Extension of a Crack by a Shear Wave[1])

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1. Introduction

The application of a sudden disturbance to the surface of an elastic body gives rise to elastic waves which propagate into the interior of the body. If the system of transient waves encounters internal flaws such as cracks, a complicated pattern of diffracted waves is generated. It is well known that upon diffraction of a wave by a crack the stress becomes singular in the vicinity of the crack tip, and it is therefore conceivable that waves generated by surface disturbances will cause propagation of existing internal flaws.

In this paper we investigate the conditions for crack propagation upon diffraction of an incident wave by a crack. We focus on some essential physical aspects of the problem, and the mathematical analysis is simplified by considering a two-dimensional geometry with a plane incident wave. The geometry is shown in Figure 1. In this first approach the analysis is concerned with a horizontally polarized incident shear wave, so that only one wave equation enters the analysis. Assuming that for a sufficiently sharp pulse the question of crack propagation is decided at the instant that the wave front strikes the crack tip, or briefly thereafter, the original length of the crack is immaterial and we may simplify the analytical work by considering a semi-infinite crack.

The analysis consists of two parts. The particle velocity behind the crack tip and the shear stress ahead of the crack tip are first determined for the diffraction of a plane transient wave by a crack which extends with an arbitrary (time-dependent) velocity.

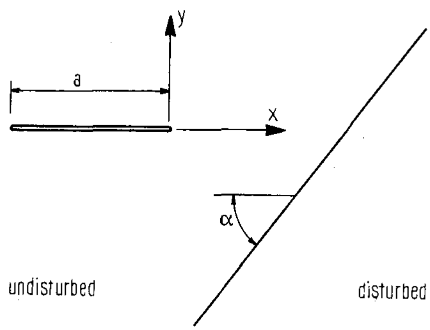

Figure 1
Horizontally polarized shear wave incident on a crack.

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The diffraction problem is solved by a method which was used by Kostrov [1] for a problem of crack propagation, and much earlier by Evvard [2] for a mathematically analogous problem of a wing in supersonic flow. Once the stress and the particle velocity have been obtained in the plane of the crack we employ a fracture criterion to investigate the conditions for crack propagation. In this paper the energy criterion is used, which states that the formation of new free surface requires energy, and fracture can thus occur only if energy is available.

The results of the analysis show that a crack in an undisturbed brittle elastic medium can be incited to extend instantaneously only if the shear stress shows a square root singularity at the wave front. If the shear stress is finite at the wave front, crack propagation may be incited a short time after the wave front has struck the crack tip. Also, if the material is already statically prestressed, crack propagation may be generated almost instantaneously by a smooth wave if the stress intensity factor of the static prestress is large enough.

Transient problems for cracks propagating at a constant velocity were investigated by Broberg [3] and Craggs [4], who were concerned with a uniformly extending central crack in a stressed body. The sudden appearance of a moving semi-infinite crack in a stressed body was investigated by Baker [5]. The elastic field of a crack extending non-uniformly under general anti-plane loading was recently studied by Kostrov [1] and Eshelby [6]. In an earlier paper the author [7] investigated in considerable detail, but by a different method, crack propagation at a constant velocity generated by an incident shear wave. Elastodynamic problems of crack propagation were reviewed by Sih [8] and by Erdogan [9], who supplied a rather complete list of references.

2. The Diffraction Problem

The propagation of horizontally polarized shear waves in a homogeneous, isotropic, linearly elastic medium is governed by the two-dimensional wave equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial s^2}, \tag{2.1a}$$

where

$$s = c\,t, \quad c = \left(\frac{\mu}{\varrho}\right)^{1/2}. \tag{2.1b}$$

In equations (2.1a) and (2.1b), $w(x, y, s)$ is the displacement normal to the xy-plane, t is the time, and μ and ϱ are the shear modulus and the mass density, respectively. The non-vanishing stresses are

$$\tau_{xz} = \mu \frac{\partial w}{\partial x}, \quad \text{and} \quad \tau_{yz} = \mu \frac{\partial w}{\partial y}. \tag{2.2a, b}$$

At $s = 0$, a plane wave of the general form

$$w_i(x, y, s) = H\,(s + x \sin\alpha - y \cos\alpha) \int_0^{s + x \sin\alpha - y \cos\alpha} g(v)\,dv\,, \tag{2.3}$$

where $H(\,)$ is the Heaviside step function, strikes the tip of a semi-infinite crack. A short time later, at $s = s_m$, the crack is assumed to start extending in the xz-plane. Thus, at a subsequent time defined by $s > s_m$, the crack tip is located at the point D defined by $x = X(s - s_m)$. It is assumed that

$$X(s - s_m) \equiv 0 \quad \text{for} \quad s \leqslant s_m\,, \tag{2.4}$$

and

$$1 > \frac{dX}{ds} \geqslant 0 \quad \text{for} \quad s \geqslant s_m\,. \tag{2.5}$$

As shown in Figure 2, the crack has then generated a reflected wave whose wave front is indicated by EB, and a cylindrical diffracted wave whose wave front is defined by $(x^2 + y^2)^{1/2} = s$.

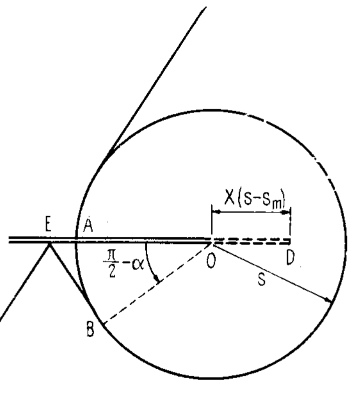

Figure 2
Incident, reflected and diffracted waves for $s > s_m$.

If there were no crack at $y = 0$, $x < X(s - s_m)$ the incident wave (2.3) would give rise to the following shear stress in the plane $y = 0$:

$$\tau_{yz} = -\mu\, g\,(s + x \sin\alpha) \cos\alpha\, H\,(s + x \sin\alpha)\,. \tag{2.6}$$

The solution to the diffraction problem is now obtained by superimposing on the incident wave the wave motion that is generated in an initially undisturbed medium by shear stresses that are equal and opposite to (2.6), and that are applied on both sides of the slit $y = 0$, $x < X(s - s_m)$. Through the superposition the surface of the crack is rendered free of tractions. Since the wave motion induced by shear tractions acting on the sides of a slit is obviously antisymmetric, the displacement vanishes for $x \geqslant X(s - s_m)$ and we need consider only the half-plane $y \leqslant 0$. For $s > 0$, the wave motion that is superimposed then must satisfy the following conditions at $y = 0$:

$$x < X(s - s_m) : \frac{\partial w}{\partial y} = g\,(s + x \sin\alpha) \cos\alpha\, H\,(s + x \sin\alpha)\,, \tag{2.7}$$

$$x \geqslant X(s - s_m) : w = 0\,. \tag{2.8}$$

In addition, we have for $s < 0$

$$w(x, y, s) = \frac{\partial w(x, y, s)}{\partial s} \equiv 0 . \tag{2.9}$$

Similar to the work of Kostrov [1], the wave propagation problem defined by equations (2.1) and (2.7)–(2.9) can be solved by employing a Green's function approach. Thus we employ the function $G(x_0 - x, y_0, s_0 - s)$, which represents the displacement for a two-dimensional line source,

$$\frac{\partial^2 G}{\partial x_0^2} + \frac{\partial^2 G}{\partial y_0^2} - \frac{\partial^2 G}{\partial s_0^2} = - \frac{1}{\mu} \delta (x_0 - x) \delta(y_0) \delta (s_0 - s) . \tag{2.10}$$

Equation (2.10) can easily be solved [10], and for $s_0 > s$ we obtain

$$G(x_0 - x, y_0, s_0 - s) = \frac{H \{(s_0 - s) - [(x_0 - x)^2 + y_0^2]^{1/2}\}}{2 \pi \mu R} , \tag{2.11}$$

where

$$R^2 = (s_0 - s)^2 - (x_0 - x)^2 - y_0^2 . \tag{2.12}$$

By symmetry considerations it follows that $2 G(x_0 - x, y_0, s_0 - s)$ is the displacement field in the half-plane $y_0 < 0$ due to a unit force applied at position x on the surface $y_0 = 0$.

If the surface shear tractions at $y = 0$ would be known, say $\tau_{yz}(x, 0, s) = \tau(x, s)$, linear superposition could be employed to write the displacement $w(x, y, s)$ in the half-plane $y_0 \leqslant 0$ in the form

$$w(x_0, y_0, s_0) = \frac{1}{\pi \mu} \iint_S \frac{\tau(x, s)}{R} dx\, ds , \tag{2.13}$$

where, as follows from equation (2.11), S is that part of the xs plane that falls inside the cone defined by

$$(s_0 - s) - [(x_0 - x)^2 + y_0^2]^{1/2} \geqslant 0 , \quad 0 \leqslant s \leqslant s_0 . \tag{2.14}$$

To employ equation (2.13), the shear stress in the plane of the crack ahead of the crack tip must, however, first be determined.

For $y_0 \equiv 0$, the region of integration S reduces to a triangular region in the xs plane, as shown in Figure 3. It is then convenient to introduce the following characteristic coordinates in the xs plane

$$\xi = \frac{s - x}{\sqrt{2}} , \qquad \eta = \frac{s + x}{\sqrt{2}} , \tag{2.15a, b}$$

whereby the denominator in (2.13) reduces to

$$(s_0 - s)^2 - (x_0 - x)^2 = 2 (\xi_0 - \xi) (\eta_0 - \eta) . \tag{2.16}$$

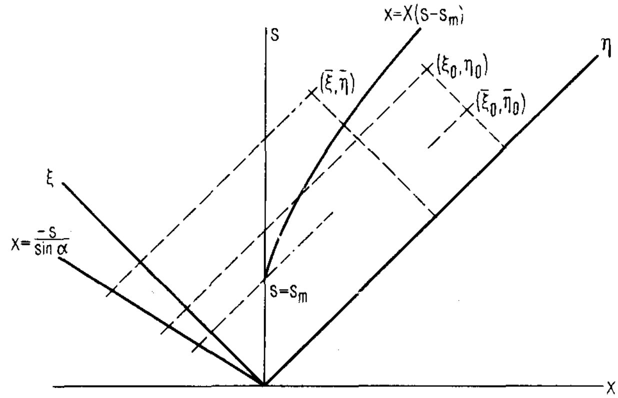

Figure 3
Regions in the xs-plane.

Let us first consider a point $\bar{x}_0$, $\bar{s}_0$, or $\bar{\xi}_0$, $\bar{\eta}_0$, see Figure 3, such that $\bar{\xi}_0 < \xi_m = s_m/\sqrt{2}$. According to equation (2.8) the displacement vanishes ahead of the crack tip, and we thus find for $y_0 = 0$, and $\bar{\xi}_0 < s_m/\sqrt{2}$

$$\cos\alpha \int_0^{\bar{\xi}_0} \frac{d\xi}{(\bar{\xi}_0 - \xi)^{1/2}} \int_{-\varkappa\xi}^{\xi} \frac{g(\xi, \eta)\, d\eta}{(\bar{\eta}_0 - \eta)^{1/2}} + \frac{1}{\mu} \int_0^{\bar{\xi}_0} \frac{d\xi}{(\bar{\xi}_0 - \xi)^{1/2}} \int_{\xi}^{\bar{\eta}_0} \frac{\tau(\xi, \eta)\, d\eta}{(\bar{\eta}_0 - \eta)^{1/2}} = 0\,, \quad (2.17)$$

where $g(\xi, \eta)$ follows from equation (2.7), and

$$\varkappa = \frac{1 - \sin\alpha}{1 + \sin\alpha}\,. \quad (2.18)$$

In equation (2.17) we have used that the medium is undisturbed ahead of the wave fronts $x = s$ and $x = -s/\sin\alpha$. Equation (2.17) is evidently satisfied if

$$\int_{\xi}^{\bar{\eta}_0} \frac{\tau(\xi, \eta)\, d\eta}{(\bar{\eta}_0 - \eta)^{1/2}} = -\,\mu\, \cos\alpha \int_{-\varkappa\xi}^{\xi} \frac{g(\xi, \eta)\, d\eta}{(\bar{\eta}_0 - \eta)^{1/2}}\,. \quad (2.19)$$

The Abel integral equation (2.19) can be solved in the standard fashion to yield

$$\tau(\xi, \eta) = -\,\frac{\mu\, \cos\alpha}{\pi\, (\eta - \xi)^{1/2}} \int_{-\varkappa\xi}^{\xi} \frac{g(\xi, u)\, (\xi - u)^{1/2}}{\eta - u}\, du\,. \quad (2.20)$$

Equation (2.20) gives the stress at a position $x > X\,(s - s_m)$ before the influence of the propagation of the crack can be detected at that position.

Now consider a point x_0, s_0, or ξ_0, η_0, such that $\xi_0 > \xi_m = s_m/\sqrt{2}$, and $\eta_0 = \eta_0$, see Figure 3. Again, since the displacement vanishes ahead of the crack, equation (2.13) yields

$$\cos\alpha \int_0^{\xi m} \frac{d\xi}{(\xi_0-\xi)^{1/2}} \int_{-\varkappa\xi}^{\xi} \frac{g(\xi,\eta)\,d\eta}{(\eta_0-\eta)^{1/2}} + \cos\alpha \int_{\xi m}^{\xi_0} \frac{d\xi}{(\xi_0-\xi)^{1/2}} \int_{-\varkappa\xi}^{N(\xi)} \frac{g(\xi,\eta)\,d\eta}{(\eta_0-\eta)^{1/2}}$$

$$+\frac{1}{\mu}\int_0^{\xi m} \frac{d\xi}{(\xi_0-\xi)^{1/2}} \int_{\xi}^{\eta_0} \frac{\tau(\xi,\eta)\,d\eta}{(\eta_0-\eta)^{1/2}} + \frac{1}{\mu}\int_{\xi m}^{\xi_0} \frac{d\xi}{(\xi_0-\xi)^{1/2}} \int_{N(\xi)}^{\eta_0} \frac{\tau(\xi,\eta)\,d\eta}{(\eta_0-\eta)^{1/2}} = 0\,, \tag{2.21}$$

where $\xi_m = s_m/\sqrt{2}$, and $N(\xi)$ is the solution of

$$\frac{N(\xi)-\xi}{\sqrt{2}} = X\left[\frac{N(\xi)+\xi}{\sqrt{2}} - s_m\right]\,. \tag{2.22}$$

Since $\bar{\eta}_0 = \eta_0$, the first and the third terms in (2.21) cancel in view of equation (2.19), and we conclude

$$\int_{N(\xi)}^{\eta_0} \frac{\tau(\xi,\eta)\,d\eta}{(\eta_0-\eta)^{1/2}} = -\,\mu\cos\alpha \int_{-\varkappa\xi}^{N(\xi)} \frac{g(\xi,\eta)\,d\eta}{(\eta_0-\eta)^{1/2}}\,. \tag{2.23}$$

The solution of (2.23) is obtained in the standard fashion as

$$\tau(\xi,\eta) = -\frac{\mu\cos\alpha}{\pi\,[\eta-N(\xi)]^{1/2}} \int_{-\varkappa\xi}^{N(\xi)} \frac{g(\xi,u)\,[N(\xi)-u]^{1/2}}{\eta-u}\,du\,. \tag{2.24}$$

Equation (2.24) is valid for $\xi > s_m/\sqrt{2}$, i.e. after the influence of the propagation of the crack can first be detected at a position $x > X\,(s - s_m)$.

By employing equations (2.13), (2.7), (2.19) and (2.23), the displacement at $y = 0$ behind the crack tip can now also be determined. If we consider the point $\bar{x}$, $\bar{s}$, or $\bar{\xi}$, $\bar{\eta}$, where $\bar{\eta} > \bar{\xi} > \xi_m = s_m\sqrt{2}$, see Figure 3, we may write

$$\left.\begin{aligned}
\pi\sqrt{2}\,w(\bar{\xi},\bar{\eta}) = {} & \cos\alpha \int_0^{\xi m} \frac{d\xi}{(\bar{\xi}-\xi)^{1/2}} \int_{-\varkappa\xi}^{\xi} \frac{g(\xi,\eta)\,d\eta}{(\bar{\eta}-\eta)^{1/2}} + \frac{1}{\mu}\int_0^{\xi m} \frac{d\xi}{(\bar{\xi}-\xi)^{1/2}} \\
& \times \int_{\xi}^{\bar{\eta}} \frac{\tau(\xi,\eta)\,d\eta}{(\bar{\eta}-\eta)^{1/2}} + \cos\alpha \int_{\xi m}^{K(\bar{\eta})} \frac{d\xi}{(\bar{\xi}-\xi)^{1/2}} \int_{-\varkappa\xi}^{N(\xi)} \frac{g(\xi,\eta)\,d\eta}{(\bar{\eta}-\eta)^{1/2}} + \frac{1}{\mu}\int_{\xi m}^{K(\bar{\eta})} \frac{d\xi}{(\bar{\xi}-\xi)^{1/2}} \\
& \times \int_{N(\xi)}^{\bar{\eta}} \frac{\tau(\xi,\eta)\,d\eta}{(\bar{\eta}-\eta)^{1/2}} + \cos\alpha \int_{K(\bar{\eta})}^{\bar{\xi}} \frac{d\xi}{(\bar{\xi}-\xi)^{1/2}} \int_{-\varkappa\xi}^{\bar{\eta}} \frac{g(\xi,\eta)\,d\eta}{(\bar{\eta}-\eta)^{1/2}}\,,
\end{aligned}\right\} \tag{2.25}$$

where

$$\frac{\bar{\eta}-K(\bar{\eta})}{\sqrt{2}} = X\left[\frac{\bar{\eta}+K(\bar{\eta})}{\sqrt{2}} - s_m\right]\,. \tag{2.26}$$

The first two terms in (2.25) cancel in view of equation (2.19). The next two terms, however, cancel because of (2.23), and equation (2.25) thus reduces to

$$w(\bar{\xi}, \bar{\eta}) = \frac{\cos\alpha}{\pi\sqrt{2}} \int_{K(\bar{\eta})}^{\bar{\xi}} \frac{d\xi}{(\bar{\xi} - \xi)^{1/2}} \int_{-\varkappa\xi}^{\bar{\eta}} \frac{g(\xi, \eta)\, d\eta}{(\bar{\eta} - \eta)^{1/2}} . \tag{2.27}$$

Thus, equation (2.27) yields the displacement for $0 \leqslant x \leqslant X\,(s - s_m)$, where $s > s_m$, and $K(\bar{\eta})$ must be computed from (2.26).

In principle the displacement at an arbitrary position can be obtained by employing equation (2.13), but the evaluation of the integrals is rather complicated for $y_0 \neq 0$.

3. The Field Variables Near the Crack Tip

To investigate whether or not the crack extends once it has been struck by the incident wave, the stress just ahead of the crack tip and the particle velocity just behind it must be known. After the crack tip has started to move, the stress ahead of the crack tip is given by equation (2.24). In terms of the physical coordinates x and s the expression for $\tau(x, s)$ can be written as

$$\tau(x, s) = -\frac{\mu\cos\alpha}{\pi\,[x - X\,(s_1 - s_m)]^{1/2}} \int_{f(s,x)}^{X(s_1 - s_m)} \frac{g\,(u, s - x + u)\,[X\,(s_1 - s_m) - u]^{1/2}}{x - u}\, du\,, \tag{3.1}$$

where

$$f\,(s, x) = -\frac{s - x}{1 + \sin\alpha}\,, \tag{3.2}$$

and s_1 is computed from

$$s - x = s_1 - X\,(s_1 - s_m)\,. \tag{3.3}$$

Some care must be exercised in extracting from equation (3.1) the singular term just ahead of the crack tip. Let us observe at $s = s^*$ a point $x = x^*$ ahead of the crack tip after the tip has started to move. At $s = s^*$ the crack tip is located at $x = X\,(s^* - s_m)$ and it is assumed that $x^* - X\,(s^* - s_m) = \varepsilon$ is very small. From equation (3.3) it is noted that at $x = X\,(s^* - s_m)$, $s = s^*$ we have $s_1 = s^*$. Consequently, at $x = x^*$ we may write

$$s_1(x^*, s^*) \simeq s^* + \frac{\partial s_1}{\partial x}\,\varepsilon\,. \tag{3.4}$$

Thus, by using equation (3.3)

$$x^* - X\,(s_1 - s_m) = s^* - s_1 \simeq -\frac{\partial s_1}{\partial x}\,\varepsilon \tag{3.5}$$

$$[x^* - X\,(s_1 - s_m)]^{1/2} \simeq \left(-\frac{\partial s_1}{\partial x}\right)^{1/2} [x^* - X\,(s^* - s_m)]^{1/2}\,. \tag{3.6}$$

From equation (3.3) we also find

$$\frac{\partial s_1}{\partial x} = -\left(1 - \frac{dX}{ds}\right)^{-1} . \tag{3.7}$$

By employing (3.1), (3.6) and (3.7) the shear stress just ahead of the crack tip may then be written as

$$\frac{\pi\,\tau(x, s)}{\mu\,\cos\alpha} = -\frac{(1 - dX/ds)^{1/2}}{[x - X(s - s_m)]^{1/2}} \int\limits_{h(s)}^{X(s-s_m)} \frac{g\,[u, s - X(s - s_m) + u]}{[X(s - s_m) - u]^{1/2}}\,du + O\,[x - X(s - s_m)]^{1/2} \tag{3.8}$$

where

$$h(s) = -\frac{s - X(s - s_m)}{1 + \sin\alpha} . \tag{3.9}$$

The particle velocity behind the crack tip can be obtained by differentiating equation (2.27), and employing

$$\frac{\partial w}{\partial s} = \frac{1}{\sqrt{2}}\left(\frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial \eta}\right) . \tag{3.10}$$

By using the limiting procedures that were discussed previously in this section we obtain after some manipulation

$$\frac{\pi}{\cos\alpha}\,\frac{\partial w}{\partial s} = \frac{dX/ds}{(1 + dX/ds)^{1/2}}\,\frac{1}{[X(s - s_m) - x]^{1/2}} \times \int\limits_{h(s)}^{X(s-s_m)} \frac{g\,[u, s - X(s - s_m) + u]}{[X(s - s_m) - u]^{1/2}}\,du + O\,[X(s - s_m) - x]^{1/2} , \tag{3.11}$$

where $h(s)$ is defined by (3.9).

4. The Balance of Rate of Energy

If we consider the problem defined by (2.7) and (2.8), and we consider an element containing the crack tip, the balance of rate of energy may be expressed in the form

$$\frac{dU}{ds} = \frac{dV}{ds} + \frac{dT}{ds} + \frac{dD}{ds} . \tag{4.1}$$

In equation (4.1), dU/ds is the instantaneous rate of work of the tractions on the bounding surfaces of the element. Also, V and T are the strain and the kinetic energy, respectively, and D represents the energy dissipation. In the present problem energy is dissipated only as surface free energy or fracture energy.

To determine the terms dU/ds, dV/ds and dT/ds, we observe that the crack propagation problem may also be thought of as a problem for a half-space which is free of tractions at $y = 0$, $x < X(s - s_m)$, but which at $y = 0$, $x \geqslant X(s - s_m)$ is subjected to shear tractions depending on x and s in such a manner that the displacement vanishes identically for $x \geqslant X(s - s_m)$. In a power balance for an element near $x = X(s - s_m)$ in the half-plane $y \leqslant 0$, the term dD/ds does not enter, but now there is an additional instantaneous rate of work term dE_c/ds which gives the rate of work of the surface tractions for $x \geqslant X(s - s_m)$ along the surface $y = 0$ as the surface is released. For an element in the half-plane the power balance may then be written

$$\frac{1}{2}\frac{dU}{ds} + \frac{1}{2}\frac{dE_c}{ds} = \frac{1}{2}\frac{dT}{ds} + \frac{1}{2}\frac{dV}{ds} . \tag{4.2}$$

Since dU/ds, dT/ds and dV/ds are the same whether one looks at the problem as a crack propagation problem or as a problem for a half-space subjected to surface tractions, we conclude from equations (4.1) and (4.2)

$$\frac{dD}{ds} = -\frac{dE_c}{ds} . \tag{4.3}$$

Equation (4.3), which was also derived by Erdogan [9], provides us with a power balance condition for crack propagation.

The particle velocity just behind the propagating crack tip and the shear stress just ahead of the crack tip are given by (3.11) and (3.8), respectively. The term dE_c/ds may now be expressed as

$$\frac{1}{2}\frac{dE_c}{ds} = \int_{X(s-s_m)-\Delta}^{X(s-s_m)+\Delta} \tau(x, 0, s)\, \frac{\partial w(x, 0, s)}{\partial s}\, dx , \tag{4.4}$$

where Δ is a small positive number.

It is noted that the stress $\tau(x, s)$ vanishes for $x < X(s - s_m)$ while the particle velocity vanishes for $x > X(s - s_m)$, and both τ and $\partial w/\partial s$ are singular at $x = X(s - s_m)$. The evaluation of (4.4) is simplified by the observation that the product of the singular terms in equations (3.8) and (3.11) actually represents a Dirac delta function. This observation, which is discussed in the Appendix, may be expressed in the form

$$\frac{H(v)}{v^{1/2}} \cdot \frac{H(-v)}{(-v)^{1/2}} = \pi\, \delta(v) . \tag{4.5}$$

By substituting (3.8) and (3.11) into (4.4) and by employing (4.5) for $v = x - X(s - s_m)$, we then arrive at the following expression for the rate of energy available to create new fracture surface

$$\frac{1}{2}\frac{dE_c}{ds} = -\frac{\mu \cos^2\alpha}{\pi} \left(\frac{1 - dX/ds}{1 + dX/ds}\right)^{1/2} \frac{dX}{ds}\, (I)^2 , \tag{4.6}$$

where

$$I = \int_{h(s)}^{X(s-s_m)} \frac{g\,[u,\, s - X\,(s - s_m) + u]}{[X\,(s - s_m) - u]^{1/2}}\, du\,. \tag{4.7}$$

For the case that $g(x, s)$ is of the form of equation (2.7), i.e. if

$$g\,[u,\, s - X\,(s - s_m) + u] = g\,[s - X\,(s - s_m) + (1 + \sin\alpha)\, u]\,, \tag{4.8}$$

it is convenient to introduce a new variable of integration

$$\zeta = s - X\,(s - s_m) + (1 + \sin\alpha)\, u\,. \tag{4.9}$$

The integral I may then be rewritten as

$$I = \frac{1}{(1 + \sin\alpha)^{1/2}} \int_0^{s + X(s-s_m)\sin\alpha} \frac{g(\zeta)\, d\zeta}{[s + X\,(s - s_m)\sin\alpha - \zeta]^{1/2}}\,. \tag{4.10}$$

Equation (4.3) must be completed by a suitable expression for the dissipation term dD/ds. Since we are considering a crack propagating in a linearly elastic solid where the only energy dissipation taking place is in a small region near the crack tip, $D(s)$ may be written as

$$D(s) = 2 \int_{s_m}^{s} \gamma_F \frac{dX}{ds}\, ds\,, \tag{4.11}$$

where γ_F is the amount of energy needed to create a unit area of fracture surface and is called the specific fracture energy of the solid. In the case of brittle fracture, the specific fracture energy has usually been identified with the surface tension of the material and thus has been assumed to be a material property, constant with respect to time and crack velocity. In that case we obtain from (4.11)

$$\frac{dD}{ds} = 2\,\gamma_F \frac{dX}{ds}\,. \tag{4.12}$$

For $s > s_m$, the balance equation (4.3) now emerges from (4.6), (4.10) and (4.12) as

$$\frac{\mu}{\pi}(1 - \sin\alpha)\left(\frac{1 - dX/ds}{1 + dX/ds}\right)^{1/2} \left\{\int_0^{s + X(s-s_m)\sin\alpha} \frac{g(\zeta)\, d\zeta}{[s + X\,(s - s_m)\sin\alpha - \zeta]^{1/2}}\right\}^2 = \gamma_F\,. \tag{4.13}$$

5. Discussion

The balance equation (4.13) represents a balance of rate of energy, which must be satisfied at all times for the fracture phenomenon to take place. For prescribed values of γ_F, α and μ, and for a given $g(\zeta)$ defining the shape of the incident wave, the path of the crack tip $X\,(s - s_m)$ can be determined from equation (4.13).

Let us first examine whether crack propagation can be generated instantaneously as the wave front strikes the crack tip, i.e., whether we can have $s_m \equiv 0$. If we set $s_m \equiv 0$, it is noted immediately that at $s = 0$ the left-hand side of (4.13) can be different from zero only if

$$g(\zeta) = W/\zeta^{1/2}\,, \tag{5.1a}$$

or

$$g(x, s) = W/(s + x \sin\alpha)^{1/2}\,. \tag{5.1b}$$

This leads to the rather interesting conclusion that a crack in an *initially undisturbed* medium will be incited to extend *instantaneously* only if the incident shear wave shows a square root singularity in the shear stress at the wave front. This conclusion is of course reached on the basis of the generally accepted assumption that in brittle elastic solids the specific fracture γ_F is independent of time and independent of the rate of crack extension dX/ds.

If $g(x, s)$ is of the form (5.1a), equation (4.13) reduces to

$$\left(\frac{1 - dX/ds}{1 + dX/ds}\right)^{1/2} (1 - \sin\alpha) = \frac{\gamma_F}{\pi\,\mu\,W^2}\,. \tag{5.2}$$

Equation (5.2) shows that the rate of crack propagation dX/ds is constant only for an incident wave defined by (5.1b).

If the shear stress is finite at the wave front the crack tip may still start to move, but only for $s_m > 0$. the value of s_m can be computed from (4.13) by considering s slightly less than s_m. At that instant $dX/ds = 0$, and we find

$$\left\{\int_0^{s_m} \frac{g(\zeta)\,d\zeta}{(s_m - \zeta)^{1/2}}\right\}^2 = \frac{\pi\,\gamma_F}{(1 - \sin\alpha)\,\mu}\,. \tag{5.3}$$

As an example we consider an incident step-stress wave,

$$g(s) = \frac{\tau_0}{\mu}\,. \tag{5.4}$$

Equation (5.3) then yields

$$s_m = \frac{\pi\,\mu\,\gamma_F}{4\,(1 - \sin\alpha)\,\tau_0^2}\,. \tag{5.5}$$

If the crack is of finite length a, the present analysis is valid for $s_m < 2\,a\,(1 + \sin\alpha)$. For $s > s_m$, the function $X\,(s - s_m)$ must be solved from the non-linear differential equation

$$\frac{4\,\tau_0^2}{\pi\,\mu}\,(1 - \sin\alpha)\,[s + X\,(s - s_m)\sin\alpha]\left(1 - \frac{dX}{ds}\right)^{1/2} = \left(1 + \frac{dX}{ds}\right)^{1/2} \gamma_F\,. \tag{5.6}$$

The equation can easily be solved for $\alpha = 0$, i.e. when the wave front is parallel to the plane of the crack. For $\alpha = 0$ we obtain for $s \geqslant s_m$

$$\frac{dX}{ds} = \frac{(4\,\tau_0^2)^2\,s^2 - (\pi\,\mu\,\gamma_F)^2}{(4\,\tau_0^2)^2\,s^2 + (\pi\,\mu\,\gamma_F)^2}\,. \tag{5.7}$$

It is noted that $dX/ds \to 1$ as $s \to \infty$.

Now let us assume that the material is already statically in a state of stress before the incident wave arrives. If a crack propagates into a statically disturbed medium the solution to the diffraction problem can be obtained by superimposing on the incident wave both the solution of the problem defined by (2.1), (2.7) and (2.8), and the solution of a problem defined by (2.1), (2.9) and the boundary conditions at $y = 0$:

$$0 < x < X\,(s - s_m): \frac{\partial w}{\partial y} = \frac{1}{\mu}\,[-\,\tau_{yz}(x, 0]_{st}\,, \tag{5.8}$$

$$x \geqslant X\,(s - s_m):\ w = 0\,. \tag{5.9}$$

To solve for this set of boundary conditions, the mathematical technique of Section 2 can be employed. In fact, the analysis of Section 2 carries through unchanged, except that $\varkappa = -1$ whenever $-\varkappa\,\xi$ appears as a lower limit in an integral. Similarly, the lower limit becomes zero in equations (3.8) and (3.10). By employing the static stress (5.8), the integral analogous to (4.7) becomes

$$I_{st} = \frac{1}{\mu} \int\limits_0^{X(s-s_m)} \frac{[-\,\tau_{yz}(u, 0)]_{st}}{[X\,(s - s_m) - u]^{1/2}}\,du\,. \tag{5.10}$$

For anti-plane shear, static fields around a crack have been investigated in great detail. It is well known that the shear stress just ahead of the crack tip is of the form

$$[\tau_{yz}(x, 0]_{st} = \frac{1}{2}\,\mu\,B\,x^{-1/2} + O(x^{1/2})\,, \tag{5.11}$$

where B is the intensity factor which depends on the static loading.

For the case that the transient wave which provides the dynamic disturbance to start crack extension is a step-stress wave of the form (5.4), the integral I, equation (4.10), becomes

$$I_w = \frac{2\,[s + X\,(s - s_m)\,\sin\alpha]^{1/2}}{(1 + \sin\alpha)^{1/2}} \left(\frac{\tau_0}{\mu}\right)^{1/2}. \tag{5.12}$$

The balance of energy equation then yields

$$\left[\frac{\mu}{\pi}\,(I_{st})^2 + \frac{2\,\mu\cos\alpha}{\pi}\,I_{st}\,I_w + \frac{\mu\cos^2\alpha}{\pi}\,(I_w)^2\right] \left(\frac{1 - dX/ds}{1 + dX/ds}\right)^{1/2} = \gamma_F\,, \tag{5.13}$$

where I_{st} and I_w are defined by (5.10) and (5.12), respectively. It can then be concluded from equation (5.13) that in a prestressed medium instantaneous crack propagation at $s = 0$ is possible. If we set $s_m = 0$ we find in view of (5.11) and (5.10)

$$[I_{st}]_{s=0} = -\frac{\pi}{2} B . \tag{5.14}$$

At $s = 0$, equation (5.13) then reduces to

$$\frac{1}{4} \mu \pi B^2 \left(1 - \frac{dX}{ds}\right)^{1/2} = \gamma_F \left(1 + \frac{dX}{ds}\right)^{1/2} . \tag{5.15}$$

The rate of crack extension at $s = 0$ can immediately be computed from (5.15). It is noted that dX/ds at $s = 0$ depends only on the static pre-stress, in particular on the intensity factor B, and it follows that instantaneous crack extension does not take place if $B^2 < 4 \gamma_F/\mu \pi$. If B is slightly less than $(4 \gamma_F/\mu \pi)^{1/2}$ the crack will, however, propagate almost instantaneously if the shear stress due to the incident wave is of the same sign as the static prestress. If the crack starts to move, the rate of extension dX/ds for $s > 0$ must be computed from a rather complicated non-linear differential equation which can be determined from (5.13).

In summary, it has been shown in this paper that a crack may be incited to extend by an incident horizontally polarized transient shear wave. As the wave front strikes the crack tip in an initially undisturbed medium, instantaneous crack propagation can occur only if the shear stress shows a square root singularity at the wave front. If the shear stress is continuous at the wave front, crack propagation may be initiated a short time after the crack tip has been struck. In a statically prestressed medium containing a crack, almost instantaneous crack propagation may occur depending on the magnitude of the stress intensity factor of the static prestress.

The analysis of this paper is based on the assumption that the material is homogeneous, isotropic and linearly elastic, and suffers brittle fracture. It should be noted that as the rate of crack extension increases, the maximum shear stress may occur in a plane other than the plane of the crack, which may give rise to branching. It is finally observed that the analysis as presented in this paper can easily be extended to the case where γ_F depends on dX/ds. All that has to be done is to modify (4.12) appropriately, which will change the right-hand side of equation (4.13).

Appendix

Equation (4.5), which was stated as

$$\frac{1}{\pi} \frac{H(v)}{v^{1/2}} \frac{H(-v)}{(-v)^{1/2}} = \delta(v) ,$$

can be proven by considering the left-hand side as the limit case for $k \to \infty$ of the function

$$s_k(v) = \begin{cases} 0 & \text{for} \quad |v| > \dfrac{1}{k} , \\[2ex] \dfrac{1}{\pi} \left(v + \dfrac{1}{k}\right)^{-1/2} \left(\dfrac{1}{k} - v\right)^{-1/2} & \text{for} \quad |v| < \dfrac{1}{k} . \end{cases}$$

It can then easily be shown that $\lim_{k\to\infty} s_k(v)$ satisfies the usual criteria for a delta function:

$$\text{(a)}\quad \lim_{k\to\infty} s_k(v) = \begin{cases} 0\,, & v \neq 0\,, \\ \infty\,, & v = 0\,, \end{cases}$$

$$\text{(b)}\quad \lim_{k\to\infty} \int_{-a}^{b} s_k(v)\,dv = 1\,,$$

$$\text{(c)}\quad \lim_{k\to\infty} \int_{-a}^{b} \varphi(v)\,s_k(v)\,dv = \varphi(0)\,,$$

where $\varphi(v)$ is integrable and continuous at $v = 0$.

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Zusammenfassung

Eine ebene, unstetige, horizontal polarisierte Schubwelle breitet sich in einem brüchigen elastischen Material aus, das einen Riss enthält. In dieser Arbeit werden die Bedingungen für den Rissfortschritt untersucht, nachdem die ankommende Welle durch den Riss gebeugt wurde. Die Untersuchung besteht aus zwei Teilen. Im ersten Teil werden die Teilchengeschwindigkeiten und Schubspannungen in der Ebene des Risses infolge Beugung der Schubwelle bestimmt. Es wird dabei angenommen, dass der Riss sich sofort, oder kurz nachdem die Wellenfront die Risskante getroffen hat, mit beliebiger Geschwindigkeit ausbreitet. Im zweiten Teil wird die Energiegleichung als Kriterium für den Rissfortschritt benützt. Es wird gezeigt, dass in einem ursprünglich ungestörten Material der Riss sich nur dann sofort ausbreitet, wenn die Schubspannung eine Quadratwurzelsingularität an der Wellenfront zeigt. Wenn die Schubspannung an der Wellenfront kontinuierlich ist, dann beginnt der Rissfortschritt, kurz nachdem die Risskante getroffen wird. In einem statisch vorgespannten Material, das einen Riss enthält, kann der Rissfortschritt sofort einsetzen, wenn der Spannungsfaktor gross genug ist.

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