On the Thermodynamics of Non-simple Elastic Materials with Two Temperatures

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1. Introduction

In [3, 5] GURTIN and WILLIAMS suggested that there are no *a priori* grounds for assuming that the second law of thermodynamics for continuous bodies involves only a single temperature; that it is more logical to assume a second law in which the entropy contribution due to heat conduction is governed by one temperature, that of the heat supply by another[1]). They showed however for an extremely general class of *simple* materials that the Clausius-Duhem inequality requires that the two temperatures be equal. CHEN and GURTIN [7] investigated the case of a *non-simple* rigid heat conductor and found that for such a material this was no longer true; significantly, dependence on the second gradient of temperature is not ruled out as it is in the single-temperature theory.

In this paper we investigate further the fact that the presence of two distinct temperatures allows a dependence on higher gradients, turning to a theory including mechanical effects. In the usual theories the presence of higher gradients of deformation than the first in elastic constitutive relations is ruled out by the second law[2]). Here, however, we see that materials of grade higher than one can occur in a thermodynamic setting provided one allows the possibility of two distinct temperatures.

Briefly, we consider a material for which the deformation gradient $\boldsymbol{F}$ and its two successive gradients and the *conductive temperature* φ and its two successive gradients at a given material point and time determine the internal energy, entropy, stress, heat flux and thermodynamic temperature at that point and time. Presuming that this last relation is invertible to yield conductive temperature as a function of thermodynamic temperature and the remaining arguments, we obtain the result that the deformation gradient and the *thermodynamic temperature* θ suffice to determine the stress, energy, and entropy and that the usual stress and entropy relations hold. However, the dependence of the heat flux and the thermodynamic temperature on the gradients of $\boldsymbol{F}$ and φ are not ruled out, and it is this dependence which gives the material its non-simple character. We further deduce certain restrictions which apply at equilibrium to the response functions of the material.

We turn next to the corresponding linearized theory and show that it is mechanically simple in the sense that strain gradients do not enter the theory. In the steady-state situations we find that the difference between the two temperatures is propor-

[1]) A similar theory was studied by MÜLLER [6] who did not require *a priori* that the entropy-flux be related to the heat flux, but rather allowed it to be specified by a separate constitutive relation.
[2]) See GURTIN [2, 4].

tional to the heat supply[3]) and that the stress differs from that predicted by classical thermoelasticity theory by a pressure which is proportional to this heat supply.

2. Constitutive Assumptions

We consider a material characterized by the five *response functions* $\hat{e}$, $\hat{\eta}$, $\hat{\boldsymbol{S}}$, $\hat{\boldsymbol{q}}$ and $\hat{\theta}$ which give the internal energy e, the entropy η, the stress $\boldsymbol{S}$, the heat flux $\boldsymbol{q}$ and the thermodynamic temperature θ when the deformation gradient $\boldsymbol{F}$, the conductive temperature φ, and the gradients

$$\boldsymbol{F}_1 = \nabla \boldsymbol{F}\,, \quad \boldsymbol{F}_2 = \nabla\nabla \boldsymbol{F}\,, \quad \boldsymbol{g} = \nabla\varphi\,, \quad \boldsymbol{G} = \nabla\nabla\varphi \tag{2.1}$$

are known:

$$\left.\begin{aligned} e &= \hat{e}(\boldsymbol{F}, \varphi, \boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2)\,, \quad \eta = \hat{\eta}(\boldsymbol{F}, \varphi, \boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2)\,, \\ \boldsymbol{S} &= \hat{\boldsymbol{S}}(\boldsymbol{F}, \varphi, \boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2)\,, \quad \boldsymbol{q} = \hat{\boldsymbol{q}}(\boldsymbol{F}, \varphi, \boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2)\,, \\ \theta &= \hat{\theta}(\boldsymbol{F}, \varphi, \boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2)\,. \end{aligned}\right| \tag{2.2}$$

We assume all quantities are referred to a configuration in which the body is homogeneous; this assumption is only for convenience. The internal energy and entropy are expressed per unit volume and the heat flux and stress per unit surface area in this reference configuration.

We assume that the partial derivative of $\hat{\theta}$ with respect to φ never vanishes; then $\hat{\theta}$ is invertible in its second argument. Writing $\bar{\varphi}$ for the inverse, we have

$$\varphi = \bar{\varphi}(\boldsymbol{F}, \theta, \boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2)\,. \tag{2.3}$$

Using $(2.2)_{1,2,3}$ and (2.3) we define the functions $\bar{e}$, $\bar{\eta}$ and $\bar{\boldsymbol{S}}$ through

$$\bar{f}(\boldsymbol{F}, \theta, \boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2) = \hat{f}(\boldsymbol{F}, \bar{\varphi}(\boldsymbol{F}, \theta, \boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2), \boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2)\,, \tag{2.4}$$

where f denotes any of e, η, $\boldsymbol{S}$. The *free-energy* ψ is given by

$$\psi = e - \theta\eta\,; \tag{2.5}$$

by (2.4),

$$\psi = \bar{\psi}(\boldsymbol{F}, \theta, \boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2) = \bar{e}(\boldsymbol{F}, \theta, \boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2) - \theta\bar{\eta}(\boldsymbol{F}, \theta, \boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2)\,. \tag{2.6}$$

3. Consequences of the Second Law

We require that the first two laws of thermodynamics

$$\dot{e} = \boldsymbol{S} \cdot \dot{\boldsymbol{F}} - \operatorname{div} \boldsymbol{q} + r\,, \tag{3.1}$$

$$\dot{\eta} \geqslant - \operatorname{div}\left(\frac{\boldsymbol{q}}{\varphi}\right) + \frac{r}{\theta}\,, \tag{3.2[4]}$$

hold at every point of the body and every time. Equation (3.2) with (2.5) and (3.1) becomes

$$\dot{\psi} + \eta\dot{\theta} - \boldsymbol{S} \cdot \dot{\boldsymbol{F}} + \left(1 - \frac{\theta}{\varphi}\right) \operatorname{div} \boldsymbol{q} + \frac{\theta}{\varphi^2}\, \boldsymbol{q} \cdot \boldsymbol{g} \leqslant 0\,. \tag{3.3}$$

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[3]) CHEN, GURTIN and WILLIAMS [8].
[4]) GURTIN and WILLIAMS [3, 5].

This inequality will be satisfied by every thermodynamic process compatible with the constitutive equations (2.2) if and only if the following five conditions hold[5]:

(a) *The response functions* $\bar{\psi}$, $\bar{\boldsymbol{S}}$, $\bar{\eta}$ *and* $\bar{e}$ *are independent of* $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{g}$ *and* $\boldsymbol{G}$; *i.e.* ψ, $\boldsymbol{S}$, η, e *are given by functions of* $\boldsymbol{F}$ *and* θ *alone*

$$\psi = \bar{\psi}(\boldsymbol{F}, \theta)\,, \quad \boldsymbol{S} = \bar{\boldsymbol{S}}(\boldsymbol{F}, \theta)\,, \quad \eta = \bar{\eta}(\boldsymbol{F}, \theta)\,, \quad e = \bar{e}(\boldsymbol{F}, \theta)\,.$$

(b) $\bar{\psi}$ *determines* $\bar{\boldsymbol{S}}$ *through the stress relation*

$$\bar{\boldsymbol{S}}(\boldsymbol{F}, \theta) = \bar{\psi}_{\boldsymbol{F}}(\boldsymbol{F}, \theta)\,.$$

(c) $\bar{\psi}$ *determines* $\bar{\eta}$ *through the entropy relation*

$$\bar{\eta}(\boldsymbol{F}, \theta) = -\bar{\psi}_{\theta}(\boldsymbol{F}, \theta)\,.$$

(d) *At each* $(\boldsymbol{F}, \varphi, \boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2)$ *either*

$$\hat{\theta}(\boldsymbol{F}, \varphi, \boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2) = \varphi\,;$$

or both

$$\hat{\boldsymbol{q}}_{\boldsymbol{G}}(\boldsymbol{F}, \varphi, \boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2) \cdot \boldsymbol{\Lambda} = 0$$

for each completely symmetric third-order tensor $\boldsymbol{\Lambda}$, *and*

$$\hat{\boldsymbol{q}}_{\boldsymbol{F}_2}(\boldsymbol{F}, \varphi, \boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2) \cdot \boldsymbol{\Omega} = 0$$

for each fifth-order tensor $\boldsymbol{\Omega}$ *that is symmetric in its final four entries.*

(e) *At each* $(\boldsymbol{F}, \varphi, \boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2)$

$$(\varphi - \hat{\theta})\,(\hat{\boldsymbol{q}}_{\boldsymbol{F}} \cdot \boldsymbol{F}_1 + \hat{\boldsymbol{q}}_{\varphi} \cdot \boldsymbol{g} + \hat{\boldsymbol{q}}_{\boldsymbol{g}} \cdot \boldsymbol{G} + \hat{\boldsymbol{q}}_{\boldsymbol{F}_1} \cdot \boldsymbol{F}_2) + \frac{\hat{\theta}}{\varphi}\,\hat{\boldsymbol{q}} \cdot \boldsymbol{g} \leqslant 0\,. \tag{3.4}$$

Here subscripts indicate partial gradients, e.g., $\hat{\boldsymbol{q}}_{\boldsymbol{F}_2}(\boldsymbol{F}, \varphi, \boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2)$ is the gradient of $\hat{\boldsymbol{q}}$ with respect to $\boldsymbol{F}_2$ (and hence is a tensor of order 5). The inner product $\boldsymbol{\Lambda} \cdot \boldsymbol{\Omega}$ of two tensors of order n is defined in the usual manner, i.e.,

$$\boldsymbol{\Lambda} \cdot \boldsymbol{\Omega} = \Lambda_{ij\ldots k}\,\Omega_{ij\ldots k}$$

in cartesian components.

In view of (b) and (c) the free energy ψ, the stress $\boldsymbol{S}$ and the entropy η obey the classical equations of thermostatics, but even though ψ, $\boldsymbol{S}$, η and e can be expressed as functions of the deformation gradient $\boldsymbol{F}$ and the thermodynamic temperature θ alone, when taken as functions of $\boldsymbol{F}$, φ, $\boldsymbol{g}$, $\boldsymbol{G}$, $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ there is no reason to suppose that they are independent of any of these arguments.

Of interest to us are the *conductivity tensor* $\boldsymbol{K}(\boldsymbol{F}, \varphi)$ and the *temperature discrepancy tensor* $\boldsymbol{A}(\boldsymbol{F}, \varphi)$. $\boldsymbol{K}(\boldsymbol{F}, \varphi)$ is the symmetric part of $-\hat{\boldsymbol{q}}_{\boldsymbol{g}}(\boldsymbol{F}, \varphi, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})$ and is assumed never to vanish, while $\boldsymbol{A}(\boldsymbol{F}, \varphi)$ is the symmetric tensor[6]

$$\boldsymbol{A}(\boldsymbol{F}, \varphi) = -\hat{\theta}_{\boldsymbol{G}}(\boldsymbol{F}, \varphi, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})\,.$$

Since the left hand side of (3.4) regarded as a function of $(\boldsymbol{g}, \boldsymbol{G}, \boldsymbol{F}_1, \boldsymbol{F}_2)$ vanishes at $\boldsymbol{g} = \boldsymbol{0}$, $\boldsymbol{G} = 0$, $\boldsymbol{F}_1 = \boldsymbol{0}$, $\boldsymbol{F}_2 = \boldsymbol{0}$, it must be a maximum at that point. Hence its

[5] We omit the proof of this assertion. It can be established with the aid of arguments due to COLEMAN and NOLL [1] and analogous to those used by CHEN and GURTIN [7].

[6] Cf. CHEN and GURTIN [7].

gradient vanishes there and its second gradient must be negative semi-definite at the point. Following CHEN and GURTIN [7] we can use these observations to obtain the following results:

(f) *At equilibrium* (i.e. *when* $\boldsymbol{g} = \mathbf{0}$, $\boldsymbol{G} = \mathbf{0}$, $\boldsymbol{F}_1 = \mathbf{0}$, and $\boldsymbol{F}_2 = \mathbf{0}$) *the thermodynamic temperature and the conductive temperature are equal and the heat flux vanishes:*

$$\hat{\theta}(\boldsymbol{F}, \varphi, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) = \varphi , \quad \hat{\boldsymbol{q}}(\boldsymbol{F}, \varphi, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}) = \mathbf{0} .$$

(g) *The derivatives*

$$\hat{\theta}_{\boldsymbol{F}}, \hat{\theta}_{\boldsymbol{F}_1}, \hat{\theta}_{\boldsymbol{F}_2}, \hat{\boldsymbol{q}}_{\boldsymbol{F}}, \hat{\boldsymbol{q}}_{\varphi}, \hat{\boldsymbol{q}}_{\boldsymbol{F}_1}, \hat{\boldsymbol{q}}_{\boldsymbol{F}_2}$$

all vanish at equilibrium.

(h) *The conductivity tensor* $\boldsymbol{K}(\boldsymbol{F}, \varphi)$ *and the temperature discrepancy tensor* $\boldsymbol{A}(\boldsymbol{F}, \varphi)$ *are linearly dependent and both are positive semi-definite.*

We omit the somewhat tedious proof of these assertions.

4. Infinitesimal Theory

In this section we present the linearized form of the general theory. We assume that the material is isotropic and consider motion relative to an undistorted stress-free reference state. We suppose that the conductive temperature φ departs only slightly from a constant reference temperature φ_0 and that the first two gradients of φ are small. We assume further that the displacement $\boldsymbol{u}$, its first three gradients, and the velocity gradient are all small. Using the results (a)–(h) of the preceding section[7]) it is not difficult to show that the constitutive equations have the following approximations:

$$\left.\begin{aligned} \boldsymbol{S} &= \lambda(\operatorname{tr}\boldsymbol{E})\,\mathbf{1} + 2\,\mu\,\boldsymbol{E} - \alpha\,(3\,\lambda + 2\,\mu)\,(\varphi - \varphi_0 - a\,\Delta\varphi)\,, \\ \boldsymbol{q} &= -k\,\nabla\varphi\,, \quad \theta = \varphi - a\,\Delta\varphi\,, \\ e &= e_0 + \alpha\,\varphi_0\,(3\,\lambda + 2\,\mu)\,(\operatorname{tr}\boldsymbol{E}) + c\,(\varphi - \varphi_0 - a\,\Delta\varphi)\,, \\ \eta &= \eta_0 + \alpha\,(3\,\lambda + 2\,\mu)\,(\operatorname{tr}\boldsymbol{E}) + c\ln\frac{\varphi - a\,\Delta\varphi}{\varphi_0}\,. \end{aligned}\right\} \tag{4.1}$$

where

$$\boldsymbol{E} = \frac{1}{2}\,(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$$

is the *infinitesimal strain tensor,*

$$c = \bar{e}_\theta(\mathbf{1}, \varphi_0)$$

is the *specific heat* (assumed to be strictly positive), and

$$e_0 = \bar{e}(\mathbf{1}, \varphi_0)\,, \quad \eta_0 = \bar{\eta}(\mathbf{1}, \varphi_0)\,.$$

In (4.1), λ, μ are the Lamé moduli, α is the coefficient of thermal expansion, k is the conductivity, and a is the temperature discrepancy. Further, since $\boldsymbol{K} = k\,\mathbf{1}$ and $\boldsymbol{A} = a\,\mathbf{1}$ are positive semi-definite and $\boldsymbol{K}$ is non-zero,

$$k > 0\,, \quad a \geqslant 0\,.$$

[7]) Cf. CHEN and GURTIN [7].

Note that the linearized constitutive relations (4.1) are *mechanically simple* in the sense that there is no dependence on $\nabla \boldsymbol{E}$ and $\nabla\nabla \boldsymbol{E}$. In fact, if $a = 0$ or $\Delta\varphi = 0$ so that the two temperatures coincide then these equations are just the classical equations of linearized thermoelasticity.

Combining the first of (4.1) with (4.2) and the equation of balance of linear momentum

$$\operatorname{div} \boldsymbol{S} + \boldsymbol{b} = \varrho\, \ddot{\boldsymbol{u}}$$

we arrive at

$$\mu\, \Delta \boldsymbol{u} + (\lambda + \mu)\, \nabla \operatorname{div} \boldsymbol{u} - \alpha\, (3\,\lambda + 2\,\mu)\, \nabla\, (\varphi - a\, \Delta\varphi) = \varrho\, \ddot{\boldsymbol{u}} - \boldsymbol{b}\,, \tag{4.3}$$

where ϱ is the density and $\boldsymbol{b}$ the body force per unit volume. On the other hand if we combine (4.1), (4.2), and (4.3) and linearize the resulting equation we are led to

$$c\, \dot{\varphi} = -\alpha\, \varphi_0\, (3\,\lambda + 2\,\mu)\, \operatorname{div} \dot{\boldsymbol{u}} + k\, \Delta\varphi + c\, a\, \Delta\dot{\varphi} + r\,. \tag{4.4}$$

Equations (4.3) and (4.4) are the linearized coupled equations governing the behavior of the fields $\boldsymbol{u}$ and φ. It is easily verified that in steady state situations with $r = 0$ the two temperatures coincide and the above equations reduce to the corresponding equations of classical thermoelasticity.

In steady situations with $r \neq 0$ (4.4) implies

$$k\, \Delta\varphi = -r\,, \tag{4.5}$$

and we conclude from $(4.1)_3$ that

$$\theta - \varphi = \frac{a}{k}\, r\,;$$

thus *the difference between the two temperatures is directly proportional to the heat supplied.* Moreover if we consider the equation for $\boldsymbol{S}$ in this case we find

$$\boldsymbol{S} = \lambda (\operatorname{tr} \boldsymbol{E})\, \mathbf{1} + 2\,\mu\, \boldsymbol{E} - \alpha\, (3\,\lambda + 2\,\mu)\, (\varphi - \varphi_0)\, \mathbf{1} - \beta\, r\, \mathbf{1}\,, \tag{4.6}$$

where β is the constant

$$\beta = \frac{\alpha\, (3\,\lambda + 2\,\mu)\, a}{k}\,.$$

Hence *the stress is equal to the classical thermoelastic value plus a pressure $p = \beta\, r$ proportional to the density of external heat supply.* This relation could possibly afford an experimental means of measuring the temperature discrepancy a.

Another possibly measurable effect which r may have upon the mechanical response of the body is as follows: We define the (infinitesimal) *change in volume* δV by

$$\delta V = \int_{\mathfrak{B}} \operatorname{tr} \boldsymbol{E}\, dV\,,$$

where $\mathfrak{B}$ is the volume occupied by the body. Then from (4.6) it follows that in steady situations

$$\delta V = \int_{\mathfrak{B}} \left\{ \frac{\operatorname{tr} \boldsymbol{S}}{3\,\lambda + 2\,\mu} + 3\,\alpha\, (\varphi - \varphi_0) \right\} dV + \frac{3\,\alpha\, a}{k}\, R\,,$$

where R is the total heat supplied to the body by radiation, i.e.,

$$R = \int_{\mathfrak{B}} r\, dV\,.$$

Thus the *volume expansion is equal its classical value plus a quantity proportional to the total external heat supply*. If, in particular, the body is undergoing a *steady-state free expansion* (zero surface tractions and body forces) then it is not difficult to show that

$$\int_{\mathfrak{B}} \operatorname{tr} \boldsymbol{S}\, dV = 0\,,$$

and hence

$$\delta V = 3\,\alpha \int_{\mathfrak{B}} (\varphi - \varphi_0)\, dV + \frac{3\,\alpha\, a}{k}\, R\,.$$

In this case φ is directly related to r by (4.5); if for instance the radiation is *uniform*, i.e. r is constant, on a cylindrical body of length l, with $\varphi = \varphi_0$ on the ends and φ uniform on cross sections the volume change is just

$$\delta V = \frac{3\,\alpha}{k} \left(a + \frac{l^2}{12}\right) R\,.$$

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Zusammenfassung

Diese Arbeit behandelt eine thermodynamische Theorie von nichteinfachen, elastischen Stoffen. Es wird gezeigt, dass Substanzen vom Grade höher als eins vorkommen können; vorausgesetzt, dass man das eventuelle Vorhandensein zweier verschiedener Temperaturen in Betracht zieht.

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