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Stability of Equilibrium of a Heavy Particle on a Rotating Surface

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1. Introduction

Apart from his fundamental work on intuitionism and on topology L. E. J. Brouwer wrote some minor articles on other subjects. In 1918 he published a paper [1] dealing with the equilibrium of a particle P moving under gravity on a surface S which is rotating with uniform angular velocity ω about a vertical axis l. If O is a point of S with a horizontal tangent plane, then if O is on l it is a point of equilibrium, and the question arises whether this equilibrium is stable or unstable.

Brouwer solves the problem first when S is smooth and then in the more difficult case of a Coulomb force with a constant coefficient of friction. At least in the first case his results are elegant and remarkable, especially if O is a saddle-point of S. The paper does not seem to have drawn much attention; a reason could be that it was published in Dutch. It is now made generally accessible by a translation in English which appeared in the second volume of Brouwer's collected works [2] and to which some notes and an interesting correspondence with Blumenthal and Hamel have been added.

In section 2 a summary of Brouwer's results in the frictionless case is given. We have added a diagram illustrating the situation. In the later sections we introduce (not Coulomb friction but) a *linear damping force* and consider its influence on the stability of the equilibrium.

In section 3 we deal with *internal* damping (proportional to the *relative* velocity of P) and in 4 with *external* damping (proportional to the *absolute* velocity of P).

2. The undamped case

A cartesian frame $OXYZ$ is fixed to the rotating surface S, with OZ upwards along l and such that XOZ and YOZ coincide with the principal normal sections of S at O, the principal curvatures being k_1, k_2 respectively. We may suppose $k_1 \geq k_2$. If k_1, k_2 are both positive S, at O, is convex from below, if $k_1 k_2 < 0$ O is a saddle-point. Taking x, y as the relative coordinates of P and restricting ourselves to motions in the neighbourhood of O we obtain the following equations of motion:

$$\ddot{x} - 2\omega\dot{y} + (gk_1 - \omega^2)\,x = 0, \qquad \ddot{y} + 2\omega\dot{x} + (gk_2 - \omega^2)\,y = 0, \tag{2.1}$$

the second terms being the components of the Coriolis force, the third arising from those of gravity (with acceleration g) and of the centrifugal force. The set 2.1 is well-known in applied mechanics: it appears in the problem concerning the stability of a rotating shaft, which leads to the concept of *critical speed* [3]. But there is an essential difference in that in the latter problem the constants analogous to k_1 and k_2 are always positive (they are the two flexural rigidities of the shaft), while in Brouwer's problem they can be negative.

Solving 2.1 in the usual way we obtain the frequency equation

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \tag{2.2}$$

with

$$\begin{aligned} &a_0 = 1, \qquad a_1 = a_3 = 0, \qquad a_2 = g(k_1 + k_2) + 2\omega^2, \\ &a_4 = (gk_1 - \omega^2)(gk_2 - \omega^2). \end{aligned} \tag{2.3}$$

The equilibrium at O is stable if and only if 2.2, a quadratic equation for λ^2, has two different negative roots. Hence the following conditions must be satisfied

$$a_2 > 0, \qquad a_4 > 0, \qquad a_2^2 - 4a_0a_4 > 0 \tag{2.4}$$

or

$$2\omega^2 + g(k_1 + k_2) > 0, \tag{2.5}$$

$$8\omega^2(k_1 + k_2) + g(k_1 - k_2)^2 > 0. \tag{2.6}$$

$$(\omega^2 - gk_1)(\omega^2 - gk_2) > 0, \tag{2.7}$$

Omitting for the sake of brevity the more trivial border-line cases ($k_1 = k_2, k_1 + k_2 = 0, k_1k_2 = 0$) we distinguish four possibilities (α), (β), (γ), (δ).

(α): $k_1 > k_2, k_1 > 0, k_2 > 0$. The inequalities 2.5 and 2.6 are now always satisfied. Hence the equilibrium is stable if either

$$\omega^2 < gk_2 \quad \text{or} \quad \omega^2 > gk_1, \tag{2.8}$$

this is the well-known result for the problem of the rotating shaft.

(β): $k_1 > 0, k_2 < 0, k_1 + k_2 > 0$. Once more 2.5 and 2.6 are satisfied. Stability takes place if and only if

$$\omega^2 > gk_1. \tag{2.9}$$

(γ): $k_1 > 0, k_2 < 0, k_1 + k_2 < 0$. We have now

$$\omega^2 > -\tfrac{1}{2}g(k_1 + k_2), \qquad \omega^2 < \tfrac{1}{8}\,g\,\frac{(k_1 - k_2)^2}{-(k_1 + k_2)}, \qquad \omega^2 > gk_1.$$

The first two inequalities are only compatible if $(3k_1 + k_2)(k_1 + 3k_2) < 0$. Since $k_1 + 3k_2 < 0$ this implies as a *necessary* condition $3k_1 + k_2 > 0$. If it is satisfied there is stability if

$$gk_1 < \omega^2 < \tfrac{1}{8}\,g\,\frac{(k_1 - k_2)^2}{-(k_1 + k_2)} \tag{2.10}$$

If $3k_1 + k_2 < 0$ the equilibrium is unstable.

(δ): $k_1 < 0, k_2 < 0$. In this case 2.5 and 2.6 are always incompatible; the equilibrium is unstable.

Brouwer's results may be illustrated by means of a diagram (Fig. 1). As case δ) gives no stability we may suppose $k_1 > 0$. We introduce the dimensionless numbers $u = \omega^2/gk_1$ ($0 < u < \infty$), and $\mathrm{v} = k_2/k_1$ ($v < 1$). The stability regions are denoted by a + sign. The curve in the diagram is an arc of the hyperbola $8u(v + 1) + (1 - v)^2 = 0$; its tangent at $(1, -3)$ is vertical. The most interesting results are β) and γ) which

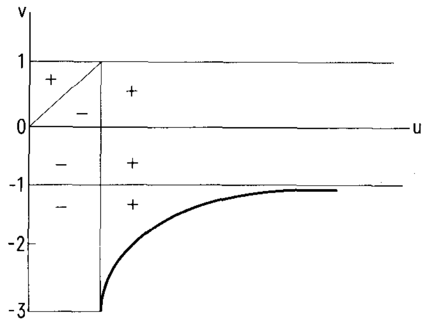

Figure 1

show that the equilibrium at a saddle-point (unstable for $\omega = 0$) may be stabilized by suitably chosen rotations (provided that $|k_2/k_1| < 3$). The stabilization of an equilibrium by *gyroscopic* forces is an old problem in dynamics, it was dealt with at length, for instance, by Thomson and Tait in their classic treatise [4]. A main theorem reads: an unstable equilibrium can be stabilized by such forces if, and only if, the number of unstable degrees of freedom is *even*. It must be kept in mind that in this theory only the Coriolis terms are taken into account and not the centrifugal forces. From Brouwer's result for the saddle-point situation (with *one* unstable coordinate) it follows that Thomson's theorem does not hold if all forces arising from the rotation are taken into account.

3. Internal damping

We deal now with Brouwer's problem in the case of a linear damping force $-c\bar{v}$ where $\bar{v}$ is the relative velocity of the particle and c a positive constant. The equations of motion are

$$\begin{aligned} &\ddot{x} - 2w\dot{y} + c\dot{x} + (gk_1 - \omega^2)x = 0, \\ &\ddot{y} + 2w\dot{x} + c\dot{y} + (gk_2 - \omega^2)y = 0, \end{aligned} \tag{3.1}$$

and the frequency equation is

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \tag{3.2}$$

with

$$\begin{aligned} &a_0 = 1, \quad a_1 = 2c, \quad a_2 = g(k_1 + k_2) + 2\omega^2 + c^2, \\ &a_3 = c\{g(k_1 + k_2) - 2\omega^2\}, \quad a_4 = (gk_1 - \omega^2)(gk_2 - \omega^2). \end{aligned} \tag{3.3}$$

The condition for stability is: all four roots of (3.2) have non-positive real parts. This is the case if $a_i > 0$ $(i = 1, 2, 3, 4)$ and moreover

$$a_1a_2a_3 > a_0a_3^2 + a_4a_1^2, \tag{3.4}$$

the well-known Routh–Hurwitz condition. Hence the following inequalities must be satisfied:

$$2\omega^2 + g(k_1 + k_2) + c^2 > 0, \tag{3.5}$$

$$-2\omega^2 + g(k_1 + k_2) > 0, \tag{3.6}$$

$$(gk_1 - \omega^2)(gk_2 - \omega^2) > 0, \tag{3.7}$$

$$-16\omega^4 + 4\omega^2[2g(k_1 + k_2) - c^2] + g^2(k_1 - k_2)^2 + 2gc^2(k_1 + k_2) > 0. \tag{3.8}$$

If $k_1 > 0, k_2 > 0$, there is stability if $\omega = 0$. Furthermore (3.5) is always satisfied; (3.6) implies $\omega^2 < \frac{1}{2}g(k_1 + k_2)$ and in view of (3.7) a necessary condition reads $\omega^2 < gk_2$. The left-hand side of (3.8) is a quadratic function Q of ω^2. The discriminant is

$$D = 4[\{c^2 + 2g(k_1 + k_2)\}^2 + 4g^2(k_1 - k_2)^2] > 0; \tag{3.9}$$

hence Q has two real zero's and $Q > 0$ for the interval in between. Furthermore

$$Q(gk_2) = g^2(k_1 - k_2)(k_1 + 7k_2) + gc^2(2k_1 + k_2) > 0, \qquad Q(0) > 0, \tag{3.10}$$

which implies $Q(\omega^2) > 0$ if $0 \leq \omega^2 < gk_2$.

The conclusion is: *if* $k_1 > 0, k_2 > 0$, *the equilibrium is stable if and only if* $\omega^2 < gk_2$.

If we compare this result with that for $c = 0$ in section 2 we conclude that the second range $(\omega^2 > gk_1)$ vanishes completely if there is any internal damping, however small. This phenomenon is not unknown in the theory of linear oscillations and depends essentially on the fact that the limit (for $c \to 0$) of condition (3.4) is different from (2.4) [5].

Suppose now $k_1 > 0, k_2 < 0$; then (3.7) is only satisfied if $\omega^2 > gk_1$ but this is incompatible with (3.6). Therefore, as a counterpart of Brouwer's result: *if O is a saddle-point the equilibrium can not be stabilized by rotation if there is any internal damping, however small.* There is an analogous thorem for systems subjected to gyroscopic forces only [6]. If, at last, $k_1 < 0, k_2 < 0$ it follows from 3.6 that stability is impossible.

4. External damping

The components of the absolute velocity of the particle are $\dot{x} - \omega y$ and $\dot{y} + \omega x$. Therefore in the case of external linear damping the equations of motion are

$$\begin{aligned} &\ddot{x} + c\dot{x} - 2\omega\dot{y} + (gk_1 - \omega^2)x - c\omega y = 0, \\ &\ddot{y} + c\dot{y} + 2\omega\dot{x} + c\omega x + (gk_2 - \omega^2)y = 0, \end{aligned} \tag{4.1}$$

$c(>0)$ being the damping factor.

The coefficients of the frequency equation are seen to be

$$\begin{aligned} a_0 &= 1, \quad a_1 = 2c, \quad a_2 = 2\omega^2 + g(k_1 + k_2) + c^2, \\ a_3 &= 2\omega^2 + g(k_1 + k_2), \quad a_4 = \omega^4 + \{-g(k_1 + k_2) + c^2\}\omega^2 + g^2 k_1 k_2. \end{aligned} \tag{4.2}$$

Stability requires $a_i > 0$, but $a_3 >$ implies $a_2 > 0$. Moreover the condition (3.4) simplifies because the term ω^4 vanishes. Therefore the equilibrium is stable if and only if the following three inequalities for ω^2, two linear and one quadratic, are satisfied

$$2\omega^2 + g(k_1 + k_2) > 0, \tag{4.3}$$

$$8(k_1 + k_2)\omega^2 + g(k_1 - k_2)^2 + 2c^2(k_1 + k_2) > 0, \tag{4.4}$$

$$Q = \omega^4 + \{-g(k_1 + k_2) + c^2\}\omega^2 + g^2 k_1 k_2 > 0. \tag{4.5}$$

We discuss once more the different cases:

(α) $k_1 > 0, \quad k_2 > 0.$

(4.3) and (4.4) are always satisfied. The discriminant of Q is

$$D = c^4 - 2g(k_1 + k_2)c^2 + g^2(k_1 - k_2)^2. \tag{4.6}$$

a quadratic function of c^2 with two positive zero's:

$$c_1^2 = g(\sqrt{k_1} - \sqrt{k_2})^2, \qquad c_2^2 = g(\sqrt{k_1} + \sqrt{k_2})^2. \tag{4.7}$$

If $c_1^2 < c^2 < c_2^2$ we have $D < 0$ and Q is positive for any ω^2. From $c^2 > c_2^2$ it follows that Q has two real zero's but in view of $-g(k_1 + k_2) + c_2^2 = 2g\sqrt{k_1 k_2} > 0$ they are both negative and therefore $Q > 0$ for $\omega^2 > 0$. If, however, $c^2 < c_1$ the two zero's $\omega_1^2, \omega_2^2, (\omega_1^2 < \omega_2^2)$ of Q are both positive and therefore $Q < 0$ for $\omega_1^2 < \omega^2 < \omega_2^2$. Summing up we have for case (α): *the equilibrium is stable for any ω^2 if the damping factor is large enough ($c^2 > c_1^2$); if $c^2 < c_1^2$ there is only stability if either $\omega^2 < \omega_1^2$ or $\omega^2 > \omega_2^2$.*

(β) $k_1 > 0, k_2 < 0, k_1 + k_2 > 0.$

(4.3) and (4.4) are again satisfied. As $k_1 k_2 < 0$ the function Q has real zero's $\omega_1^2 < 0$, $\omega_2^2 > 0$. Hence *there is stability if and only if $\omega^2 > \omega_2^2$.*

(γ) $k_1 > 0, k_2 < 0, k_1 + k_2 < 0.$

The conditions (4.3) and (4.4) are now

$$\omega^2 > \Omega_1^2 = -\tfrac{1}{2}g(k_1 + k_2), \tag{4.8}$$

$$\omega^2 < \Omega_2^2 = -\tfrac{1}{8}g\frac{(k_1 - k_2)^2}{k_1 + k_2} - \tfrac{1}{4}c^2, \tag{4.9}$$

which are only compatible if $\Omega_2^2 > \Omega_1^2$. Putting

$$H = \frac{(3k_1 + k_2)(k_1 + 3k_2)}{k_1 + k_2}, \tag{4.10}$$

this inequality implies

$$c^2 < \tfrac{1}{2}gH. \tag{4.11}$$

From this there follow two conclusions: no stability is possible if the damping factor *c surpasses a certain value* and moreover *H must be positive.* From $k_1 + k_2 < 0$ it follows $k_1 + 3k_2 < 0$; hence for stability $3k_1 + k_2 > 0$ is a necessary condition.

To investigate whether (4.5) is satisfied we remark first that Q has *one* positive zero ω_2^2. Furthermore

$$Q(\Omega_1^2) = \tfrac{1}{4}g(k_1 + k_2)(H - 2c^2),$$

hence, in view of (4.11):

$$Q(\Omega_1^2) < 0. \tag{4.12}$$

On the other hand

$$Q(\Omega_2^2) = (\Omega_2^2 - gk_1)(\Omega_2^2 - gk_2) + c^2\Omega_2^2,$$

or, after some algebra,

$$Q(\Omega_2^2) = -\tfrac{1}{64}(2c^2 - gH)(6c^2 + gH),$$

or, again in view of (4.11)

$$Q(\Omega_2^2) > 0. \tag{4.13}$$

The situation is illustrated in Fig. 2, from which it follows that the conditions are only satisfied if

$$\omega_2^2 < \omega^2 < \Omega_2^2 \tag{4.14}$$

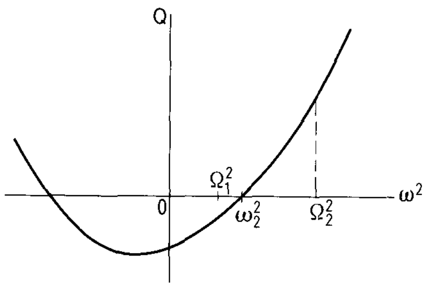

Figure 2

Summing up we have in our case γ: *a necessary condition for stability is* $c^2 < \tfrac{1}{2}gH$ (which implies $3k_1 + k_2 > 0$); *if this is satisfied there is stability if and only if (4.14) is valid.*

The influence of external damping on the stability at a saddlepoint may be derived from a comparison with the results of section 2. In all cases stability is impossible if $3k_1 + k_2 < 0$, or in other words if $|k_2| > 3k_1$.

(δ) $k_1 < 0, k_2 < 0$.

In this case (4.3) and (4.4) are incompatible; the equilibrium is unstable.

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References

[1] L. E. J. BROUWER, *Beweging van een materieel punt op den bodem eener draaiende vaas onder den invloed der zwaartekracht*, N. Arch. v. Wisk., 2e reeks, *12*, 407–419, 1918.
[2] *The Motion of a Particle on the Bottom of a Rotating Vessel under the Influence of the Gravitational Force. Collected Works*, Vol. II, 665–686, Amsterdam 1975.
[3] Cf. f.i. F. M. DIMENTBERG, Flexural Vibrations of Rotating Shafts, London, 1961.
[4] W. THOMSON and P. G. TAIT, *A Treatise on Natural Philosophy*, Vol. I, Part 1, 370–415, Cambridge, 1879.
[5] H. ZIEGLER, *Stabilitätsprobleme bei geraden Stäben und Wellen*, Z. angew. Math. Phys. *2*, 265–289 (1951); *Die Stabilitätskriterien der Elastomechanik*, Ing. Arch. *20*, 49–56 (1952); *Linear Elastic Stability*, Z. angew. Math. Phys. *4*, 1–50 (1953). O. BOTTEMA, *On the Stability of the Equilibrium of a Linear Mechanical System*, Z. angew. Math. Phys. *6*, 97–104 (1955); *The Routh-Hurwitz Condition for the Biquadratic Equation*, Proc. Kon. Ned. Ak. v. W., A, *59*, 403–406, 1956.
[6] THOMSON and TAIT, l.c. p. 391: 'Gyroscopic Forces Cannot Convert Instability into Stability if there is any Dissipativity'.

Summary

A surface S rotates with uniform velocity about the vertical axis through a point O on S with a horizontal tangent plane. A particle moves on S under gravity. In 1918 L. E. J. Brouwer investigated the stability of the equilibrium at O and especially the case of O being a saddle-point of S. Brouwer's results are discussed and extended to the case when there is either internal or external linear damping.

Zusammenfassung

Eine Fläche F dreht sich gleichmässig um die senkrechte Achse durch einen Punkt O von F mit wagrechter Berührungsebene. Ein Massenpunkt P bewegt sich zu F unter Einfluss der Schwerkraft. L. E. J. Brouwer hat 1918 die Stabilität des Gleichgewichts in O untersucht und speziell den Fall wo O ein Sattelpunkt von F ist. Die Brouwerschen Resultate werden erweitert für den Fall linearer Dämpfung, entweder proportional zur relativen oder zur absoluten Geschwindigkeit von P.

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