e^{\xi} - \varkappa_2 e^{-\xi}\right)\sin\eta,$$

where

$$\varkappa_1 = \left[1 - \frac{2a}{(a+b)\,\mu_1}\right]\left(\frac{a+b}{a-b}\right), \quad \varkappa_2 = \left[1 - \frac{2b}{(a+b)\,\mu_2}\right]\left(\frac{a+b}{a-b}\right), \quad c = \sqrt{a^2 - b^2},$$

and ξ, η are the elliptic coordinates associated with the elliptic inclusion,

$$x_1 = c\cosh\xi\cos\eta, \quad x_2 = c\sinh\xi\sin\eta,$$

the inclusion boundary being given by

$$\xi = \xi_0 = \frac{1}{2}\log\frac{a+b}{a-b},$$

with

$$c\cosh\xi_0 = a, \quad c\sinh\xi_0 = b.$$

At the ends of the major and minor axes of the ellipse the extension ratios for directions normal to the inclusion have the values

$$\frac{\mu_1(a+b) - a}{b}, \quad \frac{\mu_2(a+b) - b}{a},$$

respectively. As in the case of a circular inclusion, μ_1 and μ_2 must be greater than unity if the force resultants of the first approximation are to be tensile (for a neo-Hookean material).

In the limit $b \to 0$ ($a \to c$, $\xi_0 \to 0$), the inclusion degenerates into a line inclusion or splinter of length $2c$ along the x_1-axis. On the line $x_1 \geqslant c$, $x_2 = 0$, the principal extension ratio in the x_2-direction is μ_2 while the principal extension ratio in the x_1-direction is

$$(\mu_1 - 1) \frac{x_1}{(x_1^2 - c^2)^{1/2}} + 1 ,$$

and this tends to ∞ as the end of the splinter is approached.

The case when the major axis of the ellipse is initially inclined at an angle α to the positive x_1-axis ($|\alpha| < \pi/2$) and is at an angle β in the deformed state can also be treated and the first approximation is readily obtained. The details may be found in [8]. If the inclusion is free to rotate, the requirement of zero moment on the inclusion after deformation requires the inclination β to be given by

$$\tan\beta = \frac{a\,\mu_2 + b\,\mu_1}{a\,\mu_1 + b\,\mu_2} \tan\alpha .$$

In the limiting case of a splinter ($b \to 0$), we have, for zero torque on the splinter,

$$\tan\beta = \frac{\mu_2}{\mu_1} \tan\alpha . \tag{4.7}$$

In the pure strain which the sheet suffers at infinity, a line element initially at an angle α to the x_1-axis becomes inclined at an angle β to the x_1-axis with β given by (4.7). Thus the splinter and the line elements at infinity which were initially parallel remain parallel during the deformation, according to the first approximation.

Another example considered in [8] is a circular material inclusion in an infinite membrane under biaxial extension at infinity. The portion $r \leqslant a$ of an infinite sheet is composed of a different neo-Hookean material with material constant $\bar{C}_1$ and initial constant thickness $\bar{h}_0$. We use a bar to indicate quantities associated with the inclusion. The functions $\bar{y}_\alpha^{(1)}$, $y_\alpha^{(1)}$ of the first approximation are harmonic in the regions $r < a$, $r > a$ respectively, and they must satisfy the conditions

$$\bar{y}_\alpha^{(1)} = y_\alpha^{(1)} , \quad \bar{h}_0\,\bar{C}_1 \frac{\partial \bar{y}_\alpha^{(1)}}{\partial n^0} = h_0\,C_1 \frac{\partial y_\alpha^{(1)}}{\partial n^0} \quad \text{at} \quad r = a$$

in order to ensure continuity of traction and displacement at the interface $r = a$. At infinity the functions $y_\alpha^{(1)}$ again satisfy (3.3) with L_α^* zero. It is found that

$$y_1^{(1)} = \mu_1\,r \left[1 + \frac{(1 - m)\,a^2}{(1 + m)\,r^2}\right] \cos\theta , \quad y_2^{(1)} = \mu_2\,r \left[1 + \frac{(1 - m)\,a^2}{(1 + m)\,r^2}\right] \sin\theta , \tag{4.8}$$

$$\bar{y}_1^{(1)} = \frac{2\,\mu_1}{(1 + m)}\,x_1 , \quad \bar{y}_2^{(1)} = \frac{2\,\mu_2}{(1 + m)}\,x_2 , \tag{4.9}$$

where $m = (\bar{h}_0\,\bar{C}_1)/(h_0\,C_1)$.

We note that the inclusion material is in a state of homogeneous deformation with principal extension ratios $2\mu_1/(1+m)$, $2\mu_2/(1+m)$ along the x_1- and x_2-axes, respectively, and the form of the first approximation $y_\alpha^{(1)}$ is independent of the size of the inclusion.

The limiting case $m \to 0$ corresponds to a sheet with a circular hole under biaxial tension at infinity. Setting $m = 0$ in (4.8) we obtain

$$y_1^{(1)} = \mu_1 r \left(1 + \frac{a^2}{r^2}\right) \cos\theta, \quad y_2^{(1)} = \mu_2 r \left(1 + \frac{a^2}{r^2}\right) \sin\theta. \tag{4.10}$$

The extension ratio in the direction normal to the sheet is given by

$$\lambda^{(1)} = \frac{1}{\mu_1 \mu_2} \left[1 - \frac{a^4}{r^4}\right]^{-1}$$

and as the boundary $r = a$ is approached, $\lambda^{(1)}$ becomes infinite. This singular behavior of the first approximation in the presence of traction-free boundaries was discussed in Section 3. We return to this problem in Section 6 after developing an alternative approach in the next section for the determination of $\lambda^{(1)}$.

In order for the solution (4.8), (4.9) to be a reasonable approximation for neo-Hookean materials the transverse extension ratios $\bar{\lambda}^{(1)}$ and $\lambda^{(1)}$ in the inclusion and the exterior material must be small compared to unity. From (4.8), (4.9), this implies $\mu_1 \mu_2$ large and

$$m \ll 2\sqrt{\mu_1 \mu_2} - 1.$$

We note that the limiting case $m \to \infty$ lies outside the range of validity of the first approximation. As $m \to \infty$ expressions (4.8) do not approach the values (4.1) of the solution given previously for a rigid circular inclusion.

The case of an elliptic material inclusion in an infinite membrane under biaxial extension at infinity is treated in [8]. When the major axis of the ellipse lies along the x_1-axis it is shown in [8] that outside the ellipse

$$y_1^{(1)} = \frac{c\,\mu_1}{2} (e^\xi + s_1 e^{-\xi}) \cos\eta, \quad y_2^{(1)} = \frac{c\,\mu_2}{2} (e^\xi + s_2 e^{-\xi}) \sin\eta, \tag{4.11}$$

and in the inclusion

$$\bar{y}_1^{(1)} = \mu_1 s_3 x_1, \quad \bar{y}_2^{(1)} = \mu_2 s_4 x_2,$$

where

$$s_1 = \frac{(1 - m\,b/a)}{(1 - b/a)} s_3, \quad s_2 = \frac{(1 - m\,a/b)}{(a/b - 1)} s_4, \quad s_3 = \frac{(1 + b/a)}{(1 + m\,b/a)}, \quad s_4 = \frac{(1 + a/b)}{(1 + m\,a/b)}.$$

As in Section 4, ξ and η are elliptic coordinates and $c^2 = a^2 - b^2$, where a, b are the semi-axes, with $a > b$. The deformation of the inclusion is again homogeneous to a first approximation and this feature of the solution still applies when the principal directions of strain at infinity are inclined to the axes of the ellipse [8]. An analog in two-dimensional electrostatics exists in the problem of an elliptic dielectric placed in an electric field of uniform strength at infinity. The field produced in the dielectric is uniform.

5. Successive Substitutions for Traction-free Boundaries

When the membrane has an edge which is traction-free, it was found in Section 3 that the proposed method of successive approximations breaks down. As the edge is approached $J^{(1)} \to 0$ and the partial differential equations for $y_\alpha^{(2)}$ have a non-integrable singularity when $\lambda^{(1)}$ is taken as the inverse of $J^{(1)}$. We note that the material at a traction-free boundary is under simple tension so that on the boundary

$$\lambda_n = \lambda = (\lambda_s)^{-1/2} = \left[\left(\frac{\partial y_1}{\partial s^0}\right)^2 + \left(\frac{\partial y_2}{\partial s^0}\right)^2\right]^{-1/4}, \tag{5.1}$$

where λ_s and λ_n denote the principal extension ratios in directions tangential and normal to the edge respectively.

In this section we describe an alternative method for the determination of a first approximation $\lambda^{(1)}$ to λ which remains valid in the neighborhood of a traction-free boundary. In order to develop the method we introduce two stress functions φ and ψ and we show that the locations y_α can be found straightforwardly when φ and ψ are known. Although the functions φ and ψ can be obtained, in principle, by successive approximation, and problems solved in this manner, the main use of the stress functions here is to provide intermediate steps leading to the alternative procedure for determining a first approximation $\lambda^{(1)}$. When $\lambda^{(1)}$ is known it can be used with the first approximation $y_\alpha^{(1)}$, as before, in the equations for $y_\alpha^{(2)}$ and the method of Section 3 can then proceed as described. If the second and higher approximations are not required, the first approximation $y_\alpha^{(1)}$ being considered sufficiently accurate to describe the deformed geometry, the method here leads to values for $\lambda^{(1)}$ which accurately describe the thinning of the sheet, including both the interior and regions near traction-free edges.

The equilibrium equations (2.5) imply the existence of stress functions φ and ψ such that

$$\left.\begin{aligned} \frac{\partial \varphi}{\partial x_1} &= -\frac{\partial y_1}{\partial x_2} - \lambda^3 \frac{\partial y_2}{\partial x_1}, \\ \frac{\partial \varphi}{\partial x_2} &= \frac{\partial y_1}{\partial x_1} - \lambda^3 \frac{\partial y_2}{\partial x_2}, \end{aligned}\right\} \qquad \left.\begin{aligned} \frac{\partial \psi}{\partial x_1} &= -\frac{\partial y_2}{\partial x_2} + \lambda^3 \frac{\partial y_1}{\partial x_1}, \\ \frac{\partial \psi}{\partial x_2} &= \frac{\partial y_2}{\partial x_1} + \lambda^3 \frac{\partial y_1}{\partial x_2}, \end{aligned}\right\} \tag{5.2}$$

We then have, assuming that $\lambda \neq 1$,

$$\left.\begin{aligned} \frac{\partial y_1}{\partial x_1} &= \frac{1}{(1-\lambda^6)}\left(\frac{\partial \varphi}{\partial x_2} - \lambda^3 \frac{\partial \psi}{\partial x_1}\right), \\ \frac{\partial y_1}{\partial x_2} &= \frac{1}{(1-\lambda^6)}\left(-\lambda^3 \frac{\partial \psi}{\partial x_2} - \frac{\partial \varphi}{\partial x_1}\right), \end{aligned}\right\} \quad \left.\begin{aligned} \frac{\partial y_2}{\partial x_1} &= \frac{1}{(1-\lambda^6)}\left(\lambda^3 \frac{\partial \varphi}{\partial x_1} + \frac{\partial \psi}{\partial x_2}\right), \\ \frac{\partial y_2}{\partial x_2} &= \frac{1}{(1-\lambda^6)}\left(-\frac{\partial \psi}{\partial x_1} + \lambda^3 \frac{\partial \varphi}{\partial x_2}\right), \end{aligned}\right\} \tag{5.3}$$

where λ is given in terms of φ and ψ by

$$\lambda^{12} - j\,\lambda^7 - 2\,\lambda^6 - \lambda^4\,(|\nabla\varphi|^2 + |\nabla\psi|^2) - j\,\lambda + 1 = 0\,, \tag{5.4}$$

with

$$j = \frac{\partial \varphi}{\partial x_1}\frac{\partial \psi}{\partial x_2} - \frac{\partial \varphi}{\partial x_2}\frac{\partial \psi}{\partial x_1}.$$

Furthermore, compatibility of equations (5.3) requires

$$\left.\begin{aligned} \nabla^2\varphi &= \frac{\partial\psi}{\partial x_1}\frac{\partial\lambda^3}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\frac{\partial\lambda^3}{\partial x_1} - \frac{6\,\lambda^5}{1-\lambda^6} \\ &\qquad \times\left[\frac{\partial\lambda}{\partial x_2}\left(\frac{\partial\varphi}{\partial x_2} - \lambda^3\frac{\partial\psi}{\partial x_1}\right) + \frac{\partial\lambda}{\partial x_1}\left(\lambda^3\frac{\partial\psi}{\partial x_2} + \frac{\partial\varphi}{\partial x_1}\right)\right], \\ \nabla^2\psi &= \frac{\partial\lambda^3}{\partial x_1}\frac{\partial\varphi}{\partial x_2} - \frac{\partial\lambda^3}{\partial x_2}\frac{\partial\varphi}{\partial x_1} - \frac{6\,\lambda^5}{1-\lambda^6} \\ &\qquad \times\left[\frac{\partial\lambda}{\partial x_2}\left(\lambda^3\frac{\partial\varphi}{\partial x_1} + \frac{\partial\psi}{\partial x_2}\right) + \frac{\partial\lambda}{\partial x_1}\left(\frac{\partial\psi}{\partial x_1} - \lambda^3\frac{\partial\varphi}{\partial x_2}\right)\right]. \end{aligned}\right\} \quad (5.5)$$

Equations (5.5) constitute the governing differential equations for the functions φ and ψ.

At a boundary C_T^0 where traction components dL_α^* are prescribed, the tangential derivatives of φ, ψ are prescribed through

$$\frac{\partial\varphi}{\partial s^0}\,ds^0 = \frac{dL_1^*}{2\,h_0\,C_1}, \qquad \frac{\partial\psi}{\partial s^0}\,ds^0 = \frac{dL_2^*}{2\,h_0\,C_1} \quad \text{on} \quad C_T^0. \tag{5.6}$$

In particular, for a closed contour π^0 which is free from traction, we have

$$\varphi = \text{const.} = a_1, \quad \psi = \text{const.} = b_1 \quad \text{on} \quad \pi^0,$$

where a_1, b_1 may be set equal to zero without loss in generality if π^0 is the only traction-free contour. When there are N traction-free contours π_n^0 $(n = 1, 2, \dots, N)$ say, then

$$\varphi = a_n, \quad \psi = b_n \quad \text{on} \quad \pi_n^0 \quad (n = 1, 2, \dots, N),$$

where a_n, b_n are constants. Only one of the constant pairs (a_n, b_n) can, in general, be set equal to zero, the others being determined by the condition that the integrals y_α of equations (5.3) be single-valued. If the sheet extends to infinity in all directions and if the sheet is in uniform biaxial tension at infinity with extension ratios μ_1, μ_2 along the axes, then we have, from (5.2) and (3.3), as $r \to \infty$

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial x_1} &= \frac{1}{4\,\pi\,h_0\,C_1\,r}\left[L_1^*\sin\theta + \frac{L_2^*\cos\theta}{(\mu_1\,\mu_2)^3}\right] + o\left(\frac{1}{r}\right), \\ \frac{\partial\varphi}{\partial x_2} &= \mu_1 - \frac{\mu_2}{(\mu_1\,\mu_2)^3} + \frac{1}{4\,\pi\,h_0\,C_1\,r}\left[\frac{L_2^*\sin\theta}{(\mu_1\,\mu_2)^3} - L_1^*\cos\theta\right] + o\left(\frac{1}{r}\right), \\ \frac{\partial\psi}{\partial x_1} &= \frac{\mu_1}{(\mu_1\,\mu_2)^3} - \mu_2 + \frac{1}{4\,\pi\,h_0\,C_1\,r}\left[L_2^*\sin\theta - \frac{L_1^*\cos\theta}{(\mu_1\,\mu_2)^3}\right] + o\left(\frac{1}{r}\right), \\ \frac{\partial\psi}{\partial x_2} &= -\frac{1}{4\,\pi\,h_0\,C_1\,r}\left[L_2^*\cos\theta + \frac{L_1^*\sin\theta}{(\mu_1\,\mu_2)^3}\right] + o\left(\frac{1}{r}\right) \end{aligned}\right\} \quad (5.7)$$

where L_α^* are the components of the resultant of all external forces acting on the internal boundaries of the sheet.

Although traction boundary conditions are simplified by the use of the stress functions, boundary conditions of place are rendered more complex. From (5.3) we see that

$$\frac{\partial y_1^*}{\partial s^0} = \frac{-1}{(1-\lambda^6)}\left(\frac{\partial\varphi}{\partial n^0} + \lambda^3\frac{\partial\psi}{\partial s^0}\right), \qquad \frac{\partial y_2^*}{\partial s^0} = \frac{1}{(1-\lambda^6)}\left(-\frac{\partial\psi}{\partial n^0} + \lambda^3\frac{\partial\varphi}{\partial s^0}\right), \tag{5.8}$$

and $\partial y_\alpha^*/\partial s^0$ will be known on a boundary C_D^0 where y_α^* are prescribed.

When $\lambda \ll 1$ a procedure similar to that of Section 3 can be used to find successive approximations to the stress functions φ and ψ. The first approximations $\varphi^{(1)}$, $\psi^{(1)}$ are taken to be solutions to (5.5) with λ set equal to zero. Like $y_\alpha^{(1)}$, they are harmonic functions in the domain in question. At a boundary C_D^0 where y_α are prescribed to be the functions y_α^*, we can approximate to the boundary conditions on $\varphi^{(1)}$, $\psi^{(1)}$ by taking λ to be zero in (5.8) and requiring

$$\frac{\partial \varphi^{(1)}}{\partial n^0} = -\frac{\partial y_1^*}{\partial s^0}, \quad \frac{\partial \psi^{(1)}}{\partial n^0} = -\frac{\partial y_2^*}{\partial s^0} \quad \text{on} \quad C_D^0 .$$

In contrast to the functions $y_\alpha^{(1)}$, the functions $\varphi^{(1)}$, $\psi^{(1)}$ satisfy the exact conditions (5.6) on φ and ψ at a boundary where the traction is given. With these boundary conditions on the harmonic functions $\varphi^{(1)}$, $\psi^{(1)}$, they will be harmonic conjugates of $y_1^{(1)}$, $y_2^{(1)}$, respectively, when $y_\alpha^{(1)}$ satisfy (3.2) on C_T^0 and $y_\alpha^{(1)} = y_\alpha^*$ on C_D^0 and this is in agreement with setting $\lambda = 0$ in (5.2). This is not true, however, when the sheet extends to infinity and conditions (5.7) are imposed because in this case the boundary conditions at infinity, (5.7) on $\varphi^{(1)}$, $\psi^{(1)}$ and (3.3) on $y_\alpha^{(1)}$, do not allow $(\varphi^{(1)}, y_1^{(1)})$ and $(\psi^{(1)}, y_2^{(1)})$ to be conjugate harmonic functions.

A first approximation $\lambda^{(1)}$ to the transverse extension ratio λ is determined by using $\varphi^{(1)}$, $\psi^{(1)}$ for φ, ψ in (5.4), that is by the appropriate root of the algebraic equation

$$\lambda^{12} - j^{(1)} \lambda^7 - 2\lambda^6 - \lambda^4 [|\nabla \varphi^{(1)}|^2 + |\nabla \psi^{(1)}|^2] - j^{(1)} \lambda + 1 = 0 \tag{5.9}$$

where

$$j^{(1)} = \frac{\partial \varphi^{(1)}}{\partial x_1} \frac{\partial \psi^{(1)}}{\partial x_2} - \frac{\partial \varphi^{(1)}}{\partial x_2} \frac{\partial \psi^{(1)}}{\partial x_1} .$$

When $\lambda \ll 1$ and $j^{(1)}$ is not small, terms of λ^4 and higher in equation (5.9) can be neglected for our purposes. Near a traction-free edge of the membrane, however, $j^{(1)}$ becomes small and vanishes on the edge and the term in λ^4 must be retained in the equation even though λ is still much smaller than unity. Thus the equation

$$\lambda^4 [|\nabla \varphi^{(1)}|^2 + |\nabla \psi^{(1)}|^2] + j^{(1)} \lambda - 1 = 0 \tag{5.10}$$

will determine $\lambda^{(1)}$ with sufficient accuracy both on the boundary and inside the sheet. Equation (5.10) has only one positive root when $j^{(1)}$ is positive. We remark that the approximation (5.10) to equation (5.9) will not apply near a point in the sheet which is unstressed. At such a point $\lambda = 1$ and the derivatives of φ, ψ vanish, equation (5.9) being satisfied. Such stress-free points occur at projecting corners in a traction-free portion of the boundary.

When $(\varphi^{(1)}, y_1^{(1)})$ and $(\psi^{(1)}, y_2^{(1)})$ are conjugate functions, we can write (5.10) as

$$\lambda^4 [|\nabla y_1^{(1)}|^2 + |\nabla y_2^{(1)}|^2] + J^{(1)} \lambda - 1 = 0 \tag{5.11}$$

with

$$J^{(1)} = \frac{\partial y_1^{(1)}}{\partial x_1} \frac{\partial y_2^{(1)}}{\partial x_2} - \frac{\partial y_1^{(1)}}{\partial x_2} \frac{\partial y_2^{(1)}}{\partial x_1} .$$

When $J^{(1)}$ is not small, the term in λ^4 can be neglected and we have

$$\lambda^{(1)} = \frac{1}{J^{(1)}}$$

which is the value given for $\lambda^{(1)}$ in (3.4). On the traction-free edge $J^{(1)}$ is zero and $\lambda^{(1)}$ is determined by

$$\lambda^{(1)} = [|\nabla y_1^{(1)}|^2 + |\nabla y_2^{(1)}|^2]^{-1/4}$$

or, since the normal derivatives of $y_\alpha^{(1)}$ vanish at the edge, by

$$\lambda^{(1)} = \left[\left(\frac{\partial y_1^{(1)}}{\partial s^0}\right)^2 + \left(\frac{\partial y_2^{(1)}}{\partial s^0}\right)^2\right]^{-1/4}$$

in agreement with (5.1). It is because $\varphi^{(1)}$ and $\psi^{(1)}$ satisfy exact boundary conditions where traction is prescribed that equation (5.11) for $\lambda^{(1)}$ yields reliable results up to and including a traction-free boundary.

It may be noted that the stress functions φ, ψ introduced in this section are directly related to the Airy stress function used for two-dimensional stress fields. It is easy to show that, save for a multiplicative factor,

$$\frac{\partial \varphi}{\partial y_1} = -T_{12}, \quad \frac{\partial \varphi}{\partial y_2} = T_{11}, \quad \frac{\partial \psi}{\partial y_1} = -T_{22}, \quad \frac{\partial \psi}{\partial y_2} = T_{21} = T_{12},$$

and

$$\varphi = \frac{\partial \chi}{\partial y_2}, \quad \psi = -\frac{\partial \chi}{\partial y_1}$$

where χ is the usual Airy stress function.

6. Some Problems with Traction-free Boundaries

Several simple examples involving membranes with traction-free edges are considered in this section and the application of the modified method is illustrated.

The deformation of an infinite membrane with a circular hole of radius a subjected to uniform biaxial extension at infinity was considered in Section 4 as a limiting case of a circular material inclusion. The first approximation (4.10) shows that after deformation the hole of radius a becomes an ellipse with major and minor axes of lengths $2\,\mu_1\,a$ and $2\,\mu_2\,a$, respectively, μ_1 being the larger of the two extension ratios.

The harmonic stress functions $\varphi^{(1)}$, $\psi^{(1)}$ which satisfy exact boundary conditions at the hole $r = a$ and at infinity are found to be

$$\varphi^{(1)} = \gamma_1\, r \sin\theta \left(1 - \frac{a^2}{r^2}\right), \quad \psi^{(1)} = -\gamma_2\, r \cos\theta \left(1 - \frac{a^2}{r^2}\right) \tag{6.1}$$

where

$$\gamma_1 = \mu_1 - \frac{\mu_2}{(\mu_1\,\mu_2)^3}, \quad \gamma_2 = \mu_2 - \frac{\mu_1}{(\mu_1\,\mu_2)^3}.$$

At the edge of the hole $r = a$, we find that

$$\lambda^{(1)} = \{2\,[\gamma_1^2 + \gamma_2^2 - (\gamma_1^2 - \gamma_2^2)\cos 2\,\theta]\}^{-1/4}.$$

A special case of interest is the axi-symmetric deformation of an infinite sheet containing a circular hole. In this case $\mu_1 = \mu_2 = \mu$ and equations (4.10) assume the simple forms

$$y_1^{(1)} = \varrho^{(1)}(r)\cos\theta, \quad y_2^{(1)} = \varrho^{(1)}(r)\sin\theta; \quad \varrho^{(1)}(r) = \mu\, r\left(1 + \frac{a^2}{r^2}\right). \tag{6.2}$$

Because of the symmetry of the deformation, the exact formulation itself can be greatly simplified. As in Section 4, if we assume y_α to be given by (4.2), then $\varrho(r)$ satisfies the ordinary differential equation (4.3). The condition (4.4) on $d\varrho/dr$ at infinity remains unchanged but at $r = a$, the traction-free condition requires

$$\frac{d\varrho}{dr} = \lambda^3 \frac{\varrho}{r} \quad \text{or} \quad \left(\frac{d\varrho}{dr}\right)^2 \frac{\varrho}{r} = 1\,,$$

since

$$\lambda = \frac{r}{\varrho\,(d\varrho/dr)}$$

in this case.

Equations equivalent to (4.3) were obtained by RIVLIN and THOMAS [4] and, by successive application of Taylor series expansions starting from the edge of the hole, they were able to find numerical solutions for given values of the circumferential extension ratio ϱ/r at the edge of the hole $r = a$. A more direct method is to integrate equation (4.3) numerically, as in the rigid inclusion problem of Section 4. Figure 1

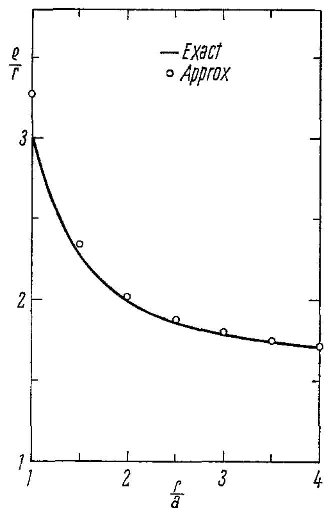

Figure 1

Variation of ϱ/r with r/a for sheet with a circular hole under extension ratio $\mu = 1.62$ at infinity; exact and first approximation values.

compares the exact numerical solution ϱ/r and the approximate solution calculated from (6.2) for all-around extension of moderate amount $\mu = 1.62$ at infinity, the corresponding circumferential extension ratio at the hole being 3.0. A discrepancy of 10% occurs at the edge of the hole but the difference diminishes rapidly as r increases and at a distance four times the radius of the hole, the difference is slight. The transverse extension ratio λ is plotted in Figure 2 against the radius, the approximate values determined from $\varphi^{(1)}$, $\psi^{(1)}$ with $\mu_1 = \mu_2 = \mu$ through (5.10) being shown as circled points near the curve for the exact values. It can be seen that (5.10) provides good estimates for λ over the whole range of r. In contrast, values for λ determined from $y_\alpha^{(1)}$ through (5.11) are much less accurate and they are shown as crosses in the figure.

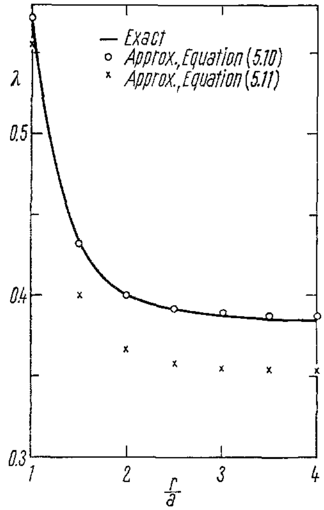

Figure 2
Variation of thickness ratio λ for $\mu = 1.62$; exact and first approximation values.

Comparisons between the exact (numerical) solution and the first approximation were also made for the case $\mu = 3.03$. Since the figures for $\mu = 3.03$ corresponding to Figures 1 and 2 for $\mu = 1.62$ would show no difference between the exact and approximate solutions for either ϱ/r or λ, the results are not shown here (the variation of ϱ/r with r for $\mu = 3.03$ is given in [4]).

The deformation of an infinite membrane with a traction-free elliptic hole of semi-axes a and b subjected to biaxial extension at infinity parallel to the axes of the ellipse can likewise be obtained by setting m equal to zero in (4.11) which then becomes

$$y_1^{(1)} = \frac{c\,\mu_1}{a-b}\,(a\cosh\xi - b\sinh\xi)\cos\eta\,, \quad y_2^{(1)} = \frac{c\,\mu_2}{a-b}\,(a\cosh\xi - b\sinh\xi)\sin\eta\,. \tag{6.3}$$

According to (6.3), the hole is again an ellipse in the deformed state with semi-axes of lengths $(a+b)\,\mu_1$ and $(a+b)\,\mu_2$. When $\mu_1 = \mu_2 = \mu$, the hole is always deformed into a circle of radius $(a+b)\,\mu$.

Stress functions $\varphi^{(1)}$, $\psi^{(1)}$ which satisfy exact boundary conditions at infinity and on the ellipse are readily determined. It can be shown that the first approximation $\lambda^{(1)}$ to the transverse extension ratio λ attains its extreme values

$$\left[\frac{a}{(a+b)\,\gamma_1}\right]^{1/2}, \quad \left[\frac{b}{(a+b)\,\gamma_2}\right]^{1/2},$$

at the ends of the major and minor axes of the ellipse. When $a\,\mu_2 = b\,\mu_1$, the hole is deformed into an ellipse of similar shape and since $a\,\gamma_2$ and $b\,\gamma_1$ are nearly equal the edge of the deformed hole has nearly constant thickness.

In the limit as b goes to zero, the hole degenerates into a crack or slit of length $2\,c$ along the x_1-axis. Setting $b = 0$ in (6.3) we have

$$y_1^{(1)} = \mu_1\,c\cosh\xi\cos\eta = \mu_1\,x_1\,, \quad y_2^{(1)} = \mu_2\,c\cosh\xi\sin\eta\,, \tag{6.4}$$

and we see that the crack $\xi = 0$ becomes an ellipse with semi-axes $\mu_1 c$ and $\mu_2 c$ in the deformed state. It may be noted also that the transverse extension ratio $\lambda^{(1)}$ has the unique limit zero as the tip of the crack is approached. Equations (6.4) can be written as

$$y_1^{(1)} = \mu_1 x_1 , \quad y_2^{(1)} = \frac{\mu_2}{\sqrt{2}} \{c^2 - x_1^2 + x_2^2 + [(c^2 - x_1^2 - x_2^2)^2 + 4 c^2 x_2^2]^{1/2}\}^{1/2} . \quad (6.5)$$

Figure 3 indicates the deformation (6.5) for the sheet with a crack when $\mu_1 = \mu_2 = 2.0$. Because of symmetry, only the first quadrant of the plane is shown. The solid lines

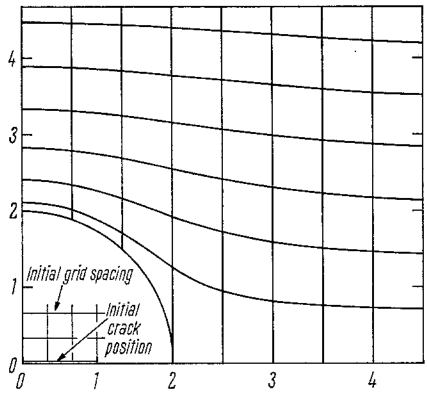

Figure 3
Deformation of a square grid near a crack; $\mu_1 = \mu_2 = 2$ at infinity.

initially formed a square grid of lines one-third of a unit apart. Initially the crack extended from -1 to 1 on the x_1-axis, and it is deformed into a circle of radius 2 units. Vertical grid lines remain vertical and the deformation is most severe at the tip of the crack, as expected. In a simple experiment, a square grid of lines approximately 0.1 cm apart was ruled on a piece of rubber sheet (cut from a cylindrical toy balloon), the ruled area being approximately 5.0 cm by 3.5 cm. A slit 0.6 cm long was made with a razor blade and the rubber was stretched so that the edges of the ruled area were close to a state of all-around extension with extension ratio 2. The deformation

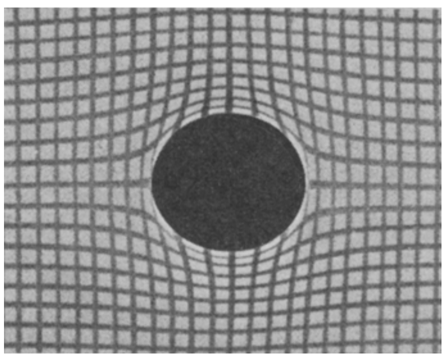

Figure 4
Deformation of a square grid near a slit in a rubber sheet under all-around stretching with extension ratio 2.

of the grid near the slit is shown in the photograph of Figure 4. The agreement between Figure 4 and the theoretical results indicated in Figure 3 is quite good, even though the neo-Hookean form of the strain-energy function is inadequate for rubber-like materials at large strains. A detailed experimental examination of the deformation near a crack tip in a rubber sheet under moderate overall extensions has been made by KNAUSS [11].

When the principal axes of strain at infinity are inclined to the x_1-, x_2-axes, we require

$$y_\alpha = c_{\alpha\beta}\, x_\beta + o\left(\frac{1}{r}\right) \quad \text{as} \quad r \to \infty\,, \tag{6.6}$$

where $c_{\alpha\beta}$ are known constants. If we define the harmonic functions z_α through

$$z_1 = \frac{c}{a-b}\,(a\cosh\xi - b\sinh\xi)\cos\eta\,, \quad z_2 = \frac{c}{a-b}\,(a\cosh\xi - b\sinh\xi)\sin\eta\,,$$

we see that $z_\alpha = x_\alpha$ at infinity and the normal derivatives of z_α vanish on the elliptic boundary. The first approximation $y_\alpha^{(1)}$ to y_α for the sheet with an elliptic hole and the deformation (6.6) at infinity is then $y_\alpha^{(1)} = c_{\alpha\beta}\, z_\beta$. Under the deformation in which a particle at the point x_α goes to z_α, the elliptic hole with semi-axes a, b becomes a circle of radius $(a+b)$. The transformation in which z_α goes to $y_\alpha^{(1)}$ subjects the whole plane to the deformation at infinity. Thus according to the first approximation, under all orientations the elliptic hole becomes an ellipse with semi-axes of lengths $(a+b)\,\mu_1$, $(a+b)\,\mu_2$ parallel to the principal directions of strain at infinity, where μ_1 and μ_2 are the principal extension ratios at infinity.

For a sheet with N holes bounded by contours π_n^0, $n = 1, 2, \ldots, N$, we introduce harmonic functions z_α which have zero normal derivatives on π_n^0 and which are such that $z_\alpha = x_\alpha$ at infinity. The first approximation to the deformation when (6.6) holds at infinity will then be $y_\alpha^{(1)} = c_{\alpha\beta}\, z_\beta$. If the contours π_n' obtained from π_n^0 by the transformation in which x_α goes to z_α are drawn on the undeformed sheet at infinity, the holes in the stretched sheet will assume the same shape, orientation and relative position as the contours π_n' drawn on the sheet at infinity.

The next example is related to the experimental determination of the stress-strain relation in pure shear [6, 7]. A short wide strip of rubber is stretched between clamps applied to the long edges of the sheet. The extension ratio in the direction of the width of the sheet is then almost unity and the sheet is in a state of pure shear if the volume remains unchanged. If the strip has width a and height b and the origin is taken at the center of the sheet with the x_1-axis along the width of the sheet, the harmonic functions $y_\alpha^{(1)}$ must satisfy the boundary conditions

$$\frac{\partial y_\alpha^{(1)}}{\partial x_1} = 0 \quad \text{on} \quad x_1 = \pm\frac{a}{2}\,,$$

and

$$y_1^{(1)} = x_1\,, \quad y_2^{(1)} = \mu\, x_2 \quad \text{on} \quad x_2 = \pm\frac{b}{2}\,,$$

where μb is the height of the deformed strip. We obtain

$$\left.\begin{aligned} y_1^{(1)} &= \frac{4a}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\,\frac{\cosh(2n+1)\,\pi\, x_2/a}{\cosh(2n+1)\,\pi\, b/2a}\,\sin\frac{2n+1}{a}\,\pi\, x_1 \\ &= x_1 - F(x_1, x_2)\,, \quad y_2^{(1)} = \mu\, x_2 \end{aligned}\right\} \tag{6.7}$$

where

$$F(x_1, x_2) = \frac{4\,b}{\pi^2} \sum_{n=0}^{\infty} \frac{(-)^n}{(2\,n+1)^2} \, \frac{\sinh\,(2\,n+1)\,\pi\,x_1/b}{\cosh\,(2\,n+1)\,\pi\,a/2\,b} \cos \frac{2\,n+1}{b} \pi\,x_2 \,.$$

According to (6.7), lines of the sheet initially horizontal remain so after deformation and the location $y_1^{(1)}$ is independent of μ. The shortening S of the line midway between the clamped edges is given by

$$S = 0.742\,\frac{b}{a} \times 100\% \,, \tag{6.8}$$

which decreases with b/a but is independent of μ. For b/a small, the extension ratio λ_1 in the direction parallel to the clamped edges of the sheet is substantially unity, and in the limit

$$\lambda_1 = 1\,, \quad \lambda_2 = \mu\,, \quad \lambda = \frac{1}{\mu}\,,$$

a state of pure shear.

Since the material at the free edges is in simple extension with extension ratio μ approximately, a better first approximation for y_1 is the harmonic function which satisfies the boundary conditions

$$y_1^{(1)} = x_1 \quad \text{on} \quad x_2 = \pm\frac{b}{2}\,, \qquad \frac{\partial y_1^{(1)}}{\partial x_1} = \frac{1}{\sqrt{\mu}} \quad \text{on} \quad x_1 = \pm\frac{a}{2}\,.$$

Hence we have

$$y_1^{(1)} = x_1 - \left(1 - \frac{1}{\sqrt{\mu}}\right) F(x_1, x_2)\,, \tag{6.9}$$

and the shortening S of the middle line is now given by

$$S = 0.742\,\frac{b}{a}\left(1 - \frac{1}{\sqrt{\mu}}\right) \times 100\%\,. \tag{6.10}$$

If we write the second approximation $y_1^{(2)}$ to y_1 as

$$y_1^{(2)} = y_1^{(1)} + \omega\,,$$

in which $y_1^{(1)}$ is given by (6.9), then ω satisfies the Poisson equation

$$\nabla^2\omega = -2\,\pi\,\varrho\,,$$

with

$$\varrho = \frac{3\,\mu}{2\,\pi}\,[J^{(1)}]^{-4}\,\frac{\partial J^{(1)}}{\partial x_1}\,, \qquad J^{(1)} = \mu\,\frac{\partial y_1^{(1)}}{\partial x_1}\,.$$

The boundary conditions on ω are

$$\omega = 0 \quad \text{on} \quad x_2 = \pm\frac{b}{2}\,, \qquad \frac{\partial\omega}{\partial x_1} = 0 \quad \text{on} \quad x_1 = \pm\frac{a}{2}\,,$$

and we see that ω will be zero on $x_1 = 0$.

Now as x_1 goes from 0 to $a/2$, $J^{(1)}$ decreases from μ to $\sqrt{\mu}$ so that $\partial J^{(1)}/\partial x_1$ is negative for $x_1 > 0$. Because $J^{(1)}$ is even in x_1, we see then that ϱ is odd in x_1 and negative for $x_1 > 0$. In the terminology of electrostatics, for $x_1 > 0$ the function ω is the potential of a distribution of (negative) charge with density ϱ in a rectangular sheet which has zero potential at three sides and zero charge line-density at the fourth.

The total 'charge' in the right half of the strip is

$$Q = \int_{-b/2}^{b/2} \int_{0}^{a/2} \varrho \, dx_1 \, dx_2 = -\frac{b}{2\pi} \left[\{J^{(1)}\}^{-3}\right]_0^{a/2} = -\frac{b}{2\pi\sqrt{\mu}} (1 - \mu^{-3/2}) \, .$$

For b/a small, the change in $J^{(1)}$ from μ at $x_1 = 0$ to $\sqrt{\mu}$ at $x_1 = a/2$ occurs mostly in a narrow band near the traction-free edge $x_1 = a/2$. For a good estimate ω^* of ω, therefore, we can assume that all of the charge is concentrated along the line $x_1 = a/2$ with charge density

$$\frac{Q}{b} = -\frac{1}{2\pi\sqrt{\mu}} (1 - \mu^{-3/2})$$

per unit length. Thus ω^* is the harmonic function which satisfies the boundary conditions

$$\omega^* = 0 \quad \text{on} \quad x_1 = 0 \, , \quad x_2 = \pm \frac{b}{2} \, , \qquad \frac{\partial \omega^*}{\partial x_1} = -\frac{1}{\sqrt{\mu}} (1 - \mu^{-3/2}) \quad \text{on} \quad x_1 = \frac{a}{2} \, .$$

It follows that

$$\omega^* = -\frac{1}{\sqrt{\mu}} (1 - \mu^{-3/2}) \, F(x_1, x_2) \, ,$$

and with this value for ω we have

$$y_1^{(2)} = x_1 - \left(1 - \frac{1}{\mu^2}\right) F(x_1, x_2) \, . \tag{6.11}$$

The same result can be obtained by means of the stress functions φ and ψ [8]. We note that the second approximation $y_\alpha^{(2)}$ to y_α differs from $y_\alpha^{(1)}$ by terms which are $O(\mu^{-2})$, in agreement with the estimate in Section 3.

According to (6.11), the shortening S of the middle line is given by

$$S = 0.742 \, \frac{b}{a} \left(1 - \frac{1}{\mu^2}\right) \times 100\% \, . \tag{6.12}$$

When $a/b = 15$, S is 4.8% when μ is 6.2. In an experiment with a strip of rubber having dimensions such that $a/b = 15$, TRELOAR [6] observed for $\mu = 6.2$ a shortening of the middle line of 12%, which is more than twice the theoretical value $S = 4.8\%$ for a neo-Hookean material. RIVLIN and SAUNDERS [7] conducted a similar experiment and they report a shortening of 3% for the extension ratio $\mu = 2.2$. The ratio a/b for the specimen employed in their experiment is not given in [7] but a figure suggests that the ratio $a/b = 20$ was used. With $a/b = 20$ and $\mu = 2.2$, formula (6.12) predicts a shortening of 2.9%.

The discrepancy between theory and experiment for the large extension ratio $\mu = 6.2$ is due to the fact that the neo-Hookean form is not a good representation for the strain energy function of the rubber for extension ratios greater than 2 or 3. A better strain energy function for rubber is the Mooney form

$$U = h_0 \, C_1 \left[\lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2 \, \lambda_2^2} + \Gamma \left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \lambda_1^2 \, \lambda_2^2\right)\right] ,$$

where $\Gamma = C_2/C_1$ and C_2, C_1 are material constants. For a state of pure shear with

$\lambda_1 = 1$, $\lambda_2 = \mu$ the stress resultant across the width of the strip is

$$T_1 = \frac{2\,h_0\,C_1}{\mu}\left[\left(1 - \frac{1}{\mu^2}\right) + \Gamma\,(\mu^2 - 1)\right].$$

Even a small value for Γ increases T_1 significantly at large values of μ. A greater curvature is then required at the traction-free edges in order to provide the resultant T_1 in the middle of the strip, and the shortening of the strip is increased. With the expression (6.11) for y_1 for a neo-Hookean material, numerical results for the case $a/b = 15$ and $\mu = 6.2$ show that straight lines initially vertical on the sheet remain quite straight except in regions very close to the traction-free edges, within a distance of the order of $a/25$. This is in contrast to the experiment of TRELOAR [6] in which appreciable curvature was observed of vertical lines initially distant $a/5$ from the traction-free edges.

In order to superpose pure shear on simple extension, the strip is stretched in simple extension in the x_2-direction with extension ratio $1/\lambda_2^2$ before the clamps are applied [7]. The clamps are then moved apart so that the extension ratio in the x_2-direction becomes μ while that in the x_1-direction is substantially λ_2 throughout the sheet. The first approximation for y_2 is

$$y_2^{(1)} = \mu\,x_2\,,$$

while a second approximation for y_1 is found to be

$$y_1^{(2)} = \lambda_2\left[x_1 - \left(1 - \frac{1}{\lambda_2^4\,\mu^2}\right) F(x_1, x_2)\right].$$

The shortening S of the middle line is then

$$S = 0.742\,\frac{b}{a}\left(1 - \frac{1}{\lambda_2^4\,\mu^2}\right) \times 100\%\,.$$

As a last example, we consider a sheet which has, in the undeformed state, a sharp corner with straight edges on one of its boundaries. The origin of the coordinate system is taken at the vertex of the corner and the x_1-axis is taken along the bisector of the corner angle. For $r \leqslant a$, say, the boundaries at the corner will be the lines $\theta = \pm\,\alpha$, where $2\,\alpha$ is the angle of the corner.

If the sides of the corner are traction-free, the harmonic functions $y_\gamma^{(1)}$ of the first approximation have zero normal derivatives on $\theta = \pm\,\alpha$. When $y_1 = \mu_1\,x_1$, $y_2 = \mu_2\,x_2$ on $r = a$, we have for $r \leqslant a$,

$$\left.\begin{aligned} y_1^{(1)} &= \frac{a\,\mu_1 \sin\alpha}{\alpha} + 2\,a\,\mu_1 \sin\alpha \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{[(n\,\pi/\alpha)^2 - 1]}\left(\frac{r}{a}\right)^{n\pi/\alpha} \cos\frac{n\,\pi}{\alpha}\,\theta\,,\\ y_2^{(1)} &= 2\,a\,\mu_2 \cos\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{[([2\,n+1]\,\pi/2\,\alpha)^2 - 1]}\left(\frac{r}{a}\right)^{(2n+1)\pi/2\alpha} \sin\frac{2\,n+1}{2\,\alpha}\,\pi\,\theta\,.\end{aligned}\right\} \qquad (6.13)$$

For $\alpha > \pi/2$ the corner is re-entrant and the derivatives $\partial y_\gamma^{(1)}/\partial x_\delta$ in (6.13) become infinite at the vertex. The corner is deformed into a smooth arc with a continuously turning tangent at the boundary point which was initially at the vertex. The radius of curvature of the deformed boundary at this point is

$$\frac{a\,\mu_2^2\cos^2\alpha}{\mu_1\sin\alpha}\;\frac{[(\pi/\alpha)^2-1]}{[1-(\pi/2\,\alpha)^2]^2}\,.$$

An example involving re-entrant corners (with $\alpha = \pi$) has been met earlier in this section where an infinite sheet with a slit was treated. Under deformation the boundary of the crack became a smooth curve.

For $\alpha < \pi/2$ the corner projects and the derivatives $\partial y_\gamma^{(1)}/\partial x_\delta$ vanish at the vertex. For the material with strain energy $U^{(1)}$ this implies that the stress resultants vanish at the corner. A neo-Hookean material would have small strains in the neighborhood of the corner so that the first approximation $y_\gamma^{(1)}$ is not a good approximation near the corner.

If a concentrated load L^* acts at the vertex of the corner in the negative x_1-direction, the derivatives $\partial y_\gamma/\partial x_\delta$ will be $O(r^{-1})$ as $r \to 0$, and we require

$$\lim_{r\to 0}\int_{-\alpha}^{\alpha}\frac{\partial y_1}{\partial r}\,r\,d\theta = \frac{L^*}{2\,h_0\,C_1}\,,\qquad \lim_{r\to 0}\int_{-\alpha}^{\alpha}\frac{\partial y_2}{\partial r}\,r\,d\theta = 0\,.$$

For $r \leqslant a$ the first approximation will be

$$\left.\begin{aligned} y_1^{(1)} &= \frac{L^*}{h_0\,C_1\,\alpha}\log\frac{r}{a} + \frac{a\sin\alpha}{\alpha} \\ &\qquad + 2\,a\sin\alpha\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{r}{a}\right)^{n\pi/\alpha}\cos\frac{n\,\pi}{\alpha}\,\theta\Big/\left[\left(\frac{n\,\pi}{\alpha}\right)^2-1\right], \\ y_2^{(1)} &= 2\,a\cos\alpha\sum_{n=0}^{\infty}(-1)^n\left(\frac{r}{a}\right)^{(2n+1)\pi/2\alpha}\sin\frac{2\,n+1}{2\,\alpha}\,\pi\,\theta\Big/\left[\left(\frac{2\,n+1}{2\,\alpha}\,\pi\right)^2-1\right], \end{aligned}\right\}\qquad (6.14)$$

when the edge $r = a$ of the membrane is held fixed. This solution is valid for all values of α less than π other than $\pi/2$. For r small, we have

$$y_1^{(1)} = \frac{L^*}{h_0\,C_1\,\alpha}\log\frac{r}{a} + O(1)\,,\quad y_2^{(1)} = O(r^{\pi/2\alpha})\,,\quad \text{as}\quad r \to 0\,.$$

Hence we see that the principal extension ratio in the x_2-direction on the line $\theta = 0$ has the limit zero as r approaches zero if $\alpha < \pi/2$, but for $\alpha > \pi/2$, the limit is infinite.

When $\alpha = \pi/2$, we have the case of a concentrated load acting normal to the straight edge of a semi-circular membrane of radius a whose curved boundary is held fixed. Expression (6.14) can then be written as

$$y_1^{(1)} = A\log\frac{r}{a} + u(r,\theta)\,,\quad y_2^{(1)} = x_2\,,\qquad (6.15)$$

where

$$A = \frac{2\,L^*}{h_0\,C_1\,\pi}\,,\qquad u = \frac{2\,a}{\pi} + 2\,a\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{r}{a}\right)^{2n}\cos 2\,n\,\theta\Big/(4\,n^2-1)\,,$$

so that $u = O(1)$ as $r \to 0$. According to (6.15), for small r straight lines θ = constant of the undeformed sheet become logarithmic curves while the circular lines r = constant become straight vertical lines in the deformed membrane.

We note that since the outer edges $\theta = \pm\pi/2$ of the membrane are free of traction, the material there is in a state of simple extension and for r small the extension ratio is found to be

$$\lambda_s = \left(1 + \frac{A^2}{r^2}\right)^{1/2} \sim \frac{A}{r}.$$

Hence on the boundary near the load, $\lambda \sim (r/A)^{1/2}$. On the line $\theta = 0$ and for r small, $J \sim A/r$ and $\lambda \sim r/A$. Since the principal extension ratio in the x_2-direction is unity in this case, we see that the central line $\theta = 0$ is in pure shear in a plane perpendicular to the (r, θ) plane. Thus, for a neo-Hookean material, the curvature of the boundary is sufficient to build up enough tensile transverse stress so that the material in the center of the band is in pure shear even though the membrane in the neighborhood of the load is stretched out into a narrow band.

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Zusammenfassung

Es werden grosse ebene Verformungen dünner elastischer Scheiben aus Neo-Hookeschem Material betrachtet und eine Methode der sukzessiven Substitutionen entwickelt, um Probleme im Rahmen der zweidimensionalen Theorie endlicher ebener Spannungszustände zu lösen. Die erste Näherung wird durch lineare Randwertprobleme für zwei harmonische Funktionen bestimmt, und sie wird asymptotisch angenähert für sehr grosse Dehnungen in der Ebene der Scheiben. Die zweite und die höheren Annäherungen werden durch Lösung Poissonscher Gleichungen gewonnen. Es werden verschiedene Beispiele behandelt, und für rotationssymmetrische Verformungen wird gute Übereinstimmung zwischen den Näherungen und den exakten Lösungen gefunden.

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