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Torque on a Sphere Inside a Rotating Cylinder

By Baltasar Mena, Dept. of Applied Mathematics, University College of Wales, Aberystwyth, Great Britain, Ernest Levinson and Bruce Caswell, Div. of Engineering, Brown University, Providence R.I., USA.

1. Introduction

The problem of the steady rotation of a sphere in an infinite expanse of fluid has received considerable attention for Newtonian and non-Newtonian liquids [1–4]. The use of this theory has been successful in the design of experimental viscometers [5]. The effect of boundary proximity upon the flow has been treated by several authors [6–10], and in particular by Brenner [10], who has examined theoretically the slow rotation of an axisymmetric body rotating symmetrically about the axis of a circular cylinder filled to a finite depth with viscous fluid. In the present paper an experimental arrangement is described in which a circular cylinder of finite dimensions is made to rotate around a sphere fixed in the centre of the cylinder. The couple on the sphere is measured over a wide range of rotational speeds for both Newtonian and non-Newtonian fluids. For the Newtonian liquids a comparison of the experimental results is made with Collins' [4] expansion of the couple as a series in even powers of the angular Reynolds number. The effect of the boundaries upon the torque is estimated following a procedure very close to the one suggested by Brenner [10] for low Reynolds numbers. The region of validity appears to be extended by correcting the angular velocity instead of the torque. The shape of the streamlines in the secondary flow is made visible.

For non-Newtonian liquids the apparatus proves to be extremely useful for an accurate determination of the zero shear rate viscosity using only a small amount of fluid.

2. Theory

The couple T_∞ required to maintain the steady rotation of a sphere of radius a with an angular velocity Ω in an infinite expanse of fluid of viscosity μ is:

$$T_\infty = 8\pi\mu a^3 \Omega. \tag{2.1}$$

If the sphere rotates about an axis of symmetry in the proximity of a boundary, the relation between the couple necessary to maintain the steady rotation in an unbounded media (T_∞) and the effective couple in the bounded media (T) is of the form [2]:

$$\frac{T_\infty}{T}=1-K\frac{T_\infty}{8\pi\mu l^3\Omega}+O\left(\frac{a}{l}\right)^5 \tag{2.2}$$

where l is a characteristic distance from the sphere to the boundary, a is the sphere radius and K is a dimensionless constant of order unity, which depends only on the container geometry.

Equation (2.2) may be inverted in order to correct the angular velocity rather than the couple[1]), i.e.:

$$\Omega_\infty=\Omega+K\frac{T}{8\pi\mu l^3}+O(\mathrm{Re}^3\, l^{-5}\ln l) \tag{2.3}$$

where Ω_∞ is the angular velocity corresponding to a torque T in an unbounded media and Ω is the angular velocity corresponding to a torque T_∞ in the bounded media.

Brenner [10] solved the problem of a sphere rotating about the longitudinal axis of a vertical cylinder of radius R_0 filled to a depth h with viscous fluid. Replacing the sphere by a point couple and using the method of reflections he obtained:

$$\frac{T_\infty}{T}=1-K\left(\frac{b_1}{h},\frac{h}{R_0}\right)\frac{T_\infty}{8\pi\mu R_0^3\Omega}+O\left[\left(\frac{a}{b_1}\right)^5,\left(\frac{a}{b_2}\right)^5,\left(\frac{a}{R_0}\right)^5\right] \tag{2.4}$$

b_1, and b_2 are, respectively, distances measured from the sphere centre to the free surface and to the bottom of the cylinder. The function $K\left(\frac{b_1}{h},\frac{h}{R_0}\right)$ is conveniently tabulated in [10].

Following these results, Mena [11] considered the case when the cylinder is closed at both ends. When the appropriate limits of Brenner's solution were taken the value of the container constant K was obtained for the experimental arrangement to be described below. With this value, (2.3) is now:

$$\Omega_\infty=\Omega+0.8062\frac{T}{8\pi\mu R_0^3}+O[\mathrm{Re}^3 R_0^{-5}\ln R_0]. \tag{2.5}$$

[1]) It can be shown that the velocity field in the unbounded case may be expressed for finite Reynolds numbers and for non-Newtonian stresses as an equation of the form:

$$V=-\frac{T\times r}{8\pi\mu_0 r^3}+O(r^{-2})$$

where μ_0 is the zero shear viscosity and r is the position vector.
The initial interaction with the container walls comes from the term of lowest order in r. However such term satisfies the linear Stokes equations; therefore the proof that (2.3) is valid for finite Reynolds numbers and for non-Newtonian stresses is based on the above properties (Caswell, private communication).

3. Experimental Apparatus

Figure 1 is a schematic description of the experimental arrangement. The facilities consist of a glass cylinder of 7.6 cm in diameter and 15.2 cm in height. The cylinder is supported from the bottom by a flat stainless steel plate. An aluminium plate is located at the top with a removable cover to permit the entrance of

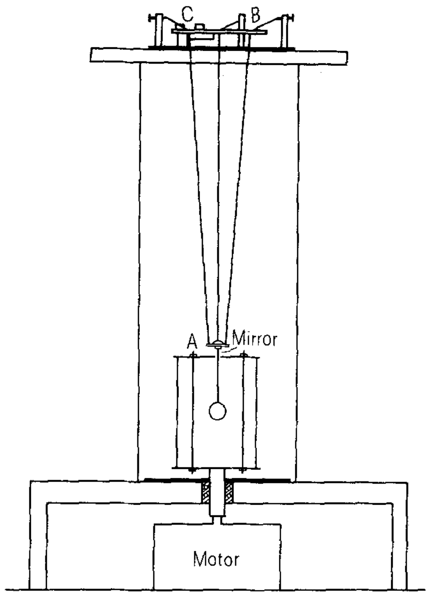


Figure 1
Schematic Description of the experimental apparatus.

the sphere. The plates are bolted together; the structure is supported by a small bearing assembly, and connected to a synchronous motor by means of a suitable gear box. The rotational speed may be varied from 0.0037 sec^{-1} to 18.8 sec^{-1} with twelve different intermediate positions, and rotates in both clockwise and anti-clockwise directions. The sphere is totally immersed in this rotating cylindrical bath and it is supported by a shaft secured to Plate A by a set screw. Three very thin wires, symmetrically located, join Plate A with Plate B which is fixed on a microscope slide and is axially aligned with the bottom plate. Plate A has a level which can be adjusted by means of a screw system C to secure the vertical plane. The distance between both plates may be varied.

An outer plexiglass cylinder houses a temperature controlled bath for the viscoelastic experiments. The variation in temperature is restricted to ± 0.1 °C.

The system employed to measure the couple is similar to that used in ballistic galvanometers. The torque induced by rotation produces an angular displacement of the sphere which is measured optically using a mirror and a telescope. Since the direction of rotation is reversible the angular displacement is amplified four times.

From static and trigonometric considerations, the torque is given by:

$$T=\frac{W r_1 r_2 \sin\theta}{[h^2-2r_1 r_2(1-\cos\theta)]^{\frac{1}{2}}}$$

where W is the net vertical force on the sphere (weight minus buoyancy force), r_1 and r_2 are the radial positions of the chords on Plates A and B respectively, h is the distance between the plates and θ is the observed deflection angle. This measured torque was corrected for two effects, the couple induced by the shaft and the couple due to the supporting wires. The former was estimated as the couple on a circular cylinder rotating with constant angular velocity Ω_∞, i.e.:

$$T=4\pi\mu a_s^2 l \Omega_\infty$$

where a_s is the shaft radius and l the wetted length. The second correction was due to the torsional resistance of the supporting wires; this resistance was determined from torsional pendulum frequency measurements [12].

4. Experiments

a) *Newtonian Fluids*

Silicone oils (Dow Corning 200) with viscosities of 0.15, 0.25, 0.498 and 1.85 poise, and Cetane (0.0304 poise) were used.

b) *Non-Newtonian Fluids*

Dilute solutions (0.05%, 0.15% and 5%) Polyisobutylene in Cetane. Densities were measured with calibrated pycnometers and viscosity measurements were verified using a Couette viscometer, a rotoviscometer and a falling-sphere viscometer, all temperature-controlled.

c) *Spheres*

Nylon spheres with radii 1,27 cm, 1.905 cm and 2.255 cm, tested for sphericity and density were used for all Newtonian and more dilute solutions and a brass sphere of 1.746 cm radius was employed for the 5% solution and for the more viscous oils.

Stainless steel shafts of 0.079 cm and 0.158 cm radii were used to examine the influence upon the viscosity measurements. In our range of values for the viscosity and angular velocity the torque due to the shaft never exceeded 2% of the total torque (this last figure being for the 5% solution, 118 poise) but other experimentalists are advised to consider shaft effects particularly for higher values of the Reynolds number, since it may not only affect the torque but it may also cause a lift force directed along the symmetry axis due to the lack of fore-and-aft symmetry [8].

5. Experimental Results

a) *Newtonian Liquids*

For low values of the angular velocity, i.e. $\Omega < 1$ no correction for wall effects was necessary and the data for all spheres and fluids obeyed Stokes formula with no observed scattering. However, for increasing velocity, the effect of the boundaries

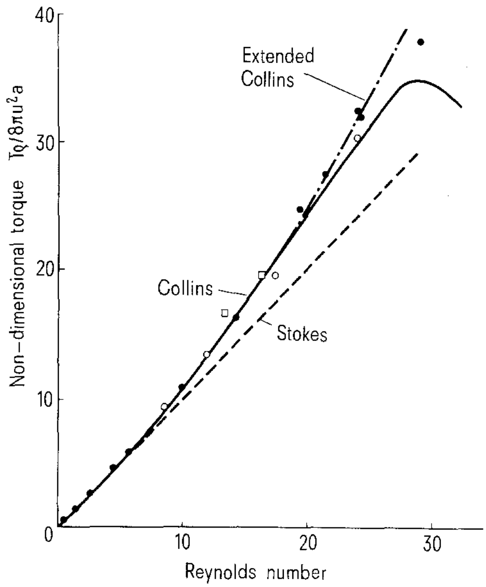


Figure 2
Experimental values of the torque for low Reynolds numbers. ● Newtonian; □ 0.05% solution; ○ 0.15% solution.

upon the torque became appreciable. Figure 2 shows a comparison between the experimental values and Collins' [4] expansion for the couple as a series in even powers of the angular Reynolds number

$$T = 8\pi\,\mu\,a^3\,\Omega_\infty\left[1 + \frac{\mathrm{Re}^2}{1200} - 7.542 \times 10^{-7}\,\mathrm{Re}^4 + O(\mathrm{Re})^6\right] \tag{5.1}$$

where

$$\mathrm{Re} = \frac{\Omega_\infty\,a^2\,\rho}{\mu}. \tag{5.2}$$

In the region of validity of this expansion ($\mathrm{Re} < 20$) excellent agreement was found with experiments; scattering in all cases being less than 2%.

Due to the availability of data beyond the range of Collins' expansion, an extension formula was obtained using a non-linear transformation of the type suggested by Shanks [13]. Under certain conditions this accelerates the convergence (or divergence) of a slowly convergent (or divergent) series which is of the 'nearly geometric' type. The 'improved' equation is:

$$\frac{T\rho}{8\pi\mu^2 a} = \mathrm{Re}\left[1 + \frac{\dfrac{\mathrm{Re}^2}{1200}}{1 + \dfrac{\mathrm{Re}^2}{1200}(1.086) + O(\mathrm{Re})^6}\right]. \tag{5.3}$$

Although (5.3) does not attempt to explain analytically the behaviour of the couple for higher Reynolds number, i.e., where inertial effects should be considered, it does illustrate a region where these effects are negligible and the perturbation theory for low Reynolds numbers is still applicable even for finite Reynolds numbers and for a rotating frame of reference. That this is true in the case of a sphere in an infinite expanse of fluid has been shown by Caswell (unpublished). However for a rotating frame of reference the proof is not simple. Figures 2 and 3 show a plot of the dimensionless torque versus the Reynolds number illustrating the region for (5.3) and the behaviour of the experimental points in the region where inertial effects are undoubtedly important (Re > 50).

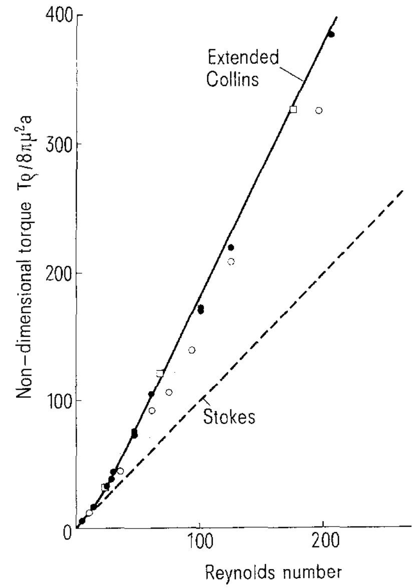
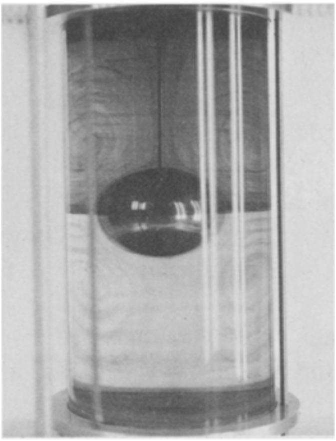


Figure 3
Experimental values of the torque for intermediate Reynolds numbers. ● Newtonian; □ 0.05 % solution; o 0.15 % solution.

b) *Non-Newtonian Liquids*

The use of a rotating sphere to determine the zero shear rate viscosity of non-Newtonian solutions is not new; Walters and Savins [5] designed a viscometer consisting of a sphere rotating in a rather large expanse of elastico-viscous fluid and were able to obtain very good qualitative as well as quantitative results. The limiting viscosity was determined by couple measurements and a parameter relating the second order normal stress differences to rate of shear was estimated by ob-


A

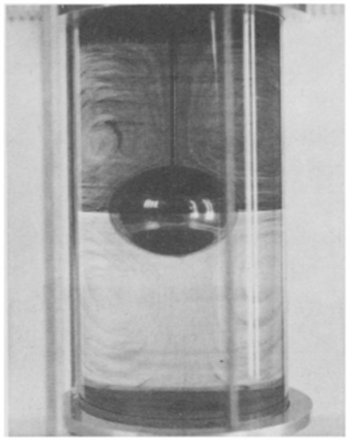

B

Plate I
Secondary flow around a sphere at low Reynolds number A (Re=5); B (Re=2).

serving the streamline patterns [2, 3]. They concluded that a sphere:bath ratio of 1:5 was necessary to give an accurate value for the zero shear rate viscosity and 1:12 for observing the streamline patterns. In the experiments described in this paper no attempt was made to observe the streamlines for non-Newtonian fluids although they were observed in the Newtonian case (Plate I). Concerning the determination of the zero shear viscosity no limit in the sphere:bath ratio is necessary since the effect of the boundaries upon the torque is taken into account. Therefore only a small amount of elastico-viscous liquid is necessary. Figure 4 shows a typical plot of $T/8\pi a^3$ for the 5% P.I.B. solution. The behaviour is linear in the region considered, as expected. The slope gives the value of the zero shear rate viscosity. A comparison of this value with that obtained by using Caswell's [9] extrapolation procedure in a falling-sphere viscometer showed a difference of 2%.

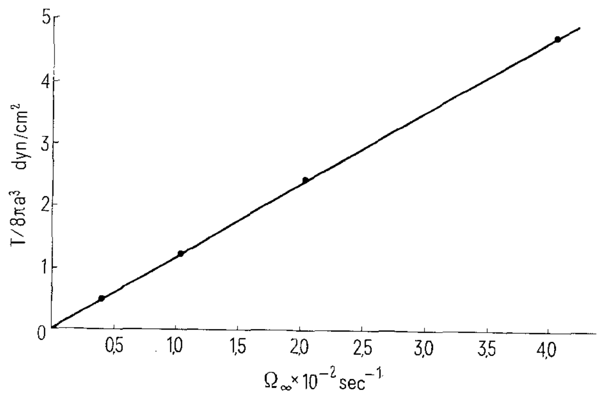


Figure 4
Experimental values of $T/8\pi a^3$ vs. Ω_∞ for 5% P. I. B. solution. The slope gives the zero shear rate viscosity.

5. Conclusions

1. The performance of the experimental apparatus as a viscometer is found to be very satisfactory, particularly for low rotational speeds. The viscosity of Newtonian fluids may be determined very accurately and a similar accuracy appears to be true in determining the limiting viscosity of viscoelastic fluids.

2. In the experimental arrangement described in this paper there is no size limitation for the sphere:bath ratio as found by some other authors [5], therefore a minimum amount of elastico-viscous fluid is necessary in determining the flow parameters for non-Newtonian fluids.

3. The experimental values for the couple acting on the sphere agree remarkably well with Collins' [4] expansion formula for the torque on a rotating sphere as a series in even powers of the angular Reynolds number.

4. It appears, from the experiments, that an extension of the results to higher Reynolds number is possible and by 'stretching' Collins' expansion by means of Shanks [13] transformation close agreement with experiments is obtained.

5. Observation of wall effects and boundary proximity upon the flow is made possible and Brenner's [10] technique proves to be very satisfactory in evaluating such effects.

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Summary

The couple on a sphere in the centre of a finite rotating circular cylinder is measured over a wide range of Reynolds numbers for both Newtonian and non-Newtonian fluids. Wall effects are calculated. Experimental results are compared with Collins' analysis. Secondary flow is made visible.

For non-Newtonian fluids the apparatus determines accurately the zero shear rate viscosity.

Résumé

Le couple sur une sphère dans un cylindre circulaire rotatoire est mesuré pour des divers nombres de Reynolds et pour des fluides Newtoniens et non-Newtoniens. L'effet des parois sur le couple est calculé. Les resultats obtenus sont comparés avec l'analyse de Collins.

Pour les fluides non-Newtoniens l'appareil determine la viscosité de zéro cisaillement.

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