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# The Fermat–Weber location problem revisited

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**Abstract**

The Fermat–Weber location problem requires finding a point in $\mathbb{R}^N$ that minimizes the sum of weighted Euclidean distances to $m$ given points. A one-point iterative method was first introduced by Weiszfeld in 1937 to solve this problem. Since then several research articles have been published on the method and generalizations thereof. Global convergence of Weiszfeld's algorithm was proven in a seminal paper by Kuhn in 1973. However, since the $m$ given points are singular points of the iteration functions, convergence is conditional on none of the iterates coinciding with one of the given points. In addressing this problem, Kuhn concluded that whenever the $m$ given points are not collinear, Weiszfeld's algorithm will converge to the unique optimal solution except for a denumerable set of starting points. As late as 1989, Chandrasekaran and Tamir demonstrated with counter-examples that convergence may not occur for continuous sets of starting points when the given points are contained in an affine subspace of $\mathbb{R}^N$. We resolve this open question by proving that Weiszfeld's algorithm converges to the unique optimal solution for all but a denumerable set of starting points if, and only if, the convex hull of the given points is of dimension $N$.

*Keywords:* Location theory; Fermat–Weber problem; Weiszfeld's iterative algorithm

## 1. Introduction

The Fermat–Weber problem requires finding a point in $\mathbb{R}^N$ which minimizes the sum of weighted Euclidean distances to $m$ given (or fixed) points. In a practical setting, the fixed points represent customers or demands, the new point denotes the unknown location of a new facility, and the weighted Euclidean distances are cost components associated with the interactions or flows between the new facility and its customers. The resulting minisum problem can be formulated as follows:

$$\text{minimize } W(x) = \sum_{i=1}^{m} w_i \| x - a_i \|, \tag{1}$$


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where $a_i=(a_{i1},\ldots,a_{iN})^{\mathrm{T}}$ is the known position of the $i$th demand point, $i=1,\ldots,m$; $x=(x_1,\ldots,x_N)^{\mathrm{T}}$ is the unknown position of the new facility; $w_i$ is a positive weighting constant which converts the distance between the new facility and demand point $i$ into a cost, $i=1,\ldots,m$; and distance is measured by the Euclidean norm:

$$\| x-a_i \| = \left[ \sum_{t=1}^{N} (x_t - a_{it})^2 \right]^{1/2}, \quad \forall x \in \mathbb{R}^N. \tag{2}$$

The Fermat–Weber problem is one of the basic models in continuous location theory. An iterative solution technique was first proposed by Weiszfeld [11], and rediscovered several years later independently by Miehle [9], Kuhn and Kuenne [8], and Cooper [4]. The algorithm is based on the first-order necessary conditions for a stationary point of the objective function which provide the following mapping of $\mathbb{R}^N$ to $\mathbb{R}^N$:

$$T(x) = \begin{cases} \dfrac{\sum_{i=1}^{m} w_i a_i / \| x-a_i \|}{\sum_{i=1}^{m} w_i / \| x-a_i \|}, & \text{if } x \notin \{a_1,\ldots,a_m\}, \\ a_i & \text{if } x = a_i \text{ for some } i=1,\ldots,m. \end{cases} \tag{3}$$

The mapping of the fixed points onto themselves ($T(a_i)=a_i$, $\forall i$) is required in order to have $T$ defined and continuous for all $x \in \mathbb{R}^N$. Weiszfeld's algorithm then consists of the following one-point iterative scheme:

$$x^{q+1} = T(x^q), \quad q=0, 1, 2, \ldots . \tag{4}$$

Global convergence of Weiszfeld's algorithm was proven by Kuhn [7] under the proviso that none of the iterates in the sequence generated by (4) coincides with a fixed point. Kuhn also concluded that whenever the fixed points, $a_1,\ldots,a_m$, are noncollinear, the sequence $\{x^q,\ q=0, 1, 2, \ldots\}$ will converge to the unique optimal solution except for a denumerable set of starting points $x^0$. This result is based on the hypothesis that for each $a_i$, $i=1,\ldots,m$, the algebraic system $T(x)=a_i$ has a finite number of roots.

In the local convergence study of Katz [6], it is shown that Weiszfeld's algorithm has a linear convergence rate when the optimal solution $x^*$ does not occur at a fixed point. Furthermore, for location in the plane ($N=2$), the upper asymptotic convergence bound takes on a value in the interval $[\frac{1}{2}, 1)$. If $x^*$ coincides with a fixed point, the convergence rate is usually linear, but may be quadratic or sublinear under special conditions. This last result is of theoretical interest only since the optimality criteria of Juel and Love [5] allow us to verify in $O(m^2)$ time if an optimal solution occurs at one of the fixed points. The global and local convergence properties of Weiszfeld's algorithm are extended to a generalized iterative procedure for $l_p$ distances in [1,2].

A flaw in the main convergence result of Kuhn [7] was pointed out recently by Chandrasekaran and Tamir [3]. In this paper two counter-examples are given which demonstrate that the system $T(x)=a_i$ may have a continuum set of solutions even when the fixed points $a_1,\ldots,a_m$ are not collinear. Thus, for the noncollinear case, the set of "bad" starting points which will terminate the algorithm at a nonoptimal vertex $a_i$ may not be denumerable as originally believed. We are therefore left wondering about the validity of Weiszfeld's algorithm.

The objective of this paper is to resolve the important question on convergence identified above. Fortunately, the validity of Weiszfeld's algorithm remains intact. We prove that a necessary and sufficient condition for the set of "bad" starting points to be denumerable is that the convex hull of the fixed points be of full dimension ($N$). In other words, the noncollinearity condition originally proposed by Kuhn is replaced by the slightly more stringent requirement conjectured by Chandrasekaran and Tamir, that the set $a_1, \dots, a_m$ not be contained in an affine subspace of $\mathbb{R}^N$.

## 2. Analysis

Referring to (3), we see that the iteration functions in the Weiszfeld procedure are given for each coordinate by

$$f_t(x) = \frac{\sum_{i=1}^m w_i a_{it} / \| x - a_i \|}{\sum_{i=1}^m w_i / \| x - a_i \|}, \quad t = 1, \dots, N. \tag{5}$$

These functions are defined and infinitely differentiable $\forall x \in \mathbb{R}^N \backslash \{a_1, \dots, a_m\}$. For purposes of the analysis, it is necessary to calculate the first-order partial derivatives of $f_t$:

$$\frac{\partial f_t(x)}{\partial x_j} = \frac{1}{s(x)} \left\{ \sum_i \frac{\partial}{\partial x_j} [y_i(x)] a_{it} - \frac{\sum_i y_i(x)\ a_{it}}{\sum_i y_i(x)} \sum_i \frac{\partial}{\partial x_j} [y_i(x)] \right\},$$

$$j = 1, \dots, N, \quad t = 1, \dots, N, \tag{6}$$

where

$$y_i(x) = \frac{w_i}{\| x - a_i \|}, \quad i = 1, \dots, m, \tag{7}$$

$$s(x) = \sum_i y_i(x), \tag{8}$$

and the summations are understood to be over the index set $\{1, \dots, m\}$. Using (5) and noting that

$$\frac{\partial y_i(x)}{\partial x_j} = \frac{-w_i(x_j - a_{ij})}{\| x - a_i \|^3}, \quad \forall i, j,$$

the first-order derivatives in (6) simplify to

$$\frac{\partial f_t(x)}{\partial x_j} = \frac{1}{s(x)} \sum_{i=1}^m \frac{w_i(x_j - a_{ij})(f_t(x) - a_{it})}{\| x - a_i \|^3}, \quad \forall j, t. \tag{9}$$

Consider the $N \times N$ Jacobian matrix

$$f'(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix}. \tag{10}$$

We obtain the following preliminary results.

**Lemma 1.** *If the set of fixed points $\{a_1, \dots, a_m\}$ is contained in an affine subspace of $\mathbb{R}^N$, the matrix $f'(x)$ is singular $\forall x \in \mathbb{R}^N \setminus \{a_1, \dots, a_m\}$.*

**Proof.** Since the set $\{a_1, \dots, a_m\}$ lies in an affine subspace of $\mathbb{R}^N$, a vector $c$ with at least one nonzero element and a scalar $c_0$ can be found such that

$$c^{\mathrm{T}} a_i = \sum_{j=1}^{N} c_j a_{ij} = c_0, \quad \forall i = 1, \dots, m. \tag{11}$$

Furthermore, for any point $y$ in the convex hull of the fixed points ($y \in \mathrm{ch}\{a_1, \dots, a_m\}$), we also have $c^{\mathrm{T}} y = c_0$. From (5) it is readily seen that $[f(x)]^{\mathrm{T}} = (f_1(x), \dots, f_N(x))$ is a convex combination of the $a_i$, so that $f(x) \in \mathrm{ch}\{a_1, \dots, a_m\}$. Hence,

$$c^{\mathrm{T}} f(x) = c_0, \quad \forall x \in \mathbb{R}^N \setminus \{a_1, \dots, a_m\}. \tag{12}$$

Consider the $j$th element of the row vector $c^{\mathrm{T}} f'(x)$:

$$\begin{aligned} \left[c^{\mathrm{T}} f'(x)\right]_j &= \sum_{t=1}^{N} c_t \frac{\partial f_t}{\partial x_j} \\ &= \sum_{t=1}^{N} c_t \left[ \frac{1}{s(x)} \sum_{i=1}^{m} \frac{w_i (x_j - a_{ij})(f_t(x) - a_{it})}{\| x - a_i \|^3} \right] \\ &= \frac{1}{s(x)} \sum_{i=1}^{m} \frac{w_i (x_j - a_{ij})}{\| x - a_i \|^3} \sum_{t=1}^{N} (c_t f_t(x) - c_t a_{it}) \\ &= 0, \quad \forall j = 1, \dots, N, \end{aligned} \tag{13}$$

where the last equality follows directly from (11) and (12). We conclude that the rows of $f'(x)$ do not form a basis in $\mathbb{R}^N$, and hence $f'(x)$ is singular $\forall x \in \mathbb{R}^N \setminus \{a_1, \dots, a_m\}$. □

**Lemma 2.** *If $\mathrm{ch}\{a_1, \dots, a_m\}$ has full dimension $N$, the matrix $f'(x)$ is invertible everywhere except at a subset of points of measure zero in $\mathbb{R}^N$.*

**Proof.** Since $f(x) \in \mathrm{ch}\{a_1, \dots, a_m\}$ and the set $\{a_1, \dots, a_m\}$ is not contained in an affine subspace of $\mathbb{R}^N$, it follows that a basis of $\mathbb{R}^N$ can be selected by taking $N$ independent weighted sums of the vectors $(f(x)-a_i)$, $i=1,\dots,m$. (Note that this is not true when $\{a_1,\dots,a_m\}$ is contained in an affine subspace of $\mathbb{R}^N$ as in the preceding lemma.)

The $j$th column of $f'(x)$ is given by

$$[f'(x)]_j = \frac{1}{s(x)} \sum_{i=1}^{m} g_{ij}(x)(f(x)-a_i), \quad j=1,\dots,N, \tag{14}$$

where

$$g_{ij}(x) = \frac{w_i(x_j - a_{ij})}{\| x-a_i \|^3}, \quad \forall i=1,\dots,m, \quad j=1,\dots,N. \tag{15}$$

We see that for all $j \in \{1,\dots,N\}$, $[f'(x)]_j$ is a weighted sum of the vectors $(f(x)-a_i)$, $i=1,\dots,m$, where each weight has a unique functional form. It readily follows that imposing a linear dependence on the columns of $f'(x)$ of the form

$$\sum_{j=1}^{N} c_j(x)[f'(x)]_j = 0 \tag{16}$$

must result in a nontrivial functional relationship on the coordinates $x$. Hence, the set of points where $f'(x)$ is singular (the columns of $f'(x)$ do not form a basis) has measure zero in $\mathbb{R}^N$. □

To illustrate the preceding results, let us consider the location problem in the plane $(N=2)$. The determinant of $f'(x)$ is given by

$$\det[f'(x)] = \frac{1}{s^2(x)} \left[ \sum_{i=1}^{m} \frac{w_i(x_1-a_{i1})(f_1(x)-a_{i1})}{\| x-a_i \|^3} \sum_{k=1}^{m} \frac{w_k(x_2-a_{k2})(f_2(x)-a_{k2})}{\| x-a_k \|^3} - \sum_{i=1}^{m} \frac{w_i(x_1-a_{i1})(f_2(x)-a_{i2})}{\| x-a_i \|^3} \sum_{k=1}^{m} \frac{w_k(x_2-a_{k2})(f_1(x)-a_{k1})}{\| x-a_k \|^3} \right].$$

First suppose that the fixed points are all collinear, and without loss of generality, let the slope of the line be given by the nonzero finite value $r$. Then,

$$f_2(x) - a_{i2} = r(f_1(x)-a_{i1}), \quad \forall x \in \mathbb{R}^N \setminus \{a_1,\dots,a_m\}, \quad i=1,\dots,m.$$

It immediately follows that $\det[f'(x)]=0$, and hence, $f'(x)$ is singular $\forall x \in \mathbb{R}^N \setminus \{a_1,\dots,a_m\}$. On the other hand, if the fixed points are not collinear, the equation $\det[f'(x)]=0$ gives a nontrivial functional relationship on the coordinates $x$. We therefore conclude that the set of points where $f'(x)$ is singular has measure zero in $\mathbb{R}^N$.

The principal result of the paper follows directly from Lemmas 1 and 2.

**Theorem.** *The set of starting points $\{x^0\}$ which will terminate the sequence generated by the Weiszfeld algorithm at some fixed point $a_i$ after a finite number of iterations is denumerable, if and only if the* ch$\{a_1,\dots,a_m\}$ *has full dimension $N$.*

**Proof.** By the fundamental inverse function theorem of calculus (e.g., see [10, p. 354]), it follows that at any point $x$ where $f'(x)$ is invertible, there exist neighbourhoods $U$ and $V$ of $x$ and $f(x)$, respectively, such that the restriction of $f$ to $U$ is a one-to-one mapping of $U$ onto $V$. Now by Lemma 2 we know that if $\mathrm{ch}\{a_1, \ldots, a_m\}$ has dimension $N$, then $f'(x)$ is invertible everywhere except at a subset of points of measure zero in $\mathbb{R}^N$. Hence, a neighbourhood $U$ can be constructed around any point $x \in \mathbb{R}^N \setminus \{a_1, \ldots, a_m\}$ such that $f(x)$ provides a one-to-one mapping $\forall x \in U$. We conclude that the set $\{x \mid f(x) = b\}$ must be denumerable for any $b \in \mathbb{R}^N$. Using the same reasoning as in [7], it follows that the set of starting points $\{x^0\}$ which will terminate the sequence generated by $T(x)$ at some $a_i$ after a finite number of iterations is also denumerable.

On the other hand, if $\mathrm{ch}\{a_1, \ldots, a_m\}$ is contained in an affine subspace of $\mathbb{R}^N$, then by Lemma 1, $f'(x)$ is singular everywhere. We conclude that the set $\{x^0\}$ defined above is no longer denumerable. □

The preceding theorem confirms an important property of the Weiszfeld algorithm. It now follows that when $\mathrm{ch}\{a_1, \ldots, a_m\}$ has full dimension $N$, the Weiszfeld algorithm will converge to the optimal solution for all but a denumerable set of starting points. The preceding results are readily extended to provide the following conclusion.

**Corollary 1.** *The iterative sequence generated by the Weiszfeld algorithm will converge to an optimal solution for all but a denumerable set of starting points* $\{x^0\}$ *whenever* $x^0$ *is restricted to the smallest affine subspace containing* $\{a_1, \ldots, a_m\}$.

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