The Plastic Indentation of a Semi-Infinite Solid by a Perfectly Rough Circular Punch

By GEORGE EASON[1]) and RICHARD T. SHIELD[2]), Newcastle upon Tyne and Providence, R. I.

1. Introduction

Although many problems of plastic flow under conditions of plane strain have been solved, there are, as yet, few solutions to axially symmetric problems. Previous work includes solutions to the problems of the partially plastic thick-walled tube (see for example [6][3]), and the problem of converging plastic flow in a conical channel [9,7]. These problems have been solved for both the Tresca and the von Mises yield conditions.

SHIELD [8] has considered the general problem of the plastic flow of metals under conditions of axial symmetry for the Tresca yield criterion and associated plastic stress-strain relations. The following work is essentially a continuation of that paper. In [8], as here, the material was assumed to be isotropic, non-hardening, rigid-plastic and a discussion was given of the types of plastic flow which could occur. The discussion and the applications in [8] showed that the plastic regimes which satisfy the hypothesis proposed by HAAR and VON KÁRMÁN [4] are of importance in the solution of axially symmetric problems. This hypothesis states that the circumferential stress is equal to one of the principal stresses in the meridional planes during plastic deformation. The problems considered under this hypothesis were the incipient necking of a circular cylinder stressed to yielding in tension, and the indentation of a semi-infinite solid by a smooth, circular flat-ended rigid punch.

In this paper, section 2 contains a résumé of the equations of [8] relevant to the present work. In section 3 a possible velocity field is obtained for the compression of a circular cylinder. In sections 4 and 5 the indentation of a semi-infinite solid by a perfectly rough, circular flat-ended rigid punch is considered. A complete solution to this problem is obtained by numerical methods, the numerical work being performed on the high speed digital computers AMOS and FERDINAND. The value of the average pressure over the surface of the punch is found to be $6{\cdot}05\,k$, where k is the maximum shearing stress permissible in the body, in contrast to the value of $5{\cdot}69\,k$ obtained in [8]

[1]) King's College, Newcastle upon Tyne.
[2]) Brown University, Providence, R. I., USA.
[3]) Numbers in brackets refer to References, page 42.

for the smooth punch. The maximum numerical value of the ratio between the shearing stress τ_{rz} on the surface of the punch to the normal stress σ_z is found to be 0·139. It follows that the condition of perfect roughness between the punch and the material can be replaced by the condition that the coefficient of friction between the punch and the material must exceed 0·139 for the solution to apply.

2. The Basic Equations

The basic equations of axially symmetric plastic flow were discussed in detail in [8] for a rigid-plastic nonhardening material obeying Tresca's maximum shear stress yield criterion. As in the application of the theory in [8], the hypothesis of Haar and von Kármán [4] is assumed for plastic states, i. e. the circumferential stress is taken to be equal to one of the other two principal stresses in the meridional planes. More specifically, the circumferential stress is taken equal to the algebraically greater of the other two principal stresses, as in the regions of deformation to be considered the radial velocity is positive and produces a tensile circumferential strain rate (plastic regime F of [8]). The Tresca yield criterion then requires the principal stresses in the meridional planes to differ by $2\,k$, where k is the maximum shearing stress permissible in the material. A short summary is now given of the relevant equations.

In a cylindrical polar co-ordinate system (r, θ, z), the only non-vanishing stress components in a stress distribution which is axially symmetric about the z-axis are σ_r, σ_θ, σ_z, τ_{rz}. The circumferential stress, σ_θ, is a principal stress, the other principal stresses σ_1, σ_2 $(\sigma_1 \geqq \sigma_2)$ in the meridional planes being given by

$$\left.\begin{aligned} \sigma_1 &= \frac{1}{2}\,(\sigma_r + \sigma_z) + \left\{\frac{1}{4}\,(\sigma_r - \sigma_z)^2 + \tau_{rz}^2\right\}^{1/2}, \\ \sigma_2 &= \frac{1}{2}\,(\sigma_r + \sigma_z) - \left\{\frac{1}{4}\,(\sigma_r - \sigma_z)^2 + \tau_{rz}^2\right\}^{1/2}. \end{aligned}\right\} \tag{1}$$

For the plastic regime under discussion we have

$$\sigma_\theta = \sigma_1 = \sigma_2 + 2\,k\,. \tag{2}$$

The equations of equilibrium take the forms

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0\,, \qquad \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0\,, \tag{3}$$

in the absence of body forces.

Equations (2), (3) provide four equations for the determination of the four stress components so that in a sense the problem is 'statically determinate'. The system of equations is hyperbolic with orthogonal characteristics in the

(r, z)-plane which coincide with the slip-lines, the lines of maximum shearing stress. The slip-lines will subsequently be referred to as α- and β-lines.

Surface elements perpendicular to the α- and β-lines are acted upon by a shearing stress k and a normal stress which will be denoted by $-p$. The stress components are given by

$$\sigma_r = -p - k\sin 2\varphi\,, \quad \sigma_z = -p + k\sin 2\varphi\,, \quad \tau_{rz} = k\cos 2\varphi\,, \quad \sigma_\theta = -p + k\,, \tag{4}$$

where φ is the inclination of the α-line to the r-axis. The equilibrium equations (3) determine p and φ, and referred to the slip-lines the equations take the forms

$$\left.\begin{aligned} dp + 2\,k\,d\varphi + k\,(\sin\varphi + \cos\varphi)\,\frac{ds_\alpha}{r} &= 0 \text{ on an } \alpha\text{-line}\,, \\ dp - 2\,k\,d\varphi - k\,(\sin\varphi + \cos\varphi)\,\frac{ds_\beta}{r} &= 0 \text{ on a } \beta\text{-line}\,, \end{aligned}\right\} \tag{5}$$

where ds_α, ds_β are elements of length along the α- and β-lines.

On the axis of symmetry the shearing stress is zero and the radial and circumferential stresses are of equal magnitude, so that on the axis

$$\varphi = \frac{3}{4}\,\pi\,. \tag{6}$$

Using this result, an expansion of p and φ for small r shows that in the neighbourhood of the axis of symmetry

$$p + 4\,k\,\varphi = \text{const on an } \alpha\text{-line}\,, \quad p - 4\,k\,\varphi = \text{const on a } \beta\text{-line}\,, \tag{7}$$

to first order.

The velocity field is assumed to have radial symmetry, although this does not necessarily follow from the symmetry of the stress field. If u, w are the velocities in the r, z directions respectively, the non-zero strain rates are given by

$$\varepsilon_r = \frac{\partial u}{\partial r}\,, \quad \varepsilon_\theta = \frac{u}{r}\,, \quad \varepsilon_z = \frac{\partial w}{\partial z}\,, \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\,, \tag{8}$$

with ε_θ a principal strain rate. The other principal strain rates ε_1, ε_2 associated with the σ_1, σ_2 directions are

$$\left.\begin{aligned} \varepsilon_1 &= \frac{1}{2}\,(\varepsilon_r + \varepsilon_z) + \frac{1}{2}\,\{\,(\varepsilon_r - \varepsilon_z)^2 + \gamma_{rz}^2\}^{1/2}, \\ \varepsilon_2 &= \frac{1}{2}\,(\varepsilon_r + \varepsilon_z] - \frac{1}{2}\,\{\,(\varepsilon_r - \varepsilon_z) + \gamma_{rz}^2\}^{1/2}. \end{aligned}\right\} \tag{9}$$

The incompressibility of the material requires, with (8) for the strain rates,

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0\,. \tag{10}$$

This equation and the isotropy condition, which can be written

$$\frac{\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r}}{\dfrac{\partial u}{\partial r} - \dfrac{\partial w}{\partial z}} = -\cot 2\varphi\,, \tag{11}$$

serve to determine the velocity components u, w when the slip-line field is known. For numerical work it is convenient to introduce the components of velocity U, W along the α- and β-lines respectively, given by

$$U = u\cos\varphi + w\sin\varphi\,, \quad W = -u\sin\varphi + w\cos\varphi\,. \tag{12}$$

Equations (10) and (11) on u, w require U, W to satisfy the equations

$$\left.\begin{aligned} dU - W\,d\varphi + \frac{u\,ds_\alpha}{2\,r} &= 0 \text{ on an } \alpha\text{-line}\,, \\ dW + U\,d\varphi + \frac{u\,ds_\beta}{2\,r} &= 0 \text{ on a } \beta\text{-line}\,. \end{aligned}\right\} \tag{13}$$

For the velocity field to be associated with the stress field the velocities must be such that

$$\Gamma \geqq \frac{u}{r} \geqq 0\,, \tag{14}$$

where

$$\Gamma = \frac{\partial U}{\partial s_\beta} + \frac{\partial W}{\partial s_\alpha} + U\,\frac{\partial\varphi}{\partial s_\alpha} - W\,\frac{\partial\varphi}{\partial s_\beta}\,. \tag{15}$$

The theory outlined in this section will now be applied to obtain a deformation mode for the compression of a circular cylinder and to obtain the solution to the problem of indentation by a rough circular punch.

3. Compression of a Circular Cylinder

In section 3 of [8] several examples were given of possible incipient deformation modes for a circular cylinder of rigid-plastic material stressed to yielding in uniaxial compression. The stress component σ_z then has the value $-2\,k$, all other stress components being zero. In this case the characteristics are straight lines inclined at an angle of $\pi/4$ to the co-ordinate axes. The cylinder occupies the region $r \leqq 1$, $z \geqq 0$, and velocity solutions were obtained which are such that the normal velocity over the end of the cylinder, $z = 0$, is constant. Here we obtain a velocity field which satisfies the further restriction that the radial velocity u be zero over the end $z = 0$ of the cylinder.

For simplicity the normal velocity is taken to be unity over the end OA of the cylinder (Figure 1). The velocity field involves a rigid body motion of region OAC in the axial direction and flow in the region ABC. As in [8] it can be shown that $u = w = 0$ on BC because of the rigidity of the material above BC. The incompressibility condition and the condition of zero shearing strain rate together with the continuity condition across AC then require u and w to be given by

$$u = \frac{1}{\pi} (1 - \tan^2\psi)^{1/2} , \quad w = \frac{1}{\pi} \cos^{-1}(\tan\psi) , \tag{16}$$

in region ABC,

$$-\frac{1}{4}\pi \leqq \psi \leqq \frac{1}{4}\pi ,$$

where the value of the inverse cosine lies between 0 and π. The angle ψ is defined as in Figure 1. The velocity field defined by (16) was given previously in a different form in [8], apart from a multiplicative constant (equations (3.3) of [8]).

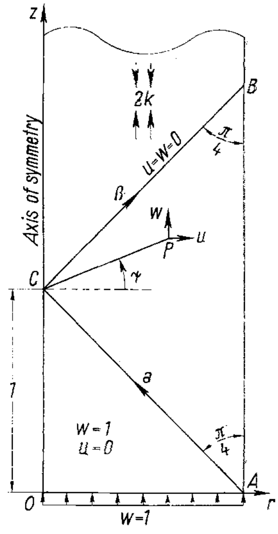

Figure 1
Cylinder stressed to yielding in compression.

4. Circular Punch: Stress Field

In [8] a complete solution to the problem of the indentation of a semi-infinite solid of rigid-plastic material by a smooth, circular flat-ended rigid punch was found. The closely allied problem of the indentation of a semi-infinite solid of rigid plastic material by a perfectly rough, circular flat-ended rigid punch is investigated here. These problems are both of importance in testing the hardness of metals (see, for example [10]). The body is assumed to occupy the region $z \geqq 0$, with the origin of co-ordinates at the centre O of the punch (Figure 2). The punch indents the portion OA of the surface of the solid, $z = 0$. The plastic stress field near the punch is assumed to satisfy (2). This assumption is shown to be correct by the fact that the solution to the problem is obtained.

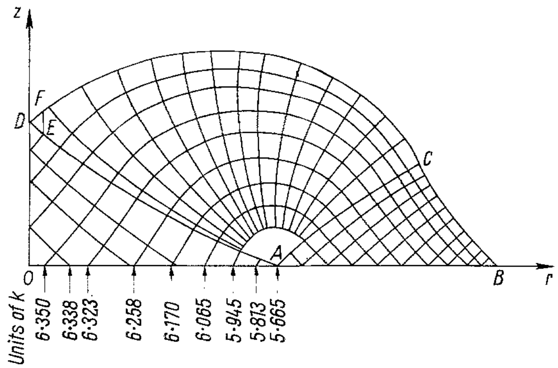

Figure 2
Slip-line field due to indentation by a rough circular punch.

Referring to Figure 2, the stress field in region ABC is determined by the stress-free surface AB. The solution for this type of region has been tabulated in [8], and the results of Table I of that paper may be applied. The line AC is an α-line and BC is a β-line. The field in region ACD is determined by the α-line AC and the singular point A, the fan being terminated by the α-line AD which meets the axis of symmetry at an angle of $\pi/4$, at D. The field in region ADO is determined by the α-line AD and the condition that the β-lines of the region meet the axis of symmetry at angles of $\pi/4$. The slip lines must meet the axis of symmetry at angles of $\pi/4$ in order that the shearing stress be zero on the axis.

The stress field in region $ACDO$ was obtained by numerical integration of (5). These differential equations were replaced by finite difference equations and the same numerical procedure used as in [8]. Equations (7) were used to determine the position of the point D. Figure 2 shows the main details of the characteristic network obtained as a result of the numerical integration. The calculated pressure distribution over the punch is shown graphically in Figure 3. The average pressure over the punch is found to be 6·05 k. For comparison, the pressure distribution found in [8] for a smooth punch is also shown in Figure 3, the average pressure over the punch in this case is found to be 5·69 k. With the distance OA as the unit of length, the distance OB, in Figure 2, was found to be 1·88, and the distance OD was found to be 0·57. The angle of the fan was found to be 116°. A coefficient of friction may be defined along OA by writing

$$\mu = -\frac{\tau_{rz}}{\sigma_z}. \tag{17}$$

The maximum value of μ along OA was found to occur at A and to have the value 0·139. Thus, the solution obtained here holds for a rough punch provided that the coefficient of friction between the material and the punch exceeds 0·139.

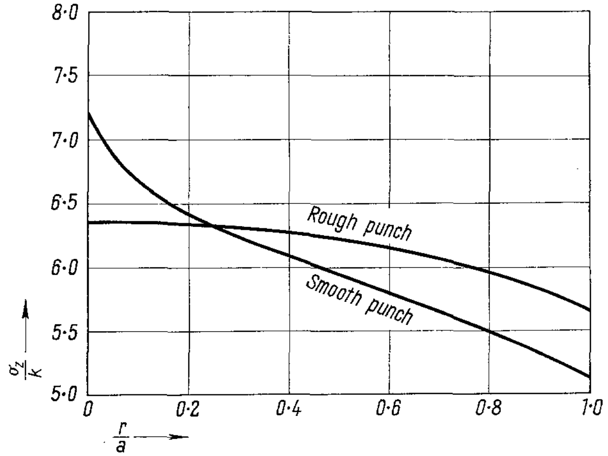

Figure 3
Pressure distribution over punch.

As in the case of the smooth punch the stress field has been extended into the rest of the body. The method of extension closely follows that used in [8] in which theorems developed by Bishop [1] for the case of plane strain were applied.

The extension of the stress field is indicated in Figure 4, and was obtained by assuming the material to be fully plastic and satisfying (2). The β-line BCD, together with the condition that slip-lines meet the axis of symmetry at angles of $\pi/4$, determines the field to the left of the α-line through B. The field to the right of the α-line through B is determined by the α-line through B and terminating the field by a stress-free surface, BJK. The α-lines begin to intersect one another at G, a point on the α-line through B, and a shock is introduced, shown by the heavy line in Figure 4. Another shock is introduced at point M, where the α-lines intersect on the α-line through C. The two shocks intersect at point H, and the stress field was continued by the introduction of a shock HL, and a fan HJ of small angle (shown by the broken line in Figure 4). The stress-free boundary is parallel to the axis of symmetry at the point K. The stress field was terminated by the introduction of a stress discontinuity KLN, above which the stress is uniaxial compression or tension parallel to the axis of symmetry. The stresses in the region are indicated in the figure, and do not violate yield.

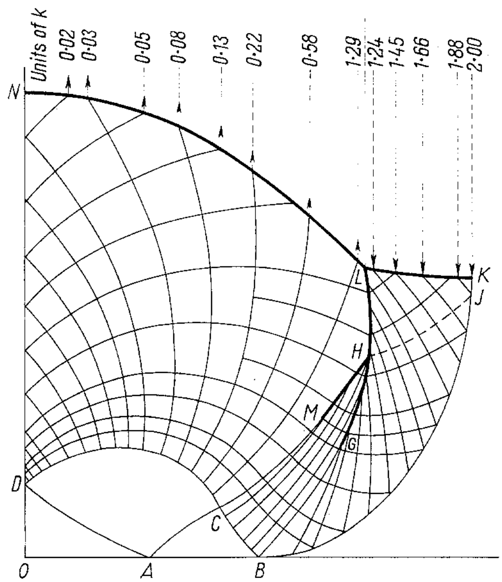

Figure 4
Extension of the slip-line field into the rigid region.

The total force acting on KLN was calculated and found to be within 3·5% of the pressure on OA. This discrepancy is almost certainly due to the inaccuracy of the finite difference forms of (5) near the axis, and of (7) away from the axis. Neither set of equations is really satisfactory in the neighbourhood of the axis of symmetry. With the distance OA as the unit of length, the point K is distant 3·59 from ON, 2·21 from OB, and N is distant 3·65 from OB.

The stress field obtained is statically admissible and the value of 6·05 k is a lower bound for the average pressure by the limit analysis theorems due to Drucker, Greenberg and Prager [3].

5. Circular Punch: Velocity Field

An incipient plastic velocity field is obtained by assuming that flow is confined to the region $ABCD$ of Figure 2. It is assumed that the punch has unit velocity parallel to the axis of symmetry, and that the region OAD moves as a rigid body in the axial direction with the velocity of the punch. The velocity field in region $ABCD$ must satisfy the conditions that the normal velocity across AD is continuous and that the normal velocity across BCD is zero, as the region above this line is assumed to be rigid. As AD is an α-line, the first of these conditions implies W continuous across AD, so that

$$W = \cos\varphi \text{ on } AD. \tag{18}$$

Along AD the first of equations (13) holds, and with (18) this takes the form

$$dU - \cos\varphi \, d\varphi + (U - \sin\varphi) \frac{dr}{2r} = 0 . \tag{19}$$

The general solution of this equation is

$$U - \sin\varphi = \frac{A}{r^{1/2}} , \tag{20}$$

where A is a constant. To avoid an infinity in the value of U at the point D, A must be taken to be zero, so that

$$U = \sin\varphi \text{ on } AD. \tag{21}$$

Thus the velocity field is seen to be continuous across AD. A similar argument to the above shows that

$$U = W = 0 \text{ along } BCD , \tag{22}$$

so that the velocity is also continuous across BCD.

The point D is a singular point of the velocity field. Near this point the slip-lines are inclined at angles of $\pi/4$ to the co-ordinate axes, and the velocity field given by (16) may be taken to hold. This velocity field was assumed to apply in the region DEF of Figure 2. Since the velocity is now known along the lines $AFECB$, the velocities within this region may be determined by integration of (13). The point A is also a singular point of the velocity field.

The velocity field in region $AFECB$ was obtained by numerical integration of (13). The same numerical procedure was adopted as in [8], the differential equations being replaced by finite difference equations. The incompressibility of the material provides a check on the accuracy of the calculation. It was found that the flow across AB was within 1% of the flow across OA. The deformation of an initially square grid which would result if this incipient velocity field were maintained for a short interval of time is shown in Figure 5.

The velocity field must also satisfy the inequalities (14) if it is to be associated with the plastic stress field. The inequalities were checked numerically at various points of the field, a sufficient number of points being taken to ensure that the inequalities held everywhere.

The velocity field is a kinematically admissible deformation mode and application of the limit analysis theorems shows that the value 6·05 k is an upper bound for the indentation pressure. The value 6·05 k is thus the actual indentation pressure, as it is also a lower bound. A theorem due to HILL [5] states that where deformation is actually occurring the stress field is unique. Also a theorem to BISHOP, GREEN and HILL [2] states that if any region of a complete solution is necessarily rigid, then it must be rigid in all complete solutions. Application of these theorems shows that the stress field of Figure 2 in region $ABCD$ is the actual plastic stress field, and plastic deformation can only occur in region $ABCD$ of Figure 5.

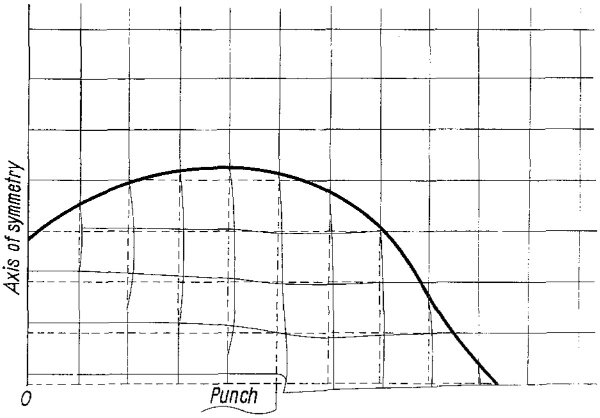

Figure 5
Resulting deformation of a square grid if the incipient velocity field were maintained for a short period of time.

REFERENCES

[1] BISHOP, J. F. W., J. Mech. Phys. Solids *2*, 43 (1953).
[2] BISHOP, J. F. W., GREEN, A. P., and HILL, R., J. Mech. Phys. Solids *4*, 256 (1956).
[3] DRUCKER, D. C., GREENBERG, H. J., and PRAGER, W., Quart. appl. Math. *9*, 381 (1952).
[4] HAAR, A., and VON KÁRMÁN, TH., Nachr. Ges. Wiss. Göttingen, Math.-phys. Kl. *1909*, 204.
[5] HILL, R., Phil. Mag. *42*, 868 (1951).
[6] KOITER, W. T., *Anniversary Volume in Applied Mechanics Dedicated to C. B. Biezeno* (H. Stam, Haarlem, Holland 1953), p. 233.
[7] SHIELD, R. T., J. Mech. Phys. Solids *3*, 246 (1955).
[8] SHIELD, R. T., Proc. Roy. Soc. [A] *233*, 267 (1955).
[9] SOKOLOVSKY, V. V., *Theory of Plasticity*, 2nd ed. (Moscow State Printing House of Technical and Theoretical Literature 1950), p. 302.
[10] TABOR, D., *The Hardness of Metals* (Clarendon Press, Oxford 1951).

Zusammenfassung

In dieser Arbeit werden das plastische Spannungsfeld und ein zulässiges Geschwindigkeitsfeld für den eben begrenzten Halbraum gegeben, der unter dem Einfluss eines ideal rauhen, starren Stempels mit kreisförmigem Querschnitt steht. Das Material ist als starr-plastisch vorausgesetzt, ohne Verfestigung, und der Fliessbedingung von TRESCA genügend. Es wird gezeigt, dass die Hypothese von HAAR und VON KÁRMÁN auf dieses Problem anwendbar ist, wonach zwei von den drei Hauptspannungen gleich sind. Es wird auch eine gültige Fortsetzung des plastischen Spannungsfeldes ins starre Gebiet in der Nähe des Stempels erhalten.

(Received: May 11, 1959.)

Flow of an Electrically Conducting Fluid Past a Porous Flat Plate in the Presence of a Transverse Magnetic Field

By ANADI SHANKAR GUPTA, Kharagpur, India[1])

1. Introduction

In recent years, the subject of hydromagnetics has attracted the attention of many authors in view not only of its own interest but also of its enormous applications to problems of geophysical and astrophysical significance.

The general equations of unified velocity and magnetic fields for the flow of an incompressible, viscous and electrically conducting fluid subject to a magnetic field have been derived by BATCHELOR [1][2]).

In this paper, we have discussed the effect of a transverse magnetic field on the steady flow of an incompressible electrically conducting fluid past an infinite porous flat plate when the plate is subjected to either suction or injection. Exact solution has been obtained for the modified Navier-Stokes and Maxwell equations under the usual assumptions of magneto-hydrodynamics. The equation of heat transfer including viscous and Joule dissipation has also been integrated and the rate of heat transfer calculated. The corresponding problem in the absence of magnetic field has been solved by GRIFFITH and MEREDITH [2].

2. Basic Equations

The fundamental equations of hydromagnetics are:

(a) MAXWELL'S equations:

$$\operatorname{div} \boldsymbol{E} = \frac{4 \pi c^2 \varrho}{\varepsilon} , \tag{1}$$

---

[1]) Indian Institute of Technology.

[2]) Numbers in brackets refer to References, page 50.