Mathematical Programming 50 (1991) 239-258
North-Holland

An O(n^3L) potential reduction algorithm for linear programming

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Received 1 August 1988
Revised manuscript received 13 February 1989

We describe a primal-dual potential function for linear programming:

$$\phi(x, s) = \rho \ln(x^{\mathrm{T}}s) - \sum_{j=1}^{n} \ln(x_j s_j)$$

where $\rho \geqslant n$, x is the primal variable, and s is the dual-slack variable. As a result, we develop an interior point algorithm seeking reductions in the potential function with $\rho = n+\sqrt{n}$. Neither tracing the central path nor using the projective transformation, the algorithm converges to the optimal solution set in $\mathrm{O}(\sqrt{n}L)$ iterations and uses $\mathrm{O}(n^3L)$ total arithmetic operations. We also suggest a practical approach to implementing the algorithm.

Key words: Linear programming, primal and dual, interior algorithms, potential functions.

1. Introduction

Since Karmarkar [17] proposed the polynomial interior algorithm for linear programming (LP), many developments have been made to the growing literature on interior algorithms: the projective algorithm, the affine scaling algorithm, and the path-following algorithm. All of these interior algorithms use the scaling technique and solve a least-squares problem at each iteration, and they are related to the classical barrier function method of Frisch [9] and Fiacco and McCormick [8] (see, for examples, Gill et al. [12] and Iri and Imai [16]).

Karmarkar first introduced the potential function to linear programming in his projective algorithm [17]. Then, Anstreicher [2], Gay [10], de Ghellinck and Vial [11], Todd and Burrell [30] and Ye and Kojima [36] proposed a primal projective algorithm using dual variables. The projective algorithm, as well as Karmarkar's original algorithm, uses potential functions to measure its iterative progress and converges in $\mathrm{O}(nL)$ iterations and $\mathrm{O}(n^{3.5}L)$ arithmetic operations, where L is the data length and n is the number of variables in LP. In practice, far fewer iterations

This paper was presented at the AMS Joint Summer Conference, held at Bowdoin College, Brunswick, ME, 1988, under the title of "A class of potential functions for linear programming".

are required when a large-sized step is taken along the descent direction of the potential function.

Barnes [3], Kortanek and Shi [19] and Vanderbei et al. [34] updated the primal affine scaling algorithm that was originally proposed by Dikin [7]. Adler et al. [1] and Monma and Morton [24] then developed and implemented the dual affine scaling algorithm. The polynomial status of the affine scaling algorithm is still unknown, but it works well in practice by taking a large-sized step along the descent direction of the objective function.

Another polynomial interior algorithm, the (dual) path-following algorithm, was introduced by Renegar [27], who established the first $O(\sqrt{n}L)$-iteration interior algorithm for LP. Using Karmarkar's rank-one technique, Gonzaga [15] and Vaidya [32] further upgraded the algorithm's complexity to $O(n^3L)$. Renegar's algorithm is related to the "analytic center" of Sonnevend [29] and the central trajectories or pathways analyzed by Bayer and Lagarias [4], Megiddo [21] and Megiddo and Shub [22]. Finally, Kojima et al. [18] and Monteiro and Adler [26] developed the primal–dual path-following algorithm, Goldfarb and Liu [13] and Ye [35] developed the primal path-following algorithm, and Ben Daya and Shetty [5] and Mehrotra and Sun [23] developed the dual path-following algorithm for linear and/or convex quadratic programming. While remaining "centered", this algorithm seeks reductions in the objective function and converges in $O(\sqrt{n}\,L)$ iterations and $O(n^3L)$ arithmetic operations. Unfortunately, the need to trace closely the central path severely limits the permissible stepsize at any iteration. (A large-step variant of the primal–dual path-following algorithm has been implemented by McShane et al. [20] with encouraging practical results, but the theoretical guarantee no longer exists.)

Recently, several efforts were made to improve the interior algorithms. Todd and Ye [31, 37] introduced a class of potential functions for linear programming and proposed a primal–dual projective algorithm, the centered projective algorithm, using a primal–dual potential function. They have shown that the step direction of this algorithm is the gradient-projection of the potential function in the projective scaling fame. If the centering condition is satisfied, then the direction is also the direction of the path-following algorithm. The algorithm is motivated by seeking reductions in the potential function, as is the case for the projective algorithms. It converges in $O(\sqrt{n}\,L)$ iterations but still has to follow the central path. Nevertheless, the approximate centering is an automatic by-product of the choice of the potential function. Monteiro et al. [25] simultaneously used the primal and dual affine scaling algorithm, resulting in an $O(nL^2)$-iteration algorithm. Gonzaga [14] used the steepest descent method for a potential function in the primal affine scaling framework, leading to an $O(n^2L)$ or $O(nL)$-iteration algorithm. His potential function is a special case of Todd and Ye's class of potential functions, and it uses the assumption of the known minimal objective value.

Therefore, the question remains open: Do we have to follow the central path to achieve $O(\sqrt{n}\,L)$-iteration convergence for linear programming, or can we obtain an $O(\sqrt{n}\,L)$-iteration algorithm based on potential function reduction?

In this paper, we further study the primal-dual potential function described by Todd and Ye [31, 37]. As a result, we develop an interior algorithm directly minimizing the potential function in the LP standard form via the scaled-gradient-projection method (see Ye [35]). The algorithm seeks reductions in a suitable potential function like the projective algorithm, but without using the projective transformation. It converges in $O(\sqrt{n}\,L)$ iterations and $O(n^3L)$ arithmetic operations like the path-following algorithms, but without tracing the central path. We present the algorithm in two forms, the primal form and the dual form. We show how our algorithm is related to the other interior algorithms mentioned above. We also discuss a practical approach to implementing the algorithm.

2. Potential function and linear programming

A linear program is usually given in the following standard form:

LP: minimize $c^{\mathrm{T}}x$

subject to $Ax=b,\ x\geqslant 0,$

where $c\in\mathbb{R}^n$, $A\in\mathbb{R}^{m\times n}$ and $b\in\mathbb{R}^m$ are given, $x\in\mathbb{R}^n$, and $^{\mathrm{T}}$ denotes the transpose. The dual to LP can be written as

LD: maximize $b^{\mathrm{T}}y$

subject to $s=c-A^{\mathrm{T}}y\geqslant 0,$

where vector $y\in\mathbb{R}^m$ and $s\in\mathbb{R}^n$. The components of s are called dual slacks. For all x and y that are feasible for LP and LD,

$$b^{\mathrm{T}}y\leqslant z^*\leqslant c^{\mathrm{T}}x, \tag{1}$$

where z^* denotes the minimal (maximal) objective value of LP (LD) (Dantzig [6]).

In this paper, the upper-case letter (X) designates the diagonal matrix of the vector (x) in lower-case. We also assume that:

(A1) The relative interiors of the feasible regions of LP and LD are nonempty.

(A2) A has full rank.

The second assumption is added merely for simplicity.

Given an interior primal solution x^0 and dual solution y^0 such that

$$Ax^0=b \quad\text{and}\quad x^0>0$$

and

$$s^0=c-A^{\mathrm{T}}y^0>0,$$

the primal affine scaling algorithm moves from x^0 along the descent direction

$$-DP_{AD}Dc, \tag{2}$$

where

$$P_{AD}=I-DA^{\mathrm{T}}(AD^2A^{\mathrm{T}})^{-1}AD,$$

and the dual affine scaling algorithm moves from y^0 along the ascent direction

$$(AD^2A^{\mathrm{T}})^{-1}b, \tag{3}$$

where D is a diagonal scaling matrix (see, for example, [1, 7, 34]).

In the primal affine scaling algorithm $D = X^0$; in the dual affine scaling algorithm $D = (S^0)^{-1}$. Note that if x^0 and y^0 satisfy

$$X^0 s^0 = e$$

then $(A(S^0)^{-2}A^{\mathrm{T}})^{-1}b$ becomes $(A(X^0)^2A^{\mathrm{T}})^{-1}b$. In general, we call (2) the primal affine direction and (3) the dual affine direction.

For the primal we consider the potential function

$$\text{(P)} \qquad \bar{\phi}(x, \underline{z}) = \rho \ln(c^{\mathrm{T}}x - \underline{z}) - \sum_{j=1}^{n} \ln(x_j)$$

and for the dual we consider the potential function

$$\text{(D)} \qquad \underline{\phi}(y, \bar{z}) = \rho \ln(\bar{z} - b^{\mathrm{T}}y) - \sum_{j=1}^{n} \ln(s_j),$$

while for the primal-dual we employ the joint potential function

$$\text{(P-D)} \quad \phi(x, s) = \rho \ln(x^{\mathrm{T}}s) - \sum_{j=1}^{n} \ln(x_j s_j),$$

where $\underline{z} \leqslant z^* \leqslant \bar{z}$ and $n \leqslant \rho < \infty$. The primal potential function was used by Karmarkar for $\rho = n+1$ (see Ye and Kojima [36]) and by Gonzaga for $\rho \geqslant n$ and $\underline{z} = z^*$ [14]. The primal-dual potential function was introduced by Todd and Ye for $\rho = n + \sqrt{n}$ [31] and $\rho = 2n$ [37].

Since $x^{\mathrm{T}}s = x^{\mathrm{T}}(c - A^{\mathrm{T}}y) = c^{\mathrm{T}}x - b^{\mathrm{T}}y$, both of the primal and dual potential functions are related to the primal-dual potential function in the following way:

$$\bar{\phi}(x, \underline{z}) = \phi(x, s) + \sum_{j=1}^{n} \ln(s_j) \tag{4a}$$

and

$$\underline{\phi}(y, \bar{z}) = \phi(x, s) + \sum_{j=1}^{n} \ln(x_j), \tag{4b}$$

and the gradient vectors of (P) and (D) are

$$\nabla \bar{\phi}(x, \underline{z}) = \nabla_x \phi(x, s) = \frac{\rho}{c^{\mathrm{T}}x - \underline{z}}\, c - X^{-1}e \tag{5a}$$

and

$$\nabla \underline{\phi}(y, \bar{z}) = \nabla_y \phi(x, s) = -\frac{\rho}{\bar{z} - b^{\mathrm{T}}y}\, b + AS^{-1}e, \tag{5b}$$

where

$$\underline{z} = b^T y \quad \text{and} \quad \bar{z} = c^T x.$$

Furthermore, the primal-dual potential function (PD) can be written in the equivalent form (Todd and Ye [31])

$$\phi(x, s) = (\rho - n) \ln(x^T s) - \sum_{j=1}^{n} \ln\left(\frac{x_j s_j}{x^T s}\right).$$

From the geometric mean - arithmetic mean inequality, we have

$$-\sum_{j=1}^{n} \ln\left(\frac{x_j s_j}{x^T s}\right) \geqslant n \ln n.$$

Hence,

$$(\rho - n) \ln(c^T x - b^T y) = (\rho - n) \ln(x^T s) \leqslant \phi(x, s) - n \ln n \leqslant \phi(x, s). \tag{6}$$

This tells the exact amount, $-(\rho - n)L$, by which ϕ should be reduced to reach

$$c^T x - b^T y \leqslant 2^{-L}.$$

Before going further, we state the following lemma, which is essentially due to Karmarkar [17].

Lemma 1. *Let $x \in \mathbb{R}^n$ and $\|x - e\|_\infty < 1$. Then*

$$\sum_{j=1}^{n} \ln x_j \geqslant (e^T x - n) - \frac{\|x - e\|^2}{2(1 - \|x - e\|_\infty)},$$

where e is the vector of all ones, and $\|\cdot\|$ (without subscript) denotes the L_2 norm.

Proof. For $1 \leqslant j \leqslant n$,

$$\begin{aligned}
\ln x_j &= \ln(1 + x_j - 1) \\
&= (x_j - 1) - \frac{(x_j - 1)^2}{2} + \frac{(x_j - 1)^3}{3} - \frac{(x_j - 1)^4}{4} + \cdots \\
&\geqslant (x_j - 1) - \frac{(x_j - 1)^2}{2} (1 + |x_j - 1| + |x_j - 1|^2 + \cdots) \\
&= (x_j - 1) - \frac{(x_j - 1)^2}{2(1 - |x_j - 1|)} \geqslant (x_j - 1) - \frac{(x_j - 1)^2}{2(1 - \|x - e\|_\infty)}.
\end{aligned}$$

Summing up the inequality over j, we have

$$\sum_{j=1}^{n} \ln x_j \geqslant (e^T x - n) - \frac{\|x - e\|^2}{2(1 - \|x - e\|_\infty)}. \qquad \square$$

Due to the concavity of the first term of the potential functions, $\ln(c^Tx-\underline{z})$ or $\ln(\bar{z}-b^Ty)$, and Lemma 1, for any two points

$$x^0>0, \qquad x^1>0 \quad \text{and} \quad \|(X^0)^{-1}(x^1-x^0)\|_\infty<1$$

we have

$$\bar{\phi}(x^1,\underline{z})-\bar{\phi}(x^0,\underline{z})\leq\nabla\bar{\phi}^T(x^0,\underline{z})(x^1-x^0) + \frac{\|(X^0)^{-1}(x^1-x^0)\|^2}{2(1-\|(X^0)^{-1}(x^1-x^0)\|_\infty)}, \tag{7a}$$

and for any two points

$$s^0=c-A^Ty^0>0, \qquad s^1=c-A^Ty^1>0 \quad \text{and} \quad \|(S^0)^{-1}(s^1-s^0)\|_\infty<1$$

we have

$$\underline{\phi}(y^1,\bar{z})-\underline{\phi}(y^0,\bar{z})\leq\nabla\underline{\phi}^T(y^0,\bar{z})(y^1-y^0) + \frac{\|(S^0)^{-1}(s^1-s^0)\|^2}{2(1-\|(S^0)^{-1}(s^1-s^0)\|_\infty)}. \tag{7b}$$

The right-hand sides of (7a) and (7b) provide quadratic over-estimators for the reduction of the primal and dual potential functions.

3. The primal form

Let $\underline{z}^0=b^Ty^0$ for some $s^0=c-A^Ty^0>0$. Then, we minimize the linearized *primal* potential function subject to the ellipsoid constraint corresponding to the second order term in (7a):

$$\begin{aligned} \text{PP:} \qquad & \text{minimize} \quad \nabla\bar{\phi}^T(x^0,\underline{z}^0)(x-x^0) \\ & \text{subject to} \quad A(x-x^0)=0, \\ & \qquad\qquad\quad \|(X^0)^{-1}(x-x^0)\|\leq\beta<1, \end{aligned}$$

and denote by x^1 the minimal solution for PP. Thus, we have

$$x^1-x^0=-\beta\frac{X^0P_{AX^0}X^0\nabla\bar{\phi}(x^0,\underline{z}^0)}{\|P_{AX^0}X^0\nabla\bar{\phi}(x^0,\underline{z}^0)\|}. \tag{8}$$

Let

$$p(\underline{z}^0)=P_{AX^0}X^0\nabla\bar{\phi}(x^0,\underline{z}^0).$$

Then

$$\nabla\bar{\phi}^T(x^0,\underline{z}^0)(x^1-x^0)=-\beta\|p(\underline{z}^0)\|.$$

Hence, due to (7a) the reduction of the primal potential function satisfies

$$\bar{\phi}(x^1,\underline{z}^0)-\bar{\phi}(x^0,\underline{z}^0)\leq-\beta\|p(\underline{z}^0)\|+\frac{\beta^2}{2(1-\beta)}. \tag{9}$$

Now, we focus on the expression of $p(\underline{z}^0)$, which from (5a) can be rewritten as

$$\begin{aligned} p(\underline{z}^0) &= P_{AX^0}X^0\left(\frac{\rho}{c^{\mathrm{T}}x^0-\underline{z}^0}c-(X^0)^{-1}e\right) \\ &= (I-X^0A^{\mathrm{T}}(A(X^0)^2A^{\mathrm{T}})^{-1}AX^0)\left(\frac{\rho}{c^{\mathrm{T}}x^0-\underline{z}^0}X^0c-e\right) \\ &= \frac{\rho}{c^{\mathrm{T}}x^0-\underline{z}^0}X^0s(\underline{z}^0)-e \end{aligned} \tag{10}$$

with

$$s(\underline{z}^0)=c-A^{\mathrm{T}}y(\underline{z}^0) \tag{11}$$

and

$$\begin{aligned} y(\underline{z}^0) &= (A(X^0)^2A^{\mathrm{T}})^{-1}AX^0\left(X^0c-\frac{c^{\mathrm{T}}x^0-\underline{z}^0}{\rho}e\right) \\ &= y_1-\frac{c^{\mathrm{T}}x^0-\underline{z}^0}{\rho}y_2 \end{aligned}$$

where

$$y_1=(A(X^0)^2A^{\mathrm{T}})^{-1}A(X^0)^2c$$

and

$$y_2=(A(X^0)^2A^{\mathrm{T}})^{-1}AX^0e=(A(X^0)^2A^{\mathrm{T}})^{-1}b.$$

It is clear that y_1 is related to the primal affine direction of (2), which corresponds to the gradient-projection of the linear objective function; y_2 is related to the dual affine direction of (3), which corresponds to the gradient-projection of the barrier function. Both directions are also closely linked to the projective algorithm for the standard LP (see, for example, Ye and Kojima [36]).

Regarding $\|p(\underline{z}^0)\|$, we have the following lemma.

Lemma 2. *Let*

$$\Delta^0=\frac{c^{\mathrm{T}}x^0-\underline{z}^0}{n}=\frac{(x^0)^{\mathrm{T}}s^0}{n}, \qquad \Delta=\frac{(x^0)^{\mathrm{T}}s(\underline{z}^0)}{n},$$

$\rho=n+\sqrt{n}$, *and* $\alpha<1$. *If*

$$\|p(\underline{z}^0)\|<\min\left(\alpha\sqrt{\frac{n}{n+\alpha^2}},1-\alpha\right), \tag{12}$$

then the following three inequalities hold

$$s(\underline{z}^0)>0, \tag{13a}$$

$$\|X^0s(\underline{z}^0)-\Delta e\|<\alpha\Delta \tag{13b}$$

and

$$\Delta<(1-0.5\alpha/\sqrt{n})\Delta^0. \tag{13c}$$

Proof. The proof is by contradiction.

(i) If the inequality of (13a) is not true, then $\exists j$ such that $s_j(\underline{z}^0) \leqslant 0$ and

$$\|p(\underline{z}^0)\| \geqslant 1 - \frac{\rho}{n\Delta^0} x_j s_j(\underline{z}^0) \geqslant 1.$$

(ii) If the inequality of (13b) does not hold, then

$$\begin{aligned}
\|p(\underline{z}^0)\|^2 &= \left\| \frac{\rho}{n\Delta^0} X^0 s(\underline{z}^0) - \frac{\rho\Delta}{n\Delta^0} e + \frac{\rho\Delta}{n\Delta^0} e - e \right\|^2 \\
&= \left(\frac{\rho}{n\Delta^0}\right)^2 \|X^0 s(\underline{z}^0) - \Delta e\|^2 + \left\| \frac{\rho\Delta}{n\Delta^0} e - e \right\|^2 \\
&\geqslant \left(\frac{\rho\Delta}{n\Delta^0}\right)^2 \alpha^2 + \left(\frac{\rho\Delta}{n\Delta^0} - 1\right)^2 n \\
&\geqslant \alpha^2 \frac{n}{n+\alpha^2},
\end{aligned} \tag{14}$$

where the last relation prevails since the quadratic term yields a minimum at

$$\frac{\rho\Delta}{n\Delta^0} = \frac{n}{n+\alpha^2}.$$

(iii) If the inequality of (13c) is violated, then

$$\frac{\rho\Delta}{n\Delta^0} \geqslant \left(1 + \frac{1}{\sqrt{n}}\right)\left(1 - \frac{0.5\alpha}{\sqrt{n}}\right) \geqslant 1,$$

which in view of (14) leads to

$$\begin{aligned}
\|p(\underline{z}^0)\|^2 &\geqslant \left(\frac{\rho\Delta}{n\Delta^0} - 1\right)^2 n \\
&\geqslant \left(\left(1 + \frac{1}{\sqrt{n}}\right)\left(1 - \frac{0.5\alpha}{\sqrt{n}}\right) - 1\right)^2 n \\
&\geqslant \left(1 - \frac{\alpha}{2} - \frac{\alpha}{2\sqrt{n}}\right)^2 \\
&\geqslant (1-\alpha)^2. \qquad \square
\end{aligned}$$

Based on the above lemmas, we have the following potential reduction theorem.

Theorem 1. *Let x^0 and y^0 be any interior feasible solutions for* LP *and* LD, *and let $\rho = n + \sqrt{n}$, $\underline{z}^0 = b^{\mathrm{T}} y^0$, x^1 be given by* (8), *and $y^1 = y(\underline{z}^0)$ and $s^1 = s(\underline{z}^0)$ of* (11). *Then, either*

$$\phi(x^1, s^0) \leqslant \phi(x^0, s^0) - \delta$$

or

$$\phi(x^0, s^1) \leq \phi(x^0, s^0) - \delta$$

where $\delta > 0.05$.

Proof. If (12) does not hold, i.e.

$$\|p(\underline{z}^0)\| \geq \min\left(\alpha\sqrt{\frac{n}{n+\alpha^2}}, 1-\alpha\right),$$

then from (9),

$$\bar{\phi}(x^1, \underline{z}^0) \leq \bar{\phi}(x^0, \underline{z}^0) - \beta \min\left(\alpha\sqrt{\frac{n}{n+\alpha^2}}, 1-\alpha\right) + \frac{\beta^2}{2(1-\beta)},$$

hence from (4a),

$$\phi(x^1, s^0) \leq \phi(x^0, s^0) - \beta \min\left(\alpha\sqrt{\frac{n}{n+\alpha^2}}, 1-\alpha\right) + \frac{\beta^2}{2(1-\beta)}.$$

Otherwise, from Lemma 2 the inequalities of (13) hold:

(i) (13a) indicates that y^1 and s^1 are *interior* dual feasible solutions.

(ii) Using (13b) and applying Lemma 1 to the vector $X^0 s^1/\Delta$, we have

$$\begin{aligned} n \ln(x^0)^{\mathrm{T}} s^1 - \sum_{j=1}^{n} \ln(x_j^0 s_j^1) &= n \ln((x^0)^{\mathrm{T}} s^1/\Delta) - \sum_{j=1}^{n} \ln(x_j^0 s_j^1/\Delta) \\ &= n \ln n - \sum_{j=1}^{n} \ln(x_j^0 s_j^1/\Delta) \\ &\leq n \ln n + \frac{\|X^0 s^1/\Delta - e\|^2}{2(1-\|X^0 s^1/\Delta - e\|_\infty)} \\ &\leq n \ln n + \frac{\alpha^2}{2(1-\alpha)} \\ &\leq n \ln(x^0)^{\mathrm{T}} s^0 - \sum_{j=1}^{n} \ln(x_j^0 s_j^0) + \frac{\alpha^2}{2(1-\alpha)}. \end{aligned}$$

(iii) According to (13c), we have

$$\sqrt{n}(\ln(x^0)^{\mathrm{T}} s^1 - \ln(x^0)^{\mathrm{T}} s^0) = \sqrt{n} \ln \frac{\Delta}{\Delta^0} \leq -\frac{\alpha}{2}.$$

Adding the two inequalities in (ii) and (iii), we have

$$\phi(x^0, s^1) \leq \phi(x^0, s^0) - \frac{\alpha}{2} + \frac{\alpha^2}{2(1-\alpha)}.$$

Thus, by choosing $\alpha = 0.43$ and $\beta = 0.3$ we have the desired result. $\square$

Theorem 1 establishes an important fact: the *primal–dual* potential function can be reduced by a constant via solving PP on the interior of LP and LD, no matter where x^0 and y^0 are. In practice, one can perform a line search to minimize the primal-dual potential function. This results in the following primal algorithm.

Primal Algorithm.

Given $Ax^0 = b$, $x^0 > 0$ and $s^0 = c - A^{\mathrm{T}}y^0 > 0$;
let $\underline{z}^0 = b^{\mathrm{T}}y^0$ and set $k = 0$;
while $c^{\mathrm{T}}x^k - b^{\mathrm{T}}y^k \geqslant 2^{-L}$ **do**
  begin
  compute $s(\underline{z}^k)$ of (11) and formulate $p(\underline{z}^k)$ of (10);
  if the inequality of (12) does not hold **then**
    $x^{k+1} = x^k - \beta^* X^k p(\underline{z}^k)$ with $\beta^* = \arg\min_{\beta \geqslant 0} \phi(x^k - \beta X^k p(\underline{z}^k), s^k)$;
    $s^{k+1} = s^k$ and $\underline{z}^{k+1} = \underline{z}^k$;
  else
    $s^{k+1} = s(\underline{z}^*)$ with $\underline{z}^* = \arg\min_{\underline{z} \geqslant \underline{z}^k} \phi(x^k, s(\underline{z}))$;
    $x^{k+1} = x^k$ and $\underline{z}^{k+1} = b^{\mathrm{T}}y(\underline{z}^*)$;
  end;
  $k = k + 1$;
end.

The performance of the primal algorithm results from the following theorem.

Theorem 2. *Let $\rho = n + \sqrt{n}$ and $\phi(x^0, s^0) \leqslant O(\sqrt{n}\,L)$. Then, the primal algorithm terminates in $O(\sqrt{n}\,L)$ iterations and each iteration uses $O(n^3)$ arithmetic operations.*

Proof. In $O(\sqrt{n}\,L)$ iterations

$$\phi(x^k, s^k) \leqslant -L\sqrt{n}.$$

Then, from (6),

$$\sqrt{n}\ln(c^{\mathrm{T}}x^k - b^{\mathrm{T}}y^k) < -L\sqrt{n},$$

i.e.

$$c^{\mathrm{T}}x^k - b^{\mathrm{T}}y^k = (x^k)^{\mathrm{T}}s^k < 2^{-L}. \qquad \square$$

The condition of the initial potential value in Theorem 2 is not critical. In fact, along the central path

$$\phi(x^0, s^0) = \sqrt{n}\ln(c^{\mathrm{T}}x^0 - b^{\mathrm{T}}y^0) + n\ln n.$$

Hence, $\phi(x^0, s^0) = O(\sqrt{n}\,L)$ while $c^{\mathrm{T}}x^0 - b^{\mathrm{T}}y^0 \leqslant 2^L$. Several papers on the path-following algorithm have shown how to transform a LP problem to an equal-sized LP problem with known centers x^0 and s^0 (see, for example, Kojima et al. [18], Renegar [27] and Ye [35]). Also note that if $x^{k+1} = x^k$ in the algorithm, the projection matrix in (8) is unchanged and should be reused for the next iteration. In practice, a *strict* lower bound $z^0 < z^*$ suffices to start the algorithm, i.e. the known y^0 and s^0 are not necessary. Moreover, a bi-directional search over β and $\underline{z}$ can be employed to update x and s simultaneously in minimizing $\phi(x, s)$.

The condition of (13b) is the centering condition crucial to path-following algorithms. While this condition is strictly enforced at any iteration of a path-following algorithm, our algorithm uses it as a signal to coordinate the movements of the primal and dual. If the algorithm is implemented as it is, the iterative solutions will visit the central path many times; however, they are not required to stay on the central path — the next iterate may not even be close to the central path. If the bi-directional search over β and $\underline{z}$ is employed, the condition (13b) may never be true, and the solution sequence may never visit the central path. By any means, the progress of the algorithm is uniquely measured by the potential function. It ignores following any particular path and concentrates on shrinking the level-set of the potential function, which is contained in the feasible set and contains the optimal solution set of the LP. We believe that the actual solution path generated by the algorithm depends on various implementation and line search strategies, which is a subject for further research.

4. The dual form

Now we describe the algorithm in the dual form. Let $\bar{z}^0 = c^T x^0$ for some $x^0 > 0$ and $Ax^0 = b$. Next, we minimize the linearized *dual* potential function subject to the ellipsoid constraint corresponding to the second order term in (7b):

$$\text{PD:} \quad \text{minimize} \quad \nabla \underline{\phi}^T(y^0, \bar{z}^0)(y - y^0)$$
$$\text{subject to} \quad \|(S^0)^{-1} A^T (y - y^0)\| \leq \beta.$$

Then, denoting by y^1 the minimal solution for PD, we have

$$y^1 - y^0 = -\beta \frac{(A(S^0)^{-2}A^T)^{-1} \nabla \underline{\phi}(y^0, \bar{z}^0)}{\sqrt{\nabla \underline{\phi}^T(y^0, \bar{z}^0)(A(S^0)^{-2}A^T)^{-1} \nabla \underline{\phi}(y^0, \bar{z}^0)}}. \tag{15}$$

Let

$$p(\bar{z}^0) = (S^0)^{-1} A^T (A(S^0)^{-2}A^T)^{-1} \nabla \underline{\phi}(y^0, \bar{z}^0).$$

Then

$$\nabla \underline{\phi}^T(y^0, \bar{z}^0)(y^1 - y^0) = -\beta \| p(\bar{z}^0) \|.$$

Hence, due to (7b) the reduction of the potential function

$$\underline{\phi}(y^1, \bar{z}^0) - \underline{\phi}(y^0, \bar{z}^0) \leq -\beta \| p(\bar{z}^0) \| + \frac{\beta^2}{2(1-\beta)}.$$

Now, we focus on the expression of $p(\bar{z}^0)$, which from (5b) can be rewritten as

$$p(\bar{z}^0) = -\frac{\rho}{\bar{z}^0 - b^T y^0} S^0 x(\bar{z}^0) + e \tag{16}$$

with

$$x(\bar{z}^0) = x_1 + \frac{\bar{z}^0 - b^T y^0}{\rho} x_2 \tag{17}$$

where

$$x_1 = (S^0)^{-2} A^T (A(S^0)^{-2} A)^{-1} b$$

and

$$x_2 = (S^0)^{-1} P_{A(S^0)^{-1}} (S^0)^{-1} c.$$

It is clear that x_1 is related to the dual affine scaling direction of (3), which corresponds to the gradient-projection of the linear objective function; x_2 is related to the primal affine scaling direction of (2), which corresponds to the gradient-projection of the barrier function. We emphasize that

$$Ax(\bar{z}^0) = b,$$

i.e. $x(\bar{z})$ satisfies the equality constraints of LP.

Parallel to Lemma 2, we have the following lemma whose proof is omitted.

Lemma 3. *Let*

$$\Delta^0 = \frac{\bar{z}^0 - b^T y^0}{n} = \frac{(s^0)^T x^0}{n}, \qquad \Delta = \frac{(s^0)^T x(\bar{z}^0)}{n},$$

$\rho = n + \sqrt{n}$, *and* $\alpha < 1$. *If*

$$\| p(\bar{z}^0) \| < \min\left(\alpha \sqrt{\frac{n}{n+\alpha^2}}, 1 - \alpha \right), \tag{18}$$

then the following three inequalities hold:

$$x(\bar{z}^0) > 0,$$

$$\| S^0 x(\bar{z}^0) - \Delta e \| < \alpha \Delta$$

and

$$\Delta < (1 - 0.5\alpha/\sqrt{n}) \Delta^0. \qquad \square$$

Similar to Theorem 1, we have:

Corollary 1. *Let* x^0 *and* y^0 *be any interior feasible solutions for* LP *and* LD, *and let* $\rho = n + \sqrt{n}$, $\bar{z}^0 = c^T x^0$, $s^1 = c - A^T y^1$ *of* (15), *and* $x^1 = x(\bar{z}^0)$ *of* (17). *Then, either*

$$\phi(x^0, s^1) \le \phi(x^0, s^0) - \delta$$

or

$$\phi(x^1, s^0) \leq \phi(x^0, s^0) - \delta$$

where $\delta > 0.05$. □

Therefore, the dual algorithm can be described as follows.

Dual Algorithm.
Given $Ax^0 = b$, $x^0 > 0$ and $s^0 = c - A^T y^0 > 0$;
let $\bar{z}^0 = c^T x^0$ and set $k = 0$;
while $c^T x^k - b^T y^k \geq 2^{-L}$ **do**
begin
compute $x(\bar{z}^k)$ of (17) and formulate $p(\bar{z}^k)$ of (16);
if the inequality of (18) does not hold **then**
$s^{k+1} = s^k + \beta^* S^k p(\bar{z}^k)$ with $\beta^* = \arg\min_{\beta \geq 0} \phi(x^k, s^k + \beta S^k p(\bar{z}^k))$;
$x^{k+1} = x^k$ and $\bar{z}^{k+1} = \bar{z}^k$;
else
$x^{k+1} = x(\bar{z}^*)$ with $\bar{z}^* = \arg\min_{\bar{z} \leq \bar{z}^k} \phi(x(\bar{z}), s^k)$;
$s^{k+1} = s^k$ and $\bar{z}^{k+1} = c^T x(\bar{z}^*)$;
end;
$k = k + 1$;
end.

The worst-case complexity of the dual algorithm is identical to that of the primal algorithm.

Corollary 2. *Let* $\rho = n + \sqrt{n}$ *and* $\phi(x^0, s^0) \leq O(\sqrt{n}\, L)$. *Then, the dual algorithm terminates in* $O(\sqrt{n}\, L)$ *iterations and each iteration uses* $O(n^3)$ *arithmetic operations.* □

Again note that if $s^{k+1} = s^k$ in the dual algorithm, the projection matrix in (15) is unchanged and should be reused for the next iteration. In practice, a *strict* upper bound $\bar{z}^0 > z^*$ suffices to start the algorithm, i.e. the known x^0 is not necessary. Moreover, a bi-directional search over β and $\bar{z}$ can be employed to update x and s simultaneously in minimizing $\phi(x, s)$.

5. Further complexity analysis

Theorem 2 indicates that the potential reduction algorithm uses $O(n^{3.5}L)$ arithmetic operations. Applying Karmarkar's lower-rank scheme, we can employ a rank-one updating technique to update the projection matrix in (8) (see, for example, Shanno [28]). This can be implemented as follows.

Replacing X^0 in PP by a positive diagonal matrix D such that

$$\frac{1}{1.1} \leqslant \frac{d_j}{x_j^0} \leqslant 1.1 \quad \text{for } j = 1, \ldots, n,$$

we have

$$x^1 - x^0 = -\beta \frac{D\hat{p}(\underline{z}^0)}{\|\hat{p}(\underline{z}^0)\|},$$

where

$$\hat{p}(\underline{z}^0) = P_{AD} D \nabla \bar{\phi}(x^0, \underline{z}^0).$$

Then

$$\nabla \bar{\phi}^{\mathrm{T}}(x^0, \underline{z}^0)(x^1 - x^0) = -\beta \|\hat{p}(\underline{z}^0)\|.$$

Hence, from (7a) the reduction of the potential function is

$$\bar{\phi}(x^1, \underline{z}^0) - \bar{\phi}(x^0, \underline{z}^0) \leqslant -\beta \|\hat{p}(\underline{z}^0)\| + \frac{(1.1\beta)^2}{2(1 - 1.1\beta)},$$

since

$$\begin{aligned} \|(X^0)^{-1}(x^1 - x^0)\| &= \|(X^0)^{-1} D D^{-1}(x^1 - x^0)\| \\ &\leqslant \|(X^0)^{-1} D\| \, \|D^{-1}(x^1 - x^0)\| \\ &\leqslant 1.1 \|D^{-1}(x^1 - x^0)\| = 1.1\beta. \end{aligned}$$

Now, $\hat{p}(\underline{z}^0)$ can be written as

$$\hat{p}(\underline{z}^0) = \frac{\rho}{c^{\mathrm{T}} x^0 - \underline{z}^0} D s(\underline{z}^0) - D(X^0)^{-1} e = D(X^0)^{-1} p(\underline{z}^0), \tag{19}$$

where the expressions of $p(\underline{z}^0)$ and $s(\underline{z}^0)$ are again given by (10) and (11) with

$$y(\underline{z}^0) = (AD^2A^{\mathrm{T}})^{-1} AD\left(Dc - \frac{c^{\mathrm{T}} x^0 - \underline{z}^0}{\rho} D(X^0)^{-1} e\right). \tag{20}$$

Thus, we have

$$\|\hat{p}(\underline{z}^0)\| = \|D(X^0)^{-1} p(\underline{z}^0)\| \geqslant \|p(\underline{z}^0)\| / \|D^{-1} X^0\| \geqslant \|p(\underline{z}^0)\| / 1.1.$$

Noting that Lemma 2 still holds for $p(\underline{z}^0)$, we only need to modify the first inequality in the proof of Theorem 1 by

$$\bar{\phi}(x^1, z^0) \leq \bar{\phi}(x^0, z^0) - \frac{\beta}{1.1}\min\left(\alpha\sqrt{\frac{n}{n+\alpha^2}}, 1-\alpha\right) + \frac{(1.1\beta)^2}{2(1-1.1\beta)}.$$

Therefore, upon choosing $\alpha = 0.43$ and $\beta = 0.25$, Theorem 1 is still valid for $\delta > 0.04$. As a result, the following modified primal algorithm can be developed.

Modified Primal Algorithm.
  Given $Ax^0 = b$, $x^0 > 0$ and $s^0 = c - A^{\mathrm{T}}y^0 > 0$;
  let $z^0 = b^{\mathrm{T}}y^0$ and $D = X^0$;
  set $\alpha = 0.43$, $\beta = 0.25$ and $k = 0$;
  while $c^{\mathrm{T}}x^k - b^{\mathrm{T}}y^k \geq 2^{-L}$ **do**
    begin
    for $j = 1, \ldots, n$, **if** $d_j/x_j^k \notin [1/1.1, 1.1]$ **then** $d_j = x_j^k$;
    using $y(z^k)$ of (20), formulate $s(z^k)$ of (11), $p(z^k)$ of (10) and $\hat{p}(z^k)$ of (19);
    if the inequality of (12) does not hold **then**
      $x^{k+1} = x^k - \beta D\hat{p}(z^k)/\|\hat{p}(z^k)\|$;
      $s^{k+1} = s^k$ and $z^{k+1} = z^k$;
    else
      $s^{k+1} = s(z^*)$ with $z^* = \arg\min_{z \geq z^k} \phi(x^k, s(z))$;
      $x^{k+1} = x^k$ and $z^{k+1} = b^{\mathrm{T}}y(z^*)$;
    end;
    $k = k+1$;
  end.

The projection matrix in (20) can be calculated using a rank-one updating technique whenever d_j is changed, and each update uses O(n^2) arithmetic operations. Due to Karmarkar [17], Gonzaga [15], Vaidya [32], and many others, the total number of updates in O($\sqrt{n}\,L$) iterations is O(nL). Therefore, we have

Theorem 3. *Let $\rho = n + \sqrt{n}$ and $\phi(x^0, s^0) \leq$ O($\sqrt{n}\,L$). Then, the modified primal algorithm terminates in O($\sqrt{n}\,L$) iterations and uses O(n^3L) total arithmetic operations.* □

One can also develop a modified dual algorithm with the same worst-case complexity. However, the modified algorithm may be of only theoretical value, since a much larger step has usually been taken in practice.

6. A practical approach

As discussed by many authors, such as McShane et al [20] and Vanderbei and Lagarias [33], a theoretical algorithm usually needs modifications to become a practically useful algorithm. In practice, the affine scaling algorithm is considered one of the "practically best" algorithms, although it has not yet proved to be a polynomial algorithm. In this section, we discuss some of these theoretical and practical considerations. We propose an approach to implementing the potential reduction algorithm: while maintaining the polynomial complexity, we make the algorithm like the affine scaling algorithm as much as possible.

Consider the gradient vectors for the potential functions in (5a). To make the gradient vector as much like the gradient vector of the true objective function as possible, we can choose either $\rho=\infty$ or $\underline{z}=c^{\mathrm{T}}x$. The latter option was discussed by Ye [35]. However, to maintain polynomial complexity, ρ has to be finite and $\underline{z}\leqslant z^*$. Therefore, to maxweight the gradient vector of the objective function, we want $\underline{z}$ as close to z^* as possible in the iterative process. This leads to a potential reduction version that is as close as possible to the affine scaling algorithm. We discuss the version in the primal form. Again we focus on solving the problem PP in Section 3.

Regarding $\|p(\underline{z}^0)\|$, we have another lemma:

Lemma 4. *Let $\underline{z}^0\leqslant z^*$. Then, for $\rho>n$:*

(1) *If $s(\underline{z}^0)=c-A^{\mathrm{T}}y(\underline{z}^0)\not>0$, then*

$$\|p(\underline{z}^0)\|\geqslant 1.$$

(2) *Else $\exists\underline{z}^1$ such that $\underline{z}^0\leqslant\underline{z}^1\leqslant z^*$, and*

$$\|p(\underline{z}^1)\|\geqslant\min\left(1,\frac{\rho-n}{\sqrt{n}}\right).$$

Proof. Case (1) follows (i) of Lemma 1. In case (2), $y(\underline{z}^0)$ is an interior feasible solution for LD. Let Δ^0 and Δ be defined in Lemma 2. Then, from (14),

$$\|p(\underline{z}^0)\|^2=\left(\frac{\rho}{n\Delta^0}\right)^2\|X^0s(\underline{z}^0)-\Delta e\|^2+\left(\frac{\rho\Delta}{n\Delta^0}-1\right)^2 n. \tag{21}$$

If

$$\underline{z}^0\geqslant b^{\mathrm{T}}y(\underline{z}^0) \tag{22}$$

or

$$\Delta\geqslant\Delta^0,$$

then let $\underline{z}^1=\underline{z}^0$ and from (21),

$$\|p(\underline{z}^1)\|\geqslant\left(\frac{\rho}{n}-1\right)\sqrt{n}=\frac{\rho-n}{\sqrt{n}};$$

otherwise, we solve the following one-dimensional LP problem over $\underline{z}$:

$$\begin{aligned} \text{ZP:} \quad & \text{maximize} && \underline{z} \\ & \text{subject to} && c - A^{\mathrm{T}} y(\underline{z}) \geq 0, \\ & && b^{\mathrm{T}} y(\underline{z}) - \underline{z} \geq 0, \end{aligned}$$

and denote by $\underline{z}^*$ the maximal solution of ZP. Then, since $y(\underline{z}^*)$ is feasible for LD,

$$\underline{z}^0 < \underline{z}^* \leq b^{\mathrm{T}} y(\underline{z}^*) \leq z^*.$$

Let $\underline{z}^1 = b^{\mathrm{T}} y(\underline{z}^*)$. Then, either Case (1) or the inequality (22) holds for $\underline{z}^1$, because $\underline{z}^1$ can not be an interior solution for ZP. This gives the desired result. $\square$

Problem ZP is a one-dimensional LP problem once y_1 and y_2 of (11) are known. In addition, since $\underline{z}^0$ is an interior feasible solution for ZP, ZP is feasible and bounded from above by z^*. Thus, ZP can be solved by a ratio test (see Ye [35] for detail). In practice, we can check ZP in both Case (1) and Case (2) to see if a closer lower bound $\underline{z}^1$ for z^* can be found, where $\underline{z}^1 = -\infty$ if ZP is infeasible. Thus, the primal version can be described in the following practical algorithm.

Practical Algorithm.

Given $Ax^0 = b$, $x^0 > 0$ and $\underline{z}^0 \leq z^*$;
set $k = 0$;
while $\bar{\phi}(x^k, \underline{z}^k) \geq -\rho L$ **do**
  begin
  compute y_1 and y_2 of (11);
  solve ZP and let $\underline{z}^{k+1} = b^{\mathrm{T}} y(\underline{z}^*)$;
  if $\underline{z}^{k+1} < \underline{z}^k$ then $\underline{z}^{k+1} = \underline{z}^k$;
  compute $y(\underline{z}^{k+1})$ of (11) and $p(\underline{z}^{k+1})$ of (10);
  $x^{k+1} = x^k - \beta^* X^k p(\underline{z}^{k+1})$ with $\beta^* = \arg\min_\beta \bar{\phi}(x^k - \beta X^k p(\underline{z}^{k+1}), \underline{z}^{k+1})$;
  $k = k + 1$;
end.

The performance of the primal algorithm results from the following theorem.

Theorem 4. *For $\rho \geq n + \sqrt{n}$, the practical algorithm generates the optimal solutions for* LP *and* LD *in* $O(\rho L)$ *iterations and each iteration uses* $O(n^3)$ *arithmetic operations.*

Proof. Clearly

$$\| p(\underline{z}^{k+1}) \| \geq 1 \quad \text{for } \rho \geq n + \sqrt{n}$$

and

$$\underline{z}^k \leqslant \underline{z}^{k+1} \leqslant z^*,$$

from (9),

$$\begin{aligned}\bar{\phi}(x^{k+1}, \underline{z}^{k+1}) - \bar{\phi}(x^k, \underline{z}^k) &\leqslant \bar{\phi}(x^{k+1}, \underline{z}^{k+1}) - \bar{\phi}(x^k, \underline{z}^{k+1}) \\ &\leqslant -\beta \| p(\underline{z}^{k+1}) \| + \frac{\beta^2}{2(1-\beta)} \\ &\leqslant -\beta + \frac{\beta^2}{2(1-\beta)}.\end{aligned}$$

Let $\beta = 0.5$, then the potential function is reduced by a constant greater than 0.25. Following the argument of Ye and Kojima [36], in $\mathrm{O}(\rho L)$ iterations

$$c^{\mathrm{T}}x^k - z^* \leqslant c^{\mathrm{T}}x^k - \underline{z}^k \leqslant 2^{-L}. \qquad \square$$

From Theorems 4, for any finite ρ, the potential function algorithm is a polynomial-time algorithm with the complexity $\mathrm{O}(\rho n^3 L)$ for some $\rho \geqslant n + \sqrt{n}$. As discussed before, when ρ approaches to ∞, the algorithm becomes the affine scaling algorithm. It will be interesting to see what is the best ρ in practice.

7. Concluding remarks

We developed an $\mathrm{O}(\sqrt{n}\,L)$-iteration and $\mathrm{O}(n^3L)$-operation potential reduction algorithm, compared with the $\mathrm{O}(nL)$-iteration and $\mathrm{O}(n^{3.5}L)$-operation Karmarkar-type projective algorithm. Our algorithm is naturally equipped with a primal–dual potential function, which is uniquely used to measure the solution's progress. It does not need to trace any particular path as do path-following algorithms, or to use the projective transformation as the projective algorithm does. There is *no* step-size-restriction during its iterative process; the greater the reduction of the potential function, the faster the convergence of the algorithm.

The algorithm itself works like a dynamic game, in which one player plays the leader and the other plays the follower. In the primal form the primal player, the leader, using only information about the current dual objective value from the dual player, reduces his potential function by a constant at each step until he is "stuck". Once the leader is "stuck", the follower can then make a move to reduce his potential function by a constant. No matter who moves, the joint primal–dual potential function is reduced by a constant. In this game, there is no winner or loser, they arrive at the optimal solution set (Cournot–Nash equilibrium set to their potential functions) together. One can also develop a symmetric, cooperative strategy so that both players, the primal (x) and the dual (s), move simultaneously at each step to reduce their joint primal-dual potential function.

Acknowledgement

The author gratefully acknowledges Professors Michael Todd and Kurt Anstreicher, the editors, and the referees for their very helpful comments and remarks.

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