A Note on a Paper of Sperb [1])

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In [7] Sperb considers the eigenvalue problem for the elastically-attached membrane

$$\begin{aligned} &\Delta u + \lambda u = 0 \quad \text{in } G, \\ &\frac{\partial u}{\partial n} + \alpha u = 0 \quad \text{on } \Gamma, \end{aligned} \tag{1}$$

where G is a bounded region of n-space with boundary Γ, Δ is the Laplacian, $\partial/\partial n$ the outer normal derivative, α a non-negative constant.

In his paper, Sperb proves a number of inequalities for the first eigenvalue $\lambda_1(\alpha)$. Two particularly interesting inequalities giving lower bounds for $\lambda_1(\alpha)$ are

$$\frac{1}{\lambda_1(\alpha)} \leq \frac{1}{\lambda_1} + \frac{1}{\alpha q_1}, \tag{40'}$$

$$\frac{1}{\lambda_1(\alpha)} \leq \frac{1}{\nu_1} + \frac{A}{\alpha L}, \tag{43}$$

where λ_1 is the first eigenvalue of the fixed membrane, q_1 the first Dirichlet eigenvalue (see [4], [6], [7]), satisfying

$$q_1 = \min_{\substack{\Delta h = 0 \\ \text{in } G}} \frac{\int_\Gamma h^2\, ds}{\int_G h^2\, dx} = \min_{\substack{\phi = 0 \\ \text{on } \Gamma}} \frac{\int_G (\Delta\phi)^2\, dx}{\int_\Gamma \left(\frac{\partial\phi}{\partial n}\right)^2 \partial s},$$

and ν_1 is defined by

$$\nu_1 = \min_{\frac{\partial g}{\partial n} = c} \frac{\int_G (\Delta g)^2\, dx}{D(g)},$$

[1]) This work supported by the Department of the Navy, Naval Ordnance Systems Command under Contract N00017-72-C-4401.

where $D(g)$ is the Dirichlet integral. These inequalities were proved in [7] by a convexity argument applied to a quadratic form associated with the Robin's function. Sperb also proves (40′) by a decomposition (Zerlegung) method. We simplify this method and then also show how (43) may be derived similarly.

To show (40′), let u be the eigenfunction associated with $\lambda_1(\alpha)$ in (1). Let $u=v+h$, where

$$\Delta v=\Delta u, \quad \Delta h=0 \quad \text{in } G$$
$$v=0 \quad h=u \quad \text{on } \Gamma.$$

Note that $D(u)=D(v)+D(h)$. Then, by the triangle inequality,

$$\begin{aligned}\sqrt{\int_G u^2\,dx} &\le \sqrt{\int_G v^2\,dx}+\sqrt{\int_G h^2\,dx}\\ &\le \sqrt{\lambda_1^{-1} D(v)}+\sqrt{q_1^{-1}\int_\Gamma h^2\,ds}\\ &\le \sqrt{\lambda_1^{-1} D(u)}+\sqrt{q_1^{-1}\int_\Gamma u^2\,ds},\end{aligned}$$

where we have used the minimum principles for λ_1 and q_1 (see [4]). Squaring and using the arithmetic-geometric mean inequality

$$\begin{aligned}\int_G u^2\,dx &\le \lambda_1^{-1} D(u)+2\sqrt{\lambda_1^{-1} q_1^{-1} D(u)\int_\Gamma u^2\,ds}+q_1^{-1}\int_\Gamma u^2\,ds\\ &\le \left(\lambda_1^{-1}+(\alpha q_1)^{-1}\right) D(u)+\left(\alpha\lambda_1^{-1}+q_1^{-1}\right)\int_\Gamma u^2\,ds\\ &=\left(\lambda_1^{-1}+(\alpha q_1)^{-1}\right)\left(D(u)+\alpha\int_\Gamma u^2\,ds\right)\\ &=\left(\lambda_1^{-1}+(\alpha q_1)^{-1}\right)\lambda_1(\alpha)\int_G u^2 dx,\end{aligned}$$

which gives (40′).

To show (43), first note that, by the Principle of Duality [3], v_1 can also be characterized by

$$v_1=\min_{\int_\Gamma v\,ds=0}\frac{D(v)}{\int_G v^2\,dx}. \tag{$*$}$$

Now let $c=L^{-1}\int_\Gamma u\,ds$, where u is still the eigenfunction associated with $\lambda_1(\alpha)$. We proceed exactly as above. By the triangle inequality,

$$\sqrt{\int_G u^2\,dx}\le\sqrt{\int_G (u-c)^2\,dx}+\sqrt{c^2 A}\le\sqrt{v_1^{-1} D(u)}+\sqrt{c^2 A},$$

by (*). Squaring and using the arithmetic-geometric mean inequality,

$$\begin{aligned}\int_G u^2\,dx &\le \nu_1^{-1} D(u) + 2\sqrt{c^2 A \nu_1^{-1} D(u)} + c^2 A\\ &\le \left(\nu_1^{-1} + A(\alpha L)^{-1}\right) D(u) + \left(1+\alpha L (A\nu_1)^{-1}\right) c^2 A\\ &= \left(\nu_1^{-1} + A(\alpha L)^{-1}\right)\left(D(u)+\alpha L c^2\right)\\ &\le \left(\nu_1^{-1} + A(\alpha L)^{-1}\right)\left(D(u)+\alpha \int_\Gamma u^2\,ds\right)\\ &= \left(\nu_1^{-1} + A(\alpha L)^{-1}\right)\lambda_1(\alpha)\int_G u^2\,dx,\end{aligned}$$

which gives (43).

We conclude with a couple of remarks.

From (*) it follows, as noted by Sperb, that $\nu_1 \le \mu_2$ (see also [1]), where μ_2 is the second eigenvalue of the free membrane, and equality holds if and only if

$$\int_\Gamma w\,ds = 0 \tag{**}$$

when w is the free membrane eigenfunction associated with μ_2. Now w must have exactly one nodal line [2], which cannot be a closed curve [6]. If G has two axes of symmetry, the nodal line must be one of those axes of symmetry, by the argument of [5], and so (**) holds. Thus, if G has two axes of symmetry, $\nu_1 = \mu_2$ and

$$\frac{1}{\lambda_1(\alpha)} \le \frac{1}{\mu_2} + \frac{A}{\alpha L}. \tag{43'}$$

We also remark that (40′) can be written as

$$\frac{1}{q_1} \ge \alpha\left[\frac{1}{\lambda_1(\alpha)} - \frac{1}{\lambda_1}\right],$$

which can be used to give upper bounds for q_1 with a nearly optimal choice of α. For example, if G is a square of side a, and $\alpha = \pi/2a$, then

$$\lambda_1 = 2\pi^2/a^2, \qquad \lambda_1(\pi/2a) = \pi^2/2a^2,$$

and

$$a q_1 \le \tfrac{4}{3}\pi = 4.1888,$$

as compared with the bound 4.7530 given in Table 2 of [4].

References

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Summary

Two lower bounds of Sperb for the first eigenvalue of an elastically attached membrane are proved in an elementary way.

Zusammenfassung

Zwei untere Schranken von Sperb für den tiefsten Eigenwert einer elastisch gestützten Membran werden in einfacher Weise nachgewiesen.

(Received: December 28, 1972; revised: March 9, 1973)