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Zusammenfassung

Die vorliegende Notiz behandelt das Problem der Spannungskonzentration an einem kreisförmigen Loch in der Theorie der mikropolaren Elastizität. Es wird gezeigt, dass der Konzentrationsfaktor im Gegensatz zum klassischen Fall nicht konstant ist, sondern von drei dimensionslosen Parametern abhängt, welche mit den Materialkonstanten verknüpft sind. Bei passender Wahl dieses Parameters stellt sich heraus, dass der Konzentrationsfaktor um wenig vom klassischen Wert abweicht.

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Note on Wave Propagation in Linearly Viscoelastic Media

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Introduction

The study of propagating surfaces of discontinuity in continuous media was apparently initiated at the end of the last century. The main reference from that period is the classical treatise by HADAMARD [1][1]). In recent years the subject has been given renewed attention by COLEMAN et al. [2, 3] in two valuable papers on the propagation of wave fronts in non-linear viscoelastic solids. In the latter papers the attention was restricted, however, to the velocity of the propagating waves, and to the growth or decay of a propagating discontinuity at the wave front.

In the present note we use the analytical tools of the theory of propagating surfaces of discontinuity to determine solutions that are also valid after the wave front has passed. To show the method in its most elegant simplicity we limit ourselves to one-dimensional wave propagation, small deformations and linear material behavior. The present method may, however, prove to be useful for non-linear wave propagation problems.

The viscoelastic material that is considered here satisfies the most general linear stress-strain relation. The field quantities, such as stresses and particle velocities, are treated as functions of time, while the spatial coordinate is treated as a parameter. At an arbitrary location the solutions are obtained in the form of a Taylor expansion about the time of arrival of the wave front. The coefficients of the expansion, which depend on the parameter x, are obtained with a minimum of effort as solutions of linear first order ordinary differential equations.

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[1]) Numbers in brackets refer to References, page 144.

Analysis

We consider a thin viscoelastic rod ($0 \leqslant x < \infty$), which is assumed to be initially undisturbed. At time $t = 0$ a time-dependent stress is applied at $x = 0$. For infinitesimal strains and linear material behavior the ensuing wave motion in the rod is governed by the following two equations

$$\frac{\partial \sigma}{\partial x} = \varrho \frac{\partial^2 u}{\partial t^2} \tag{1}$$

$$\frac{\partial u}{\partial x} = J_0\, \sigma(x, t) + \int_{0^+}^{t} J^1 (t - s)\, \sigma(x, s)\, ds \tag{2}$$

where $\sigma(x, t)$ and $u(x, t)$ are the uniaxial stress and the uniaxial displacement, respectively, and ϱ is the mass-density. In (2) $J(t)$ is the uniaxial creep function; a superscript denotes differentiation with respect to the argument. We also define

$$J_0 = J(0) \quad \text{and} \quad J_0^n = \left.\frac{d^n J(s)}{ds^n}\right|_{s=0}. \tag{3}$$

In a viscoelastic material disturbances will propagate with a finite velocity if the material shows initial elastic behavior. For a linear viscoelastic material the propagation velocity depends, moreover, only on the modulus that governs the initial elastic response. From (1) and (2) the wave velocity is easily obtained as

$$c = \left(\frac{1}{\varrho J_0}\right)^{1/2}. \tag{4}$$

At a fixed position x along the rod, the material is at rest until a disturbance arrives at time $t = x/c$. In this note we propose to seek solutions for the field variables at position x in the form of a Taylor series about the time of arrival of the disturbance. Thus for fixed x and for $t \geqslant x/c$:

$$\sigma(x, t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(t - \frac{x}{c}\right)^n \left.\frac{\partial^n \sigma}{\partial t^n}\right|_{t=x/c}. \tag{5}$$

For an initially undisturbed rod the derivatives $\partial^n \sigma(x, t)/\partial t^n$ are identically zero before the wavefront arrives, and they have finite values after the wave front has passed. These derivatives, or at least many of them, are, therefore, propagating discontinuities. By using a familiar notation for discontinuities (5) can then be rewritten as

$$\sigma(x, t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(t - \frac{x}{c}\right)^n \left[\frac{\partial^n \sigma}{\partial t^n}\right]. \tag{6}$$

It is shown in the sequel that the discontinuities in the time-derivatives of the stress can be obtained directly as the solutions of inhomogeneous, ordinary differential equations of the first order.

Consider a function $f(x, t)$ which is discontinuous and has discontinuous derivatives across a plane that moves with velocity c. Basic to the study of the magnitudes of propagating discontinuities in one dimension is the kinematical condition of compatibility [4]

$$\frac{d}{dt} [f] = \left[\frac{\partial f}{\partial t}\right] + c \left[\frac{\partial f}{\partial x}\right]. \tag{7}$$

The general rule (7) may now be applied to the derivatives $\partial^n \sigma/\partial t^n$ and $\partial^{n+1} u/\partial t^{n+1}$. We obtain

$$\frac{d}{dt} \left[\frac{\partial^n \sigma}{\partial t^n}\right] = \left[\frac{\partial^{n+1} \sigma}{\partial t^{n+1}}\right] + c \left[\frac{\partial^{n+1} \sigma}{\partial x\, \partial t^n}\right] \tag{8}$$

$$\frac{d}{dt} \left[\frac{\partial^{n+1} u}{\partial t^{n+1}}\right] = \left[\frac{\partial^{n+2} u}{\partial t^{n+2}}\right] + c \left[\frac{\partial^{n+2} u}{\partial x\, \partial t^{n+1}}\right] \tag{9}$$

where c is defined by (4).

Let us differentiate the constitutive Equation (2) $n + 1$ times with respect to t:

$$\frac{\partial^{n+2} u}{\partial x\, \partial t^{n+1}} = J_0 \frac{\partial^{n+1} \sigma}{\partial t^{n+1}} + \sum_{i=1}^{n+1} J_0^i \frac{\partial^{n+1-i} \sigma}{\partial t^{n+1-i}} + \int_{0^+}^{t} J^{n+2} (t - s)\, \sigma(x, s)\, ds\,. \tag{10}$$

Since the integral is continuous at the wave front (10) yields the following relation between finite discontinuities

$$\left[\frac{\partial^{n+2} u}{\partial x\, \partial t^{n+1}}\right] = J_0 \left[\frac{\partial^{n+1} \sigma}{\partial t^{n+1}}\right] + \sum_{i=1}^{n+1} J_0^i \left[\frac{\partial^{n+1-i} \sigma}{\partial t^{n+1-i}}\right]. \tag{11}$$

After differentiating the equation of motion (1) n times with respect to t, we obtain for the finite discontinuities

$$\left[\frac{\partial^{n+1} \sigma}{\partial x\, \partial t^{n}}\right] = \varrho \left[\frac{\partial^{n+2} u}{\partial t^{n+2}}\right]. \tag{12}$$

We now form the sum $(1/\varrho)$ (12) $+ c$ (11), where c is defined by (4). By employing the relations (8) and (9) this sum reduces to

$$c\, \varrho \frac{d}{dt} \left[\frac{\partial^{n+1} u}{\partial t^{n+1}}\right] = \frac{d}{dt} \left[\frac{\partial^{n} \sigma}{\partial t^{n}}\right] + 2 \sum_{i=1}^{n+1} \alpha_i \left[\frac{\partial^{n+1-i} \sigma}{\partial t^{n+1-i}}\right] \tag{13}$$

where

$$\alpha_i = \frac{J_0^i}{2\, J_0}\,. \tag{14}$$

By employing the equation of motion (1) and the kinematical condition of compatibility (8) we can write for $n \geqslant 1$:

$$\frac{d}{dt} \left[\frac{\partial^{n-1} \sigma}{\partial t^{n-1}}\right] = \left[\frac{\partial^{n} \sigma}{\partial t^{n}}\right] + \varrho\, c \left[\frac{\partial^{n+1} u}{\partial t^{n+1}}\right]. \tag{15}$$

The first term of (13) can now be eliminated with (15), and, for $n \geqslant 1$, we obtain the following inhomogeneous ordinary differential equation for $[\partial^n \sigma / \partial t^n]$:

$$\frac{d}{dt} \left[\frac{\partial^{n} \sigma}{\partial t^{n}}\right] + \alpha_1 \left[\frac{\partial^{n} \sigma}{\partial t^{n}}\right] = F(t) \tag{16}$$

where

$$F(t) = \frac{1}{2} \frac{d^2}{dt^2} \left[\frac{\partial^{n-1} \sigma}{\partial t^{n-1}}\right] - \sum_{i=1}^{n} \alpha_{i+1} \left[\frac{\partial^{n-i} \sigma}{\partial t^{n-i}}\right]. \tag{17}$$

For $n = 0$ we employ the well-known kinematic relation

$$[\sigma] = - c\, \varrho \left[\frac{\partial u}{\partial t}\right]. \tag{18}$$

Substitution of (18) into (13) yields

$$\frac{d}{dt} [\sigma] + \alpha_1 [\sigma] = 0\,. \tag{19}$$

The solutions of (16) and (19) are determined subject to initial conditions that depend on the stress at $x = 0$. Suppose that the stress at $x = 0$ can be expanded in a MacLaurin series

$$\sigma(0, t) = \sum_{n=0}^{\infty} \sigma_n \frac{t^n}{n!}\,. \tag{20}$$

The solution of (19) is then obtained as

$$[\sigma] = \sigma_0\, e^{-\alpha_1 t}\,. \tag{21}$$

For $n \geqslant 1$ the solution of (16) is:

$$\left[\frac{\partial^{n} \sigma}{\partial t^{n}}\right] = e^{-\alpha_1 t} \int_0^t F(s)\, e^{\alpha_1 s}\, ds + \sigma_n\, e^{-\alpha_1 t}\,. \tag{22}$$

For an arbitrary position x the coefficients in the expansion (6) are obtained by replacing t by x/c in (21) and (22). The first three coefficients are

$$[\sigma] = \sigma_0 \, e^{-\alpha_1 x/c} \tag{23}$$

$$\left[\frac{\partial \sigma}{\partial t}\right] = \left\{\left(\frac{1}{2}\,\alpha_1^2 - \alpha_2\right)\left(\frac{x}{c}\right)\sigma_0 + \sigma_1\right\} e^{-\alpha_1 x/c} \tag{24}$$

$$\left.\begin{aligned}\left[\frac{\partial^2 \sigma}{\partial t^2}\right] = \Bigg\{\frac{1}{2}\left(\frac{1}{2}\,\alpha_1^2 - \alpha_2\right)^2 \left(\frac{x}{c}\right)^2 \sigma_0 - \left(\frac{1}{2}\,\alpha_1^3 - \alpha_1\,\alpha_2 + \alpha_3\right)\left(\frac{x}{c}\right)\sigma_0 \\ + \left(\frac{1}{2}\,\alpha_1^2 - \alpha_2\right)\left(\frac{x}{c}\right)\sigma_1 + \sigma_2\Bigg\} e^{-\alpha_1 x/c} \,.\end{aligned}\right\} \tag{25}$$

It should be pointed out that for a linear problem the expansion (6) can be obtained by means of the Laplace transform technique. The Laplace transform of the stress is easily obtained as

$$\bar{\sigma}(x, p) = \bar{\sigma}(0, p) \exp\{-\,(\varrho\, p^3\, \overline{J(p)})^{1/2}\, x\}\,. \tag{26}$$

A binomial expansion of the exponent, combined with an expansion of the exponential and term by term inversion, yields the same result as is obtained in a less cumbersome manner in this note. The first term of the expansion was obtained in that way by CHU [5].

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Zusammenfassung

Die analytischen Mittel der Theorie der sich fortpflanzenden Diskontinuitätsflächen werden verwendet, um die vollständige Lösung eines eindimensionalen linearen Problems der Wellenfortpflanzung aufzustellen. Diese Lösung wird erhalten als eine Entwicklung in eine Taylor-Reihe um die Zeit des Antreffens der Wellenfront. Die Entwicklungskoeffizienten sind Lösungen von gewöhnlichen Differentialgleichungen erster Ordnung.

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