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Résumé

L'étude porte sur la théorie de l'écoulement non permanent d'un fluide visqueux incompressible dans de canaux rectangulaires d'allongement divers, sous l'influence d'un gradient de pression arbitraire, dépendant du temps. Des solutions ont été obtenues dans 4 cas particuliers: 1. gradient de pression impulsif, 2. gradient de pression constant et établi brusquement, 3. gradient de pression en fonction harmonique du temps, 4. gradient de pression à une composante constante et une composante harmonique. On donne les répartitions de vitesse, les coefficients de frottement et la dissipation d'énergie par unité de longueur.

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Nozzle Flow of a Fully Ionized Plasma Based on Two Fluid Theory[1])

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I. Introduction

Most of the investigation of magneto-gasdynamic channel flow is based on 'classical' single fluid theory in which the generalized OHM's law is used instead of the exact differential equation of electrical current density and the difference of temperatures of ions and electrons is ignored. One way to improve the results of classical magneto-gasdynamics is to use multi-fluid theory in which the effect of various forces – both gasdynamic and electromagnetic – on the electric current density has been treated exactly from the macroscopic point of view and the behaviors of electrons and ions

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are treated separately. In this paper, we study the one dimensional steady channel flow of a fully ionized plasma by two-fluid theory and compare the results with those of classical single fluid theory.

There are two types of one dimensional channel flow based on the classical single fluid theory [1] [2]) the approximate one dimensional flow and the strict one-dimensional flow, which are based on different approximations on MAXWELL's equations. In this paper we consider only the strict one-dimensional flow in which all quantities are functions of the longitudinal coordinate x and strictly independent of the transverse coordinate y and z and MAXWELL's equations must be obeyed.

Most of the calculations of magneto-gasdynamic channel flow were carried out under the special restrictions [2] [3] such as constant velocity channel, constant cross-sectional area channel, constant temperature, etc. This means that the flow configurations will be very special. We shall follow the analysis of ordinary gasdynamic by considering the flow in a nozzle of given shape under various flow rates and applied electromagnetic fields. In this manner, the similarity and deviations between the results of single fluid theory and two fluid theory can be clearly observed.

We consider a fully ionized plasma consisting of electrons and singly charged ions flowing in a nozzle of slowly varying cross-section $A(x)$ which is a given function of the longitudinal coordinate x. There is a constant transverse externally applied magnetic field H_y in the y-direction and a constant transverse externally applied electric field E_z in the z-direction. For simplicity, we may consider that the nozzle is of rectangular cross-section and that the walls perpendicular to the y-axis are insulated walls and those perpendicular to the z-axis are perfectly conducting walls. We shall assume that the Reynolds number of the flow is high so that both viscosity and heat-conduction are negligible. However, we shall assume that the electrical conductivity of the plasma is finite so that the friction coefficient is finite.

II. Basic Assumptions and Fundamental Equations

The variable in the two-fluid theory are as follows:

$$\left.\begin{aligned} \boldsymbol{q}_\alpha &= i\, u_\alpha(x) + j\, v_\alpha(x) + k\, w_\alpha(x)\,, \\ & p_\alpha(x)\,; \quad \nu_\alpha(x)\,; \quad T_\alpha(x)\,, \\ \boldsymbol{E} &= i\, E_x(x) + j\, E_y(x) + k\, E_z(x)\,, \\ \boldsymbol{H} &= i\, H_x(x) + j\, H_y(x) + k\, H_z(x)\,, \end{aligned}\right\} \tag{1}$$

where $\alpha = 1$ or 2 and the subscript 1 refers to the values for ions and the subscript 2 refers to the values for electrons. The flow velocity vector $\boldsymbol{q}_\alpha$ for each species has the components u_α, v_α and w_α in the x-, y- and z-axis respectively; p_α is the partial pressure of α-species, ν_α is the number density of α-species, and T_α is the temperature of α-species. The electric field strength $\boldsymbol{E}$ has the components E_x, E_y, and E_z while the magnetic field strength $\boldsymbol{H}$ has components H_x, H_y, and H_z along the x-, y-, and z-axis respectively. The unit vectors along x-, y-, and z-axis are i, j, and k respectively.

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[2]) Numbers in brackets refer to References, page 369.

The fundamental equations which govern the 18 variables of Equation (1) are [4] [5] [7]:

1. Equation of state for α-species is

$$p_\alpha = R_A \nu_\alpha T_\alpha , \tag{2}$$

where R_A is the universal gas constant.

2. Equation of conservation of mass of α-species is

$$\nu_\alpha u_\alpha A = Q_\alpha = \text{constant} , \tag{3}$$

where the mass source term is zero because we consider only the fully ionized plasma.

3. Equation of motion of α-species is

$$m_\alpha \nu_\alpha u_\alpha \frac{d\boldsymbol{q}_\alpha}{dx} = -i \frac{dp_\alpha}{dx} + e_\alpha \nu_\alpha \boldsymbol{E} + e_\alpha \nu_\alpha \mu_e (\boldsymbol{q}_\alpha \times \boldsymbol{H}) + \alpha_{12} (\boldsymbol{q}_\alpha - \boldsymbol{q}_\beta) , \tag{4}$$

where m_α is the mass of a particle of α-species and $m_1 \gg m_2$; e_α is the electrical charge on a particle of α-species and $e_1 = -e_2 = e$ where e is the absolute electric charge; μ_e is the magnetic permeability; $\boldsymbol{B} = \mu_e \boldsymbol{H}$ (i.e. $B_x = \mu_e H_x$ etc.) is the magnetic induction, and α_{12} is the friction coefficient.

4. Energy Equation of α-species is

$$\frac{3}{2} u_\alpha \frac{dp_\alpha}{dx} + \frac{5}{2} \frac{p_\alpha u_\alpha}{\nu_\alpha} \frac{d\nu_\alpha}{dx} = \frac{3 \alpha_{12}}{m_0} k (T_\alpha - T_\beta) - \frac{m_\beta}{m_0} \alpha_{12} (\boldsymbol{q}_\alpha - \boldsymbol{q}_\beta)^2 , \tag{5}$$

where the ratio of specific heats of each species is 5/3 in our problem and $J_\alpha = \nu_\alpha e_\alpha q_\alpha$ is the electric current density of α-species, and $m_0 = m_\alpha + m_\beta$.

Equations (2) to (5) are the twelve gasdynamic equations of two fluid theory.

5. Electromagnetic equations.

The first MAXWELL's equation of electromagnetic field $\nabla \times \boldsymbol{E} = 0$ gives in our problem

$$E_y = \text{constant and } E_z = \text{constant}. \tag{6}$$

The conservation of electrical charge gives

$$\frac{1}{A} \frac{dA\, E_x}{dx} = \frac{e}{\varepsilon} (\nu_1 - \nu_2) , \tag{7}$$

where ε is the inductive capacity.

The second Maxwell equation $\nabla \times \boldsymbol{H} = \boldsymbol{J}$ gives

$$J_x = e (u_1 \nu_1 - u_2 \nu_2) = 0 , \tag{8a}$$

$$\frac{dH_z}{dx} = -e (v_1 \nu_1 - v_2 \nu_2) = -J_y , \tag{8b}$$

$$\frac{dH_y}{dx} = e (w_1 \nu_1 - w_2 \nu_2) = J_z . \tag{8c}$$

The divergence of magnetic field gives

$$A H_x = \text{constant}. \tag{9}$$

Equations (6), (7), (8b), (8c) and (9) give the relations of the electromagnetic fields and the gasdynamic variables. It should be noticed that Equation (9) is not an independent relation. We may use either Equation (8a) or (9) as a basic equation.

We should solve the 18 variables of Equations (1) from the 18 Equations (2), (3), (4), (5), (6), (7), (8b), (8c), and (9) for a given nozzle $A(x)$ and given initial and final conditions at the two ends of the nozzles.

III. Comparison Between Single Fluid Equations and Two Fluid Equations

In the single fluid theory we deal with the gross variables of the plasma as a whole which are related to the partial variables of Equation (1) by the following relations:

$$\left.\begin{aligned} &\nu = \nu_1 + \nu_2 , \quad \varrho = m\,\nu = m_1\,\nu_1 + m_2\,\nu_2 = \varrho_1 + \varrho_2 , \\ &p = p_1 + p_2 , \quad \nu\,T = \nu_1\,T_1 + \nu_2\,T_2 , \quad \varrho\,q = \varrho_1\,q_1 + \varrho_2\,q_2 , \end{aligned}\right\} \tag{10}$$

where the quantities without subscript refer to those of the plasma as a whole.

The equations of state and of continuity for the single fluid theory are obtained exactly by the summation of the corresponding equations of partial variables. However, some approximations have usually been made in the equations of motion, of electrical current and of energy for the single fluid theory so that these equations are not exactly in comparison with the two fluid theory. We shall point out the difference between the two theories in the following sections.

IV. Some Simple Solutions of the Two Fluid Theory

The general solution of the two fluid theory is usually very complicated, particularly when the number densities of electrons and ions are not the same. Preliminary computation shows that if ν_1 is not equal to ν_2, the interaction force between ions and electrons is so large that all the other forces are negligible. Hence the plasma has the tendency to be neutral. Hence we shall only consider the case for

$$\nu_1 = \nu_2 . \tag{11}$$

It should be noticed that Equation (11) means no charged separation which is usually made in the single fluid theory.

From Equations (11) and (8a), we have

$$u_1 = u_2 = u . \tag{12}$$

From Equation (11) and (7), we have

$$A\;E_x = \text{constant} . \tag{13}$$

For simplicity, we shall assume

$$H_x = 0 , \quad H_z = 0 , \quad E_x = 0 , \quad E_y = 0 . \tag{14}$$

The values H_x and H_z are the x- and z-induced magnetic field components. Since we do not have the externally applied magnetic field components in both x and z

direction, the induced magnetic field may be neglected for engineering problems in which the magnetic Reynolds number is small. The x- and y-components of electric field are assumed to be zero and they are also assumed in single fluid theory analysis in literature.

From Equations (8b), (10), (11) and (14), we have

$$v_1 = v_2 = v = 0\,,$$

because for one dimensional flow $v = 0$.

Under the assumptions (11) and (13), we need to solve simultaneously the seven variables $u, w_1, w_2, p_1, p_2, \nu_1$, and H_y. If we have p_1, p_2, and ν_1, we may calculate the temperatures T_1 and T_2 from equations of state (2). From Equations (3) to (8), we obtain the following simplified equations for our seven variables.

$$m_1 \nu_1 u \frac{du}{dx} = -\frac{dp_1}{dx} - e \nu_1 w_1 B_y\,, \tag{15a}$$

$$m_2 \nu_1 u \frac{du}{dx} = -\frac{dp_2}{dx} + e \nu_1 w_2 B_y\,, \tag{15b}$$

$$m_1 \nu_1 u \frac{dw_1}{dx} = e \nu_1 E_z + e \nu_1 u B_y + \alpha_{12} (w_1 - w_2)\,, \tag{15c}$$

$$m_2 \nu_1 u \frac{dw_2}{dx} = -e \nu_1 E_z - e \nu_1 u B_y + \alpha_{12} (w_2 - w_1)\,, \tag{15d}$$

$$\frac{3}{2} u \frac{dp_1}{dx} + \frac{5}{2} p_1 \frac{du}{dx} + \frac{5}{2} \frac{p_1 u}{A} \frac{dA}{dx} = \frac{3\,\alpha_{12}}{m_0 \nu_1} (p_1 - p_2) - \frac{m_2}{m_0} \alpha_{12} (w_1 - w_2)\,, \tag{15e}$$

$$\frac{3}{2} u \frac{dp_2}{dx} + \frac{5}{2} p_2 \frac{du}{dx} + \frac{5}{2} \frac{p_2 u}{A} \frac{dA}{dx} = -\frac{3\,\alpha_{12}}{m_0 \nu_1} (p_1 - p_2) - \frac{m_1}{m_0} \alpha_{12} (w_1 - w_2)^2\,, \tag{15f}$$

$$\frac{dH_y}{dx} = e \nu_1 (w_1 - w_2)\,, \tag{15g}$$

where $B_y = \mu_e H_y$ and $E_z =$ constant, $m_0 = m_1 + m_2$.

The initial conditions of our problem are at far upstream, say $x = x_0$

$$q_1 = q_2 = 0\,, \quad p_1 = p_2 = p_0\,, \quad T_1 = T_2 = T_0\,, \quad \nu_1 = \nu_2 = \nu_0\,, \quad B_y = B_{y0}\,. \tag{16}$$

Some simple relations may be deduced from Equation (15).

From Equations (15c) and (15d), we have

$$m_1 w_1 + m_2 w_2 = \text{constant} = m\, w = 0\,, \tag{17}$$

because there is no mass flow in the z-direction and the initial condition (16) shows that the constant in Equation (17) must be zero.

Eliminating dp_1/dx from Equations (15a) and (15e) and dp_2/dx from Equations (15b) and (15f), we have respectively the followings equations

$$\left.\begin{aligned} m_1 \nu_1 (a_1^2 - u^2) \frac{du}{dx} &= -\frac{5}{3} \frac{p_1 u}{A} \frac{dA}{dx} + e \nu_1 w_1 B_y u + \frac{2\,\alpha_{12}}{m_0 \nu_1} (p_1 - p_2) \\ &\quad - \frac{2}{3} \frac{m_2}{m_0} \alpha_{12} (w_1 - w_2)^2\,, \end{aligned}\right\} \tag{18a}$$

$$\left.\begin{aligned} m_2 \nu_1 (a_2^2 - u^2) \frac{du}{dx} = -\frac{5}{3} \frac{p_2 u}{A} \frac{dA}{dx} - e \nu_1 w_2 B_y u - \frac{2 \alpha_{12}}{m_0 \nu_1} (p_1 - p_2) \\ - \frac{2}{3} \frac{m_1}{m_0} \alpha_{12} (w_1 - w_2)^2 , \end{aligned}\right\} \quad (18\text{b})$$

where

$$a_1 = \left(\frac{5}{3} \frac{p_1}{m_1 \nu_1}\right)^{1/2} = \text{ion sound speed}; \quad a_2 = \left(\frac{5}{3} \frac{p_2}{m_2 \nu_1}\right)^{1/2} = \text{electron sound speed}.$$

Now multiplying (18a) by m_1 and (18b) by m_2 and subtracting the resultant equations, we have

$$\left.\begin{aligned} [(m_1^2 a_1^2 - m_2^2 a_2^2) - (m_1^2 - m_2^2) u^2] \frac{du}{dx} = -\frac{u}{A} (m_1^2 a_1^2 - m_2^2 a_2^2) \frac{dA}{dx} \\ + \frac{2 \alpha_{12}}{\nu_1} K (T_1 - T_2) . \end{aligned}\right\} \quad (19\text{a})$$

Equations (19a) may be written as follows:

$$\frac{du}{dx} = \frac{-(u\, a_e^2/A)\ (dA/dx) + (2\, \nu_1/\sigma)\ (e/m_1)^2\ K\ (T_2 - T_1)}{a_e^2 - u^2} , \quad (19\text{b})$$

where

$$a_e = \left[\frac{m_1^2 a_1^2 - m_2^2 a_2^2}{m_1^2 - m_2^2}\right]^{1/2} = \text{effective sound speed of a fully ionized plasma} , \quad (20)$$

$$\sigma = -\frac{\nu_1^2 e^2}{\alpha_{12}} = \text{electrical conductivity of the plasma} . \quad (21)$$

A similar expression for the single fluid has been obtained for infinite conductivity case by the senior author [5].

$$\frac{du}{dx} = \frac{-u\, a_{es}^2}{A\, (a_{es}^2 - u^2)} \frac{dA}{dx} , \quad (22)$$

where

$$a_{es} = \left(a^2 + \frac{\mu_e H_y^2}{\varrho}\right)^{1/2} . \quad (23)$$

For $\sigma = \infty$, Equation (19b) reduces to a similar form as Equation (22) but with a quite different value of critical speed. The case of infinite conductivity is not important in engineering problems. We shall not discuss it any more.

For finite electric conductivity, Equation (19b) shows that at subsonic speed, the maximum value of u will not occur at the neck dA/dx. We shall discuss this point further in our numerical example section.

Another formula for du/dx may be obtained by adding Equations (18a) and (18b) and using Equations (15c) and (15d) to eliminate $(w_1 - w_2)$, we have

$$\left.\begin{aligned} \frac{du}{dx} = \frac{1}{(1 - M^2)} \\ \times \left\{-\frac{u}{A} \frac{dA}{dx} + \frac{\sigma}{p} \left(E_z + u\, B_y + \frac{m_2 u}{e} \frac{dw_2}{dx}\right) \left(\frac{2}{5} E_z + u\, B_y + \frac{2}{5} \frac{m_2 u}{e} \frac{dw_2}{dx}\right)\right\} , \end{aligned}\right\} \quad (24)$$

where $M = u/a =$ Mach number of the plasma as a whole and

$$a = \left(\frac{5}{3} \frac{p}{m_0 \nu}\right)^{1/2} = \text{sound speed of the plasma} .$$

If the inertial terms with dw_2/dx are neglected, Equation (24) reduces to the formula of single fluid theory obtained by Sears and Resler. In our numerical example, we find that these inertial terms are indeed negligible. Hence the results on the x-wise velocity u given by the single fluid will be the same as that given by the two fluid theory. However, the main difference lies in the results of temperature as we shall see more clearly in the following numerical example.

V. Numerical Example

No simple close form solution has been found for Equations (15) because these equations are non-linear. In order to illustrate some features of the results of these equations, the following numerical example has been calculated by the high speed computing machine IBM 7090 at the Computer Science Center of the University of Maryland.

The cross-section area of the nozzle is assumed to be given by the formula

$$A = \frac{1}{24} x^2 - \frac{1}{150} x + 0.0008 , \tag{25}$$

where x in meters is measured from an arbitrary initial section $x = 0$ and $A_0 = 0.0008$ m^2 at which the gas is assumed to be fully ionized with the following properties

$$p_1 = p_2 = 1013 \text{ Newton/m}^2$$
$$T_1 = T_2 = 20000^\circ \text{ K} .$$

The externally applied electric and magnetic fields are

$$E_z = 10 \text{ volt/m}$$
$$B_{y0} = 10 \times 10^{-2} \text{ weber/m}^2 .$$

Since the plasma is fully ionized, we use Spitzer's formula for the electrical conductivity [6]

$$\sigma = \frac{7 \cdot 7 \times 10^3 \, T^{3/2}}{\ln \Lambda} \, m\,h\,o/m , \tag{26}$$

where

$$\Lambda = \frac{3\,k\,T}{2\,e^3} \left(\frac{k\,T}{\pi\,\nu_2}\right)^{1/2} \tag{27}$$

and T should be the kinetic temperature of the electrons. In single fluid theory, the gross temperature of the plasma as a whole is used. From the value of the electrica. conductivity σ, we may calculate the friction coefficient α_{12} from the Equation (21)1 It is evident that if the electron temperature T_2 is different from the plasma temperature T, different values of electrical conductivity and friction coefficient will be obtained. As a result, the flow variables given by the two fluid theory would be different from that by the single fluid theory.

Our results of numerical examples are given in figures 1 to 5.

The axial velocity distributions in a nozzle at various rates of discharge Q are shown in Figure 1 where Q is ions or electrons per cubic meter per second. When the rate of discharge is low, say $Q < 3 \times 10^{21}$ ions or electrons per cubic meter per second,

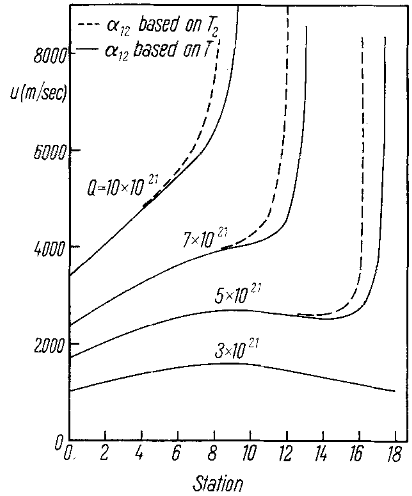

Figure 1
Velocity distributions along the nozzle at various rates of discharge Q (Low subsonic case).

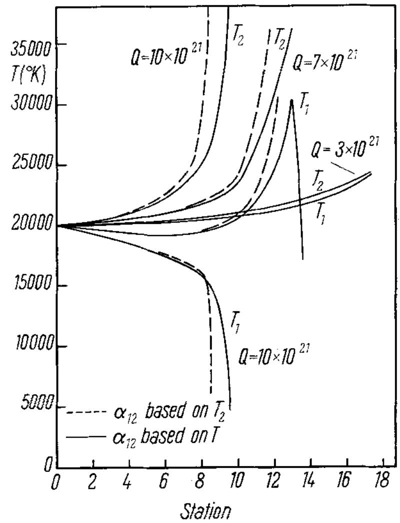

Figure 2
Temperature distributions along the nozzle at various rates of discharge Q.

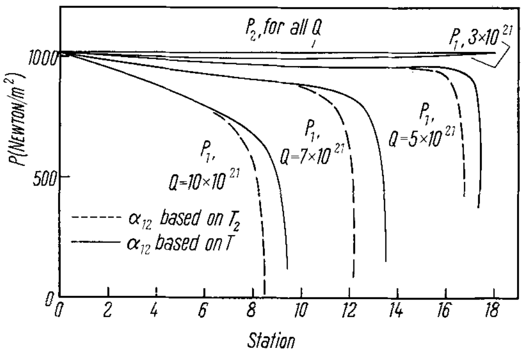

Figure 3
Pressure distributions along the nozzle at various rates of discharge Q.

smooth flow occurs in the whole nozzle. Both the single fluid theory and the two fluid theory give the same velocity distribution at low speed. One of the effects due to electromagnetic field on the x-component of velocity is that the maximum velocity occur at a position down stream from the neck of the nozzle i.e. $dA/dx = 0$. As Q increases, the location of U_{max} for a given Q moves further downstream. As the value of Q increases up to and above a critical value, the Mach number of the flow in the nozzle reaches unity at certain point of the nozzle downstream from the neck where the slope du/dx is infinite and the analysis of inviscid fluid breaks down. The critical point at $du/dx = \infty$ moves upstream as Q increases. At high speeds, the deviation of electron temperature T_2 from the plasma temperature T increases with the speed.

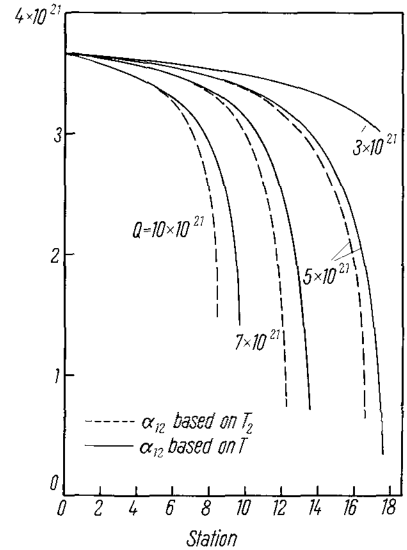

Figure 4
Number density distributions along the nozzle at various rates of discharge Q.

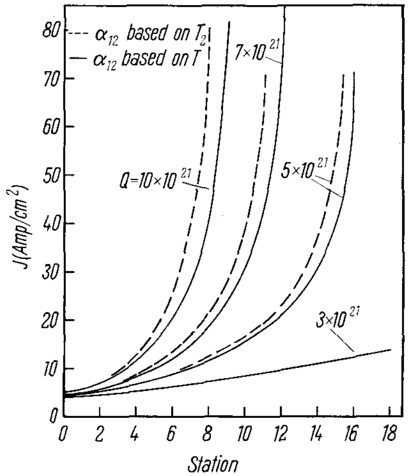

Figure 5
Electrical current density distributions along the nozzle at various rates of discharge Q.

Hence the values of electrical conductivity σ and the friction coefficient α_{12} are different depending whether we use T_2 or T in the formula (27). As a result, the velocity distribution along the nozzle will be different according to the variation of electrical conductivity. In Figures 1 to 5, the dotted curves are based on electron temperature which represents the correct values of two fluid theory, while the solid curves represent the single fluid result in which we assume $T = (T_1 + T_2)/2$. The velocity based on two fluid theory is higher than that based on single fluid theory.

Figure 2 shows the temperature distributions for various value of rate of discharge Q. The interesting result is that at low values of Q, the difference between the temperatures of electrons and ions is negligible while at high values of Q, this difference is appreciable. However, the temperature of the plasma as a whole T given by the single fluid theory is the same as that by the two fluid theory $T = (T_1 + T_2)/2$. The temperature of electrons is higher than that of ions. The most interesting result is that for high rate of discharge where the local Mach number of plasma as a whole reaches unity, the ion temperature T_1 drops to zero and so is the ion pressure at this critical location but the velocity u reaches a finite value. Hence we may consider that the ions reaches its terminal or maximum possible velocity at this critical point for a given initial pressure. This result has not been obtained previously. At low values of Q, both the ion temperature and the electron temperature increases downstream in our example but for high values of Q, the electron temperature increases and ion temperature decreases downstream. In the intermediate case e.g. $Q = 10 \times 10^{21}$ ions or electrons per cubic meter per second, the ion temperature may first increase downstream and drop suddenly near the critical point $du/dx = \infty$ or $M = 1$.

Figure 3 shows the pressure distributions along the nozzle at various values of rate of discharge Q. Again, at low values of Q, the pressures of electrons and ions are about the same while at high values of Q, they are different. The pressure of the electrons is almost constant for all values of Q while the pressures of ions varies considerably with Q. For the case where local Mach number reaches unity, the local pressure of ions drops to zero. This fact is consistent with our previous statement that the ions reach their terminal or maximum possible velocity for a given initial pressure. The effect on pressure due to the values of σ based on T or T_2 is also shown in Figure 3.

Figure 4 shows the distribution of number density, ions or electrons per cubic meter, along the nozzle at various rates of discharge Q. The number density decreases as x increases. At the critical point $M = 1$, the number density has a finite value.

Finally, Figure 5 shows the distributions of electrical current density along the nozzle at various rates of discharge Q.

VI. Conclusions

From our investigation, we found that at low rate of discharge, the single fluid theory gives almost the same result as those by two fluid theory. However, the two fluid theory gives a much more detailed picture of the flow field of the plasma. For instance, it shows that the temperature of electrons is much larger than that of ions at high rate of discharge. Since the temperature of electrons determines the electrical conductivity of the plasma, the single fluid theory may underestimate the electrical conductivity.

At the critical point $M = 1$, $du/dx = \infty$ the ion temperature and ion pressure both drop to zero, and ion velocity reaches its terminal value for a given initial pressure. It is a new result.

A special numerical example of subsonic nozzle flow of a fully ionized gas is presented. Our result may be regarded as an illustrating example to show that even though the single fluid theory may give reasonably good results of gross variables of a plasma, the multi-fluid theory will definitely give better results, particularly about the electron temperature which is essential to evaluating the over-all performance of the flow of a plasma.

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Zusammenfassung

Die streng eindimensionale stetige Strömung eines völlig ionisierten Plasmas in einer Düse mit langsam veränderlichem Querschnitt unter dem Einfluss eines transversalen elektrischen und magnetischen Feldes ist nach der Zweikomponenten-Theorie analysiert worden, und die Ergebnisse wurden mit der klassischen Einkomponenten-Theorie der Magneto-Gasdynamik verglichen. Es wurde festgestellt, dass die Einkomponenten-Theorie zwar die Werte der Variablen für den gesamten Vorgang des Plasmas liefert, dass jedoch die Zweikomponenten-Theorie anzuwenden ist, um ein Bild der Einzelheiten des Strömungsfeldes zu erhalten, vor allem vom Temperaturunterschied zwischen Elektronen und Ionen. Dieser Temperaturunterschied wächst mit steigender Temperatur. Dies dürfte wichtig sein für die Bestimmung der elektrischen Leitfähigkeit im Falle einer technischen Anwendung. In einem numerischen Beispiel wird die Unterschallströmung des völlig ionisierten Plasmas in einer Düse berechnet.

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Structure of Axisymmetric Force-Free Magnetic Fields in a Dissipative Plasma

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1. Introduction

Among the mechanisms by which a current in an ionized gas may become constricted the ordinary (or Bennett-) pinch has been the subject of extensive study. ALFVÉN [1][2]) has described a second type of filamentary current-field structure that can exist in a more rarefied plasma when electrons (e) and ions (i) are free to gyrate many times before suffering appreciable momentum changes due to collisional interaction. If the latter is characterized by suitably defined relaxation times τ_s ($s =$ e, i), and the cyclotron frequencies are $\omega_s = q_s H/c\, m_s$ (q_s/m_s being the charge/mass ratio, H the magnetic field), then conditions of this kind correspond to $\omega_s \tau_s \gg 1$. In this limit the conductivity tensor for a magnetic plasma reduces to one where the conductivity parallel to the magnetic field dominates to order $(\omega_s \tau_s)^2$ over the transverse component, and to order $\omega_s \tau_s$ over the off-diagonal terms. As the current density vector $\boldsymbol{j}$ becomes everywhere parallel to $\boldsymbol{H}$ the Lorentz force $\boldsymbol{j} \cap \boldsymbol{H}$ vanishes identically. Using $\boldsymbol{j} = (c/4\pi)$ curl $\boldsymbol{H}$ and div $\boldsymbol{H} = 0$ from Maxwell's equations[3]), the condition for a magnetic field to be force-free in this sense can be expressed as

$$-\frac{4\pi}{c}\, \boldsymbol{j} \cap \boldsymbol{H} = \boldsymbol{H} \cap \operatorname{curl} \boldsymbol{H} = 0\,, \tag{1.1}$$

or

$$\operatorname{curl} \boldsymbol{H} = \alpha \boldsymbol{H}\,, \quad \boldsymbol{H} \operatorname{grad} \alpha = 0\,, \tag{1.2}$$

where α is a scalar function of position.

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[2]) Numbers in brackets refer to References, page 385.

[3]) Gaussian c.g.s. units are employed throughout.