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Zusammenfassung

Der Aufsatz beweist die Existenz einer Drei-Parameter-Familie von inhomogenen Deformationen mit konstanten Deformationsinvarianten. Diese Deformationen lassen sich in jedem anfänglich homogenen, transversalisotropen, inkompressiblen elastischen Material durch Oberflächenkräfte aufrechterhalten.

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Stress Functions for Plane Problems with Couple Stresses

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1. Introduction

For plane problems and in the absence of body forces and body couples, the stress equations of equilibrium for a continuum which can support couple stresses may be written as

$$\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{yx}}{\partial y} = 0\,, \quad \frac{\partial t_{xy}}{\partial x} + \frac{\partial t_{yy}}{\partial y} = 0\,, \quad \frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + t_{xy} - t_{yx} = 0 \qquad (1.1)$$

when referred to rectangular Cartesian coordinates. In (1.1) t_{xx}, t_{yy}, t_{xy}, and t_{yx} are the components of the *stress tensor*, and m_{xz} and m_{yz} are the components of the *couple stress tensor*. The domain of these functions is some bounded region R of the x,y-plane. A direct and elementary derivation of (1.1) has been given by MINDLIN [1][1]). They can also be reached by specialization of the corresponding equations for the three-dimensional case [2].

It is the purpose of the present paper to give the general solution of (1.1) in terms of arbitrary functions (stress functions). In the non-polar case (identically zero couple stresses), the well-known general solution was given by AIRY [3]. It is important to emphasize that since we are dealing only with the stress equations of equilibrium, all of our results (except those in Section 4) are independent of any constitutive equations which the stresses may be required to satisfy.

In Section 2 the stress function solution is derived and shown to be complete. The degree of arbitrariness of the stress functions for a given set of stresses is then examined.

In Section 3 the stress functions are interpreted in terms of the resultant force and moment transmitted across an arc in the body. This leads to necessary and sufficient conditions for the stresses to satisfy in order that the stress functions be single-valued.

Finally, in Section 4 the stress function solution given by MINDLIN [1] for the case of linearized elasticity is obtained from our general solution.

2. Stress Function Solution

Here and in what follows, we will not state explicitly the smoothness requirements. They may be readily inferred from well-known theorems of calculus; see, for example, COURANT [4].

[1]) Numbers in brackets refer to References, page 792.

The following theorem, which supplies the stress function solution of (1.1), may be confirmed by direct substitution.

Theorem 2.1. *Define stresses through*

$$t_{xx} = \frac{\partial F}{\partial y}, \quad t_{yy} = -\frac{\partial G}{\partial x}, \quad t_{xy} = \frac{\partial G}{\partial y}, \quad t_{yx} = -\frac{\partial F}{\partial x}, \quad m_{xz} = \frac{\partial H}{\partial y} - F, \quad m_{yz} = -\frac{\partial H}{\partial x} - G, \tag{2.1}$$

where F, G, and H are arbitrary functions. Then these stresses satisfy (1.1).

The proof of the next theorem, which shows that every solution of (1.1) can be represented in the form (2.1), is actually the motivation for representation (2.1).

Theorem 2.2. *Let the stresses satisfy* (1.1). *Then there exist functions F, G, and H on R such that the stresses can be represented by* (2.1). *Furthermore, if R is simply connected, then the stress functions F, G, and H will be single-valued.*

Proof. According to the theory of total differentials [4], $(1.1)_1$ implies the existence of a function F (single-valued if R is simply connected) such that

$$t_{xx} = \frac{\partial F}{\partial y}, \quad t_{yx} = -\frac{\partial F}{\partial x}.$$

Similarly, by $(1.1)_2$ there exists a function G such that

$$t_{xy} = \frac{\partial G}{\partial y}, \quad t_{yy} = -\frac{\partial G}{\partial x}.$$

Then $(1.1)_3$ can be written as

$$\frac{\partial}{\partial x}(m_{xz} + F) + \frac{\partial}{\partial y}(m_{yz} + G) = 0,$$

and hence there is a function H such that

$$m_{xz} + F = \frac{\partial H}{\partial y}, \quad m_{yz} + G = -\frac{\partial H}{\partial x}.$$

This completes the proof.

It is of some interest to know to what extent the stress functions are determined by the stresses they represent. This information is contained in the following theorem.

Theorem 2.3. *Let a given set of stresses which meet* (1.1) *be represented according to* (2.1) *by the stress functions F, G, H and also by the stress functions F′, G′, H′. Then*

$$F - F' = F_0, \quad G - G' = G_0, \quad H - H' = -G_0 x + F_0 y + H_0,$$

where F_0, G_0, H_0, are constants.

Proof. By (2.1)

$$\frac{\partial}{\partial x}(F - F') = \frac{\partial}{\partial y}(F - F') = 0, \quad \frac{\partial}{\partial x}(G - G') = \frac{\partial}{\partial y}(G - G') = 0,$$

$$\frac{\partial}{\partial x}(H - H') = -(G - G'), \quad \frac{\partial}{\partial y}(H - H') = (F - F').$$

Hence

$$F - F' = \text{const.} = F_0, \quad G - G' = \text{const.} = G_0,$$

$$H - H' = -G_0 x + F_0 y + \text{const.} = -G_0 x + F_0 y + H_0;$$

and the proof is complete.

It is worth noting that (2.1) may be regarded as a special case of GÜNTHER's [5, 6] stress function solution of the three-dimensional equilibrium equations. Also, it is a trivial matter to write (2.1) in invariant form.

3. Resultant Force and Moment

Guided by TRUESDELL's and TOUPIN's [2] treatment of MICHELL's [7] work in the non-polar case, we now give a physical interpretation of the stress functions in terms of the resultant force and moment on an arc in the body.

Consider an element of arc $i\,dx + j\,dy$, where i and j are the unit vectors along the x- and y-axes, respecitvely. Since a unit normal to this element of arc is $i\,dy/ds - j\,dx/ds$, the components of the *stress vector* and the *couple stress vector* on the element of arc are given by

$$t_x = t_{xx}\frac{dy}{ds} - t_{yx}\frac{dx}{ds}\,, \qquad t_y = t_{xy}\frac{dy}{ds} - t_{yy}\frac{dx}{ds}\,, \tag{3.1}$$

and

$$m_z = m_{xz}\frac{dy}{ds} - m_{yz}\frac{dx}{ds}\,, \tag{3.2}$$

respectively. Equations (3.1), (3.2), and (2.1) yield

$$t_x = \frac{dF}{ds}\,, \qquad t_y = \frac{dG}{ds}\,, \qquad m_z - y\,t_x + x\,t_y = \frac{d}{ds}\,(H - y\,F + x\,G)\,. \tag{3.3}$$

Next let (x_1, y_1) and (x_2, y_2) be any two points in R, and let Γ be any simple arc contained in R directed from (x_1, y_1) to (x_2, y_2). Then the components of the *resultant force* and the *resultant moment* (about the origin) transmitted across Γ measured per unit thickness of the body are

$$T_x(\Gamma) = \int_\Gamma t_x\,ds\,, \qquad T_y(\Gamma) = \int_\Gamma t_y\,ds\,, \tag{3.4}$$

and

$$M_z(\Gamma) = \int_\Gamma (m_z - y\,t_x + x\,t_y)\,ds\,, \tag{3.5}$$

respectively. The following two theorems are immediate consequences of (3.3), (3.4), and (3.5).

Theorem 3.1. *Let Γ be a simple arc directed from (x_1, y_1) to (x_2, y_2) as above. Then*

$$F\big|_{(x_2,\,y_2)} - F\big|_{(x_1,\,y_1)} = T_x(\Gamma)\,, \qquad G\big|_{(x_2,\,y_2)} - G\big|_{(x_1,\,y_1)} = T_y(\Gamma)\,,$$

$$(H - y\,F + x\,G)\big|_{(x_2,\,y_2)} - (H - y\,F + x\,G)\big|_{(x_1,\,y_1)} = M_z(\Gamma)\,.$$

Theorem 3.2. *A necessary and sufficient condition that the stress functions F, G, and H be single-valued is that the stresses be totally self-equilibrated in the sense that*

$$T_x(C) = T_y(C) = M_z(C) = 0$$

for every simple closed contour C in R.

Of course it follows immediately from (1.1), (3.1), (3.2), (3.4), (3.5), and the divergence theorem, that if the boundary of R consists of a number of simple closed contours; then the stresses will be totally self-equilibrated if and only if the resultant force and resultant moment on each of the bounding contours vanishes.

4. Linear Elasticity and Mindlin's Solution

In particular theories of materials, the stresses are required to satisfy certain conditions of compatibility as well as the equilibrium Equations (1.1). In the linear theory of elasticity [1], one of the compatibility equations is

$$\frac{\partial m_{yz}}{\partial x} = \frac{\partial m_{xz}}{\partial y}\,. \tag{4.1}$$

Equation (4.1) implies the existence of a function K such that

$$m_{xz} = \frac{\partial K}{\partial x}, \qquad m_{yz} = \frac{\partial K}{\partial y}. \tag{4.2}$$

By (4.2) and (2.1)

$$F = \frac{\partial H}{\partial y} - \frac{\partial K}{\partial x}, \qquad G = -\frac{\partial H}{\partial x} - \frac{\partial K}{\partial y}. \tag{4.3}$$

Thus from (4.2), (4.3), and (2.1) the complete solution of the equilibrium Equations (1.1) and the compatibility Equation (4.1) is

$$t_{xx} = \frac{\partial^2 H}{\partial y^2} - \frac{\partial^2 K}{\partial x\,\partial y}, \qquad t_{yy} = \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 K}{\partial x\,\partial y},$$

$$t_{xy} = -\frac{\partial^2 H}{\partial x\,\partial y} - \frac{\partial^2 K}{\partial y^2}, \qquad t_{yx} = -\frac{\partial^2 H}{\partial x\,\partial y} + \frac{\partial^2 K}{\partial x^2}, \qquad m_{xz} = \frac{\partial K}{\partial x}, \qquad m_{yz} = \frac{\partial K}{\partial y},$$

where the stress functions H and K are arbitrary. This is MINDLIN's [1] solution. Of course if the rest of the compatibility equations are taken into account, then H and K will have to satisfy certain differential equations.

Note Added in Proof, November 4, 1966: After this paper had gone to the printer, Professor SCHAEFER informed me that some of its results are contained in his *Versuch einer Elastizitätstheorie des zweidimensionalen ebenen Cosserat-Kontinuums*, Miszellaneen der Angewandten Mechanik, 277–292, Akademie-Verlag, Berlin 1962. However, I have not yet been able to obtain this work and thus do not know to what extent the two papers overlap.

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Zusammenfassung

Die Lösung der zweidimensionalen Gleichgewichtsbedingungen für ein Kontinuum, das Spannungsmomente aufnehmen kann, wird mit Hilfe von willkürlichen Spannungsfunktionen gegeben. Die Spannungsfunktionen werden mit Hilfe der resultierenden Einzelkraft und des resultierenden Momentes gedeutet, welche durch einen Bogen im Körper übertragen werden.

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