

4. Any theory of elasticity must proceed from a relation between the tensors of stress and relative deformation or strain. The tensor of stress can have the covariant or contravariant form, and expressions in these forms are therefore required also for the tensor of strain. Usually the dimensions of the body are given in the initial state, and the strain tensor must be expressed in such state. However, the initial state is not one of equilibrium; this is the property of the final state. In some problems the necessity then arises of expressing the strain tensor in the final state. These conditions make it desirable to make use of the eight expressions listed in the present paper.

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## Phases of Elastic Materials

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### Introduction

The nature of a simple material may be, in large part, described by its symmetry group<sup>1)</sup>, which is the set of all transformations of reference configuration which leave invariant the response of the material to all deformation-temperature histories. Clearly this description is no longer adequate in the case of materials which suffer basic structural changes due to deformation and temperature, changes which arise, for example, in the case of phase transitions. Thus it is implicit in most treatments of the subject that constitutive equations are to be taken as defined for a single phase of the material. More in keeping with the general attitude of modern continuum mechanics, however, is to regard the constitutive functional as fixed and to consider changes of the material under deformation to be reflected in changes in symmetry. This is a rather simple concept, but one which we believe may be useful in describing material response. We here demonstrate its application to the simplest non-trivial case: an elastic material for which the phase is a function only of the density and temperature. We begin by introducing a group, called the phase symmetry group, which we allow to be a function of the density  $\varrho$  and temperature<sup>2)</sup>  $\theta$ . For a given  $\varrho$  and  $\theta$  this group is the set of all density preserving transformations of the reference configuration which are indistinguishable by an experiment in which the density and the temperature after the deformation are  $\varrho$  and  $\theta$  respectively. Using this group we are able to give physically meaningful definitions of such notions as solid phase, liquid phase, mesomorphic phase and crystalline phase. Further we show that most of the results known previously for the symmetry group have immediate counterparts for the phase symmetry group.

<sup>1)</sup> We prefer this term to "isotropy group".

<sup>2)</sup> Wang and Bowen [1966] in considering the "instantaneous isotropy group" of a quasi-elastic material allow the group to depend upon the present temperature.

### Elastic Materials. The Symmetry Group $\mathfrak{G}_M$

An elastic material is described by a constitutive equation relating the stress tensor  $S$  to the deformation gradient  $F$  and temperature  $\theta$ :

$$S = \hat{S}_M(F, \theta).$$

Here  $M$  is the (local) reference configuration relative to which  $F$  is taken<sup>3)</sup>. The function  $\hat{S}_M$  is called the response function of the material relative to  $M$ . We assume that  $\hat{S}_M(F, \theta)$  is defined for every non-singular tensor  $F$ , every positive scalar  $\theta$  and every configuration  $M$ , and is objective in the sense that

$$Q \hat{S}_M(F, \theta) Q^T = \hat{S}_M(Q F, \theta)$$

for every orthogonal tensor  $Q$ .

The response functions  $\hat{S}_M$  and  $\hat{S}_N$  relative to two different configurations  $M$  and  $N$  are not independent; they are related by

$$\hat{S}_M(F, \theta) = \hat{S}_N(F G^{-1}, \theta),$$

where  $G$  is the deformation gradient from  $M$  to  $N$ , i.e.,  $N = GM$ .

The *symmetry group*  $\mathfrak{G}_M$  of the material relative to the configuration  $M$  is the group of all non-singular tensors  $H$  for which the identity

$$\hat{S}_M(F, \theta) = \hat{S}_M(F H, \theta)$$

holds for all  $F$  and  $\theta$ . Thus  $H$  belongs to  $\mathfrak{G}_M$  if and only if the response from the configuration  $N = HM$  is the same as that from  $M$ , i.e.,

$$\hat{S}_N(F, \theta) = \hat{S}_M(F, \theta)$$

for all  $F$  and  $\theta$ . Also, if  $M$  and  $N$  are any two configurations and if  $G$  is the deformation gradient from  $M$  to  $N$ , then

$$\mathfrak{G}_N = G \mathfrak{G}_M G^{-1}.$$

The only restriction<sup>4)</sup> we place on the group  $\mathfrak{G}_M$  is that it be contained in the unimodular group  $\mathfrak{U}$ . The material is called a *solid* if for some  $M$ ,  $\mathfrak{G}_M$  is a subgroup of the orthogonal group  $\mathfrak{O}$ ; a *fluid* if for some (and hence every)  $M$ ,  $\mathfrak{G}_M = \mathfrak{U}$ <sup>5)</sup>.

### The Phase Symmetry Group $\mathfrak{P}_M(\varrho, \theta)$

Let  $\varrho > 0$  and  $\theta > 0$  be given values of the density and temperature. The *phase symmetry group*  $\mathfrak{P}_M(\varrho, \theta)$  at  $(\varrho, \theta)$  relative to a configuration  $M$  is the group of all unimodular tensors  $H$  for which the identity

$$\hat{S}_M(F, \theta) = \hat{S}_M(F H, \theta)$$

holds whenever the configuration  $FM$  has density  $\varrho$ , i.e., whenever  $\varrho |\det F| = \varrho_M$ , where  $\varrho_M$  is the density in  $M$ . Thus a unimodular tensor  $H$  belongs to  $\mathfrak{P}_M(\varrho, \theta)$  if and only if

$$\hat{S}_N(F, \theta) = \hat{S}_M(F, \theta)$$

whenever  $N = HM$  and  $\varrho |\det F| = \varrho_M (= \varrho_N)$ . Clearly the mapping  $(\varrho, \theta) \rightarrow \mathfrak{P}_M(\varrho, \theta)$  generates the symmetry group  $\mathfrak{G}_M$  by means of the following relation:

$$\mathfrak{G}_M = \bigcap_{\substack{\text{all} \\ \varrho, \theta}} \mathfrak{P}_M(\varrho, \theta);$$

<sup>3)</sup> For precise definitions of the concepts used in this section see Truesdell and Noll [1965].

<sup>4)</sup> Conditions under which this restriction is necessary are discussed by Gurtin and Williams [1966].

<sup>5)</sup> This classification of materials is due to Noll [1958]. See also Coleman and Noll [1964] and Truesdell and Noll [1965].

i.e.,  $H$  belongs to  $\mathfrak{G}_M$  if and only if  $H$  belongs to  $\mathfrak{P}_M(\varrho, \theta)$  for every  $\varrho$  and  $\theta$ . If  $M$  and  $N$  are any two configurations and  $G$  the deformation gradient from  $M$  to  $N$ , then

$$\mathfrak{P}_N(\varrho, \theta) = G \mathfrak{P}_M(\varrho, \theta) G^{-1}.$$

We now utilize the group  $\mathfrak{P}_M(\varrho, \theta)$  to define as follows the various phases of the material.

(i) The material is in a *solid phase* at  $(\varrho, \theta)$  if for some  $M$ ,  $\mathfrak{P}_M(\varrho, \theta)$  is a subgroup of  $\mathfrak{D}$ ; in this case the configuration  $M$  is called *undistorted*.

(ii) The material is in the *fluid phase* at  $(\varrho, \theta)$  if for some (and hence every)  $M$ ,  $\mathfrak{P}_M(\varrho, \theta) = \mathfrak{U}$ .

(iii)<sup>6)</sup> The material is in a *mesomorphic* (or *liquid crystal*) phase at  $(\varrho, \theta)$  if it is neither in a solid phase nor in the fluid phase at  $(\varrho, \theta)$ .

(iv) The material is in an *isotropic phase* at  $(\varrho, \theta)$  if for some  $M$ ,  $\mathfrak{D}$  is a subgroup of  $\mathfrak{P}_M(\varrho, \theta)$ ; in this case  $M$  is called *undistorted*.

Trivially each fluid phase is isotropic. We now list other consequences of the above definitions, omitting the proofs of those results which are direct counterparts of known theorems.

( $\alpha$ )<sup>7)</sup> Every isotropic phase is either a fluid phase or a solid phase.

( $\beta$ )<sup>8)</sup> An orthogonal tensor  $Q$  belongs to  $\mathfrak{P}_M(\varrho, \theta)$  if and only if

$$Q \hat{S}_M(F, \theta) Q^T = \hat{S}_M(Q F Q^T, \theta)$$

for every  $F$  with  $\varrho | \det F | = \varrho_M$ .

( $\delta$ )<sup>9)</sup> If the material is in an isotropic phase at  $(\varrho, \theta)$ , then the residual stress at the temperature  $\theta$  in any undistorted local configuration with density  $\varrho$  is a hydrostatic pressure. (The residual stress in a local configuration  $M$  is  $\hat{S}_M(I, \theta)$ .)

( $\epsilon$ )<sup>10)</sup> If the material is in an isotropic phase at  $(\varrho, \theta)$ , then for every  $F$  satisfying  $\varrho | \det F | = \varrho_M$  we have

$$\hat{S}_M(F, \theta) = -p I + \mu B + \nu B^2,$$

where  $B = FF^T$ , and  $p = p(B, \theta)$ ,  $\mu = \mu(B, \theta)$  and  $\nu = \nu(B, \theta)$  are scalar functions of the principal invariants of  $B$ . If the material is in the fluid phase  $\mu = \nu = 0$  and  $p = p(\varrho, \theta)$ .

( $\varphi$ )<sup>11)</sup> Any two undistorted configurations of an isotropic phase are connected by a deformation gradient of the form  $\alpha Q$ , where  $Q$  is orthogonal, and every such deformation gradient carries an undistorted configuration into another.

( $\gamma$ )<sup>12)</sup> If the material is in the solid phase at  $(\varrho, \theta)$  and  $M$  and  $N$  are undistorted configurations, then

$$\mathfrak{P}_N(\varrho, \theta) = Q \mathfrak{P}_M(\varrho, \theta) Q^T,$$

where  $Q$  is the orthogonal tensor corresponding to the polar decomposition of the deformation gradient from  $M$  to  $N$ . Thus  $\mathfrak{P}_N(\varrho, \theta)$  and  $\mathfrak{P}_M(\varrho, \theta)$  are conjugate within the orthogonal group.

A solid phase is called *aeolotropic*, or *anisotropic*, if the corresponding phase symmetry group, relative to some (and hence every) undistorted configuration, is a *proper* subgroup of the orthogonal group  $\mathfrak{D}$ . By ( $\gamma$ ) the intrinsic symmetry of an aeolotropic solid phase is described by an equivalence class of conjugate proper subgroups of  $\mathfrak{D}$ . Eleven such equivalence classes correspond to the crystal classes<sup>13)</sup> and may be used to define the various possible *crystalline phases*. Another type of aeolotropic phase is defined by equivalence to a group of the form  $\{Q \in \mathfrak{D} | Q \mathbf{v} = \mathbf{v} \text{ or } Q \mathbf{v} = -\mathbf{v}\}$  where  $\mathbf{v}$  is some unit vector; such a phase is called *transversely isotropic*.

<sup>6)</sup> Cf. Coleman [1965], Wang [1965].

<sup>7)</sup> Cf. Truesdell and Noll [1965], p. 84.

<sup>8)</sup> Cf. Noll [1958], Theorem 5; Coleman and Noll [1964], Proposition 4.

<sup>9)</sup> Cf. Coleman and Noll [1964], Proposition 6.

<sup>10)</sup> Cf. Coleman and Noll [1964], Proposition 7; Truesdell and Noll [1965], p. 140.

<sup>11)</sup> Cf. Coleman and Noll [1964], Proposition 8.

<sup>12)</sup> Cf. Coleman and Noll [1964], pp. 97–98.

<sup>13)</sup> See, for example, Truesdell and Noll [1965], p. 83.

The next two results demonstrate the connection between the notion of a material being a solid or fluid and that of being in the solid or fluid phase.

(i) If the material is a fluid then every phase is a fluid phase.

(ii) If the material is an isotropic solid then every phase is either a fluid phase or an isotropic solid phase.

The proof of (i) is immediate: since  $\mathfrak{G}_M \subset \mathfrak{P}_M(\rho, \theta) \subset \mathfrak{U}$ ,  $\mathfrak{G}_M = \mathfrak{U}$  implies  $\mathfrak{P}_M(\rho, \theta) = \mathfrak{U}$ . To establish (ii) we assume  $\mathfrak{G}_M = \mathfrak{D}$  for some  $M$ . Then  $\mathfrak{D} = \mathfrak{G}_M \subset \mathfrak{P}_M(\rho, \theta) \subset \mathfrak{U}$  which implies  $\mathfrak{P}_M(\rho, \theta) = \mathfrak{D}$  or  $\mathfrak{P}_M(\rho, \theta) = \mathfrak{U}$  for every  $(\rho, \theta)$ , since the orthogonal group is maximal in the unimodular group<sup>14</sup>).

*Remark;* VARLEY and DAY<sup>15</sup>) consider phase transformations in a different manner. Given an isotropic elastic solid they consider deformations of the material which can occur at constant temperature and pressure. They find that if for a given hydrostatic pressure and temperature the Gibbs free energy function has a (possibly constrained) stationary point then there exist deformations (corresponding to that stationary value) other than a uniform dilatation which may occur at this pressure and temperature. They then note that the symmetry group of the material relative to the second configuration is not the full orthogonal group, i.e., that this configuration is not undistorted; they call such a configuration a non-isotropic phase of the material.

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### Zusammenfassung

Es wird eine Methode beschrieben für die Charakterisierung von Stoff-Phasen durch Angabe der Symmetrie-Gruppe als Funktion der Deformation und Temperatur. Wenn in einem elastischen Stoff die Phase durch Dichte und Temperatur bestimmt wird, dann gelten die meisten der üblichen Resultate für die Gesamt-Symmetrie-Gruppe auch für die Gruppe, die einer gegebenen Phase entspricht.

<sup>14</sup>) Brauer [1965], Noll [1965].

<sup>15</sup>) Varley and Day [1966].