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Zusammenfassung

Die Thermodynamik der richtungsorientierten Medien wird mit Hilfe der Clausius-Duhem-Ungleichheit und des Prinzips der materiellen Objektivität untersucht. Ein besonderer Fall der Materialsymmetrie wird diskutiert.

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Extremal Principles and Isoperimetric Inequalities for some Mixed Problems of Stekloff's Type

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1. We consider a plane domain G with boundary Γ. The 'mixed Stekloff problem' we are here concerned with is that of a vibrating homogeneous membrane without masses in G, but carrying masses along Γ: linear density $\varrho(s) \geqslant 0$; the 'total mass' is $M = \oint_\Gamma \varrho\, ds$; moreover, we suppose that our membrane is elastically supported along Γ: elastic coefficient $k(s)$. – We have the eigenvalue problem:

$$\Delta u = 0 \quad \text{in} \quad G\,, \quad \frac{\partial u}{\partial n} + [k(s) - \lambda\, \varrho(s)]\, u = 0 \quad \text{along} \quad \Gamma\,. \tag{1}$$

($\partial/\partial n$ is the *outer* normal derivative.)

2. For the classical Stekloff problem with $k(s) \equiv 0$, the eigenvalues are noted μ rather than λ; the first is $\mu_1 = 0$. This problem has been considered by several authors [10, 16, 23]. Some closely related problems have been considered by TROESCH [19, 20] and by WEHAUSEN and LAITONE [21].

WEINSTOCK [23] showed that, among all simply connected domains with analytic boundary and assigned total mass $M = \oint_\Gamma \varrho\, ds$, the circles with constant linear density ϱ along Γ yield the largest second eigenvalue μ_2, i.e.

$$\mu_2 \leqslant \frac{2\,\pi}{M}\,.$$

His proof uses conformal mapping and is very similar to that of SZEGÖ [17] for the corresponding isoperimetric inequality concerning free simply connected membranes with *homogeneous mass distribution in their interior:* $\mu_2 \leqslant \pi p^2/M$ with $p \simeq 1.8412$. – WEINBERGER [22] avoided the use of conformal mapping and thus extended Szegö's inequality to multiply connected membranes and to higher dimensions; as he indicates in the same paper, Szegö and Weinberger jointly noted that Szegö's proof (based on conformal mapping) actually yields, for simply connected plane free homogeneous membranes, the stronger isoperimetric inequality $\mu_2^{-1} + \mu_3^{-1} \geqslant 2\,M/\pi p^2$.

Exactly in the same way, we now remark that, for the Stekloff problem, WEINSTOCK's proof [23] of the inequality $\mu_2 \leqslant 2\,\pi/M$ actually yields the sharper isoperimetric result

$$\mu_2^{-1} + \mu_3^{-1} \geqslant \frac{M}{\pi}:$$

Among all simply connected domains with analytic boundary carrying an assigned total mass M, the circles with constant linear mass density along their circumference yield the smallest value of $\mu_2^{-1} + \mu_3^{-1}$.

Indeed: Given a *two-dimensional* linear space of functions, which we call L_2, its 'inverse Rayleigh trace' $TRinv[L_2]$ is by definition [8] equal to $R[v_1]^{-1} + R[v_2]^{-1}$, where v_1 and v_2 are any two functions in L_2, 'orthogonal in the Dirichlet-metric': $D(v_1, v_2) = 0$, $R[v]$ is the Rayleigh quotient $D(v)/\oint_\Gamma \varrho(s)\, v^2\, ds$ and $D(v)$ is the Dirichlet integral $\iint_G \operatorname{grad}^2 v\, dA$; $dA = dx\, dy$ is the element of area. – Now

$$\mu_2^{-1} + \mu_3^{-1} = \operatorname{Max}_{L_2} TRinv[L_2]$$

if we restrict L_2 by the condition that each function v in it should be orthogonal to the constant 1 'in the ϱ-norm', i.e. $\oint_\Gamma \varrho(s)\, v\, ds = 0$. – As Weinstock (using Szegö's method) has shown, there exists a conformal mapping $w(z)$ of the domain $G = G_z$ onto the unit circle $|w| < 1$ such that $\oint_{\Gamma_z} \varrho\, w\, ds = 0$ $(ds = |dz|)$; let $w = U + i\,V$; the cartesian coordinates U and V of the w-plane are themselves eigenfunctions of the Stekloff problem in the circle with $\varrho \equiv 1$, corresponding to the double eigenvalue $\mu_2^0 = \mu_3^0 = 1$; $U(z)$ and $V(z)$ are the 'transplanted' [14] functions in G_z, and they satisfy all orthogonality conditions:

$$\oint_{\Gamma_z} \varrho\, U\, ds = 0\,, \quad \oint_{\Gamma_z} \varrho\, V\, ds = 0\,, \quad D_{G_z}(U, V) = D_{|w|<1}(U, V) = 0\,.$$

Therefore

$$\mu_2^{-1} + \mu_3^{-1} \geqslant TRinv\,[L(U, V)] = R[U]^{-1} + R[V]^{-1} = \frac{\oint_{\Gamma_z} \varrho\, U^2\, ds}{D_{G_z}(U)} + \frac{\oint_{\Gamma_z} \varrho\, V^2\, ds}{D_{G_z}(V)}$$

$$= \frac{1}{\pi} \oint_{\Gamma_z} \varrho\, (U^2 + V^2)\, ds = \frac{1}{\pi} \oint_{\Gamma_z} \varrho\, ds = \frac{M}{\pi}.$$

3. We now come back to our 'mixed Stekloff problem' (1) with elastic support $k(s)$ at the boundary. Then the Rayleigh quotient is

$$R[v] = \frac{D(v) + \oint_\Gamma k(s)\, v^2\, ds}{\oint_\Gamma \varrho(s)\, v^2\, ds}\,.$$

The first eigenvalue λ_1 is no longer zero; it is characterized by Rayleigh's principle: $\lambda_1 = \mathrm{Min}_v\, R[v]$.

Following PICARD's lines (cf. [13, 18]), we now prove elementarily that *an eigenfunction u* **of constant sign** *necessarily realizes the Minimum of the Rayleigh quotient* $R[v]$.

Indeed, let v be any continuous and piecewise derivable function;

$$\mathrm{grad}^2 v - \mathrm{div}\left(\frac{v^2}{u}\,\mathrm{grad}\, u\right) = \mathrm{grad}^2 v + \frac{v^2}{u^2}\,\mathrm{grad}^2 u - 2\,\frac{v}{u}\,\mathrm{grad}\, v \cdot \mathrm{grad}\, u$$

$$= \left(\mathrm{grad}\, v - \frac{v}{u}\,\mathrm{grad}\, u\right)^2 \geqslant 0\,;$$

whence by integration in G:

$$0 \leqslant D(v) - \oint_\Gamma \frac{v^2}{u}\,\frac{\partial u}{\partial n}\,ds = D(v) + \oint_\Gamma k(s)\, v^2\, ds - \lambda \oint_\Gamma \varrho(s)\, v^2\, ds\,. \tag{2}$$

This last inequality is true regardless of the signs of $k(s)$ and of $\varrho(s)$; if $\varrho(s) \geqslant 0$, it follows that $R[v] \geqslant \lambda = R[u]$, thus $\lambda = \lambda_1$ and $u = u_1$ is the *first* eigenfunction. – We used essentially that u is *harmonic* and has *constant sign*.

Remark. – Let u be any *harmonic* function of *constant sign* in G; we have by (2):

$$D(v) \geqslant \oint_\Gamma \frac{v^2}{u}\,\frac{\partial u}{\partial n}\,ds\,. \tag{3}$$

This inequality contains *Dirichlet's principle:* if $v = u$ on the boundary Γ, then $D(v) \geqslant \oint u\, \partial u/\partial n\, ds = D(u)$. – (On the other hand, the admissible choice $v = \mathrm{const}$ implies $\oint_\Gamma 1/u\, \partial u/\partial n\, ds \leqslant 0$, which follows also from the fact that $\Delta \ln u = \mathrm{div}\,(\mathrm{grad}\, u/u) = -\,\mathrm{grad}^2 u/u^2 \leqslant 0$, $\ln u$ is superharmonic.)

Furthermore, if u and $\partial u/\partial n$ have the same sign on the part of Γ where $v \neq 0$, then inequality (3) also implies *Thomson's principle* for vector fields $\boldsymbol{p} = \mathrm{grad}\, u$ without sources: by Schwarz' inequality, we then have

$$D(v) \geqslant \oint_\Gamma \frac{v^2}{u}\,\frac{\partial u}{\partial n}\,ds \geqslant \frac{\left(\oint_\Gamma v\,\frac{\partial u}{\partial n}\,ds\right)^2}{\oint_\Gamma u\,\frac{\partial u}{\partial n}\,ds} = \frac{\left(\oint_\Gamma v\,\boldsymbol{p}\cdot\boldsymbol{n}\,ds\right)^2}{\iint_G \boldsymbol{p}^2\,dA}$$

($\boldsymbol{n}$ is the outer normal).

Inequality (3) can be transformed into a somewhat sharper form by introducing into it the function $v + c$ (c = constant) instead of v; the optimal constant is

$$c_{opt} = -\oint_\Gamma \frac{v}{u}\,\frac{\partial u}{\partial n}\,ds \Big/ \oint_\Gamma \frac{1}{u}\,\frac{\partial u}{\partial n}\,ds$$

and it gives the inequality

$$D(v) \geqslant \oint_\Gamma \frac{v^2}{u}\,\frac{\partial u}{\partial n}\,ds + \frac{\left(\oint_\Gamma \frac{v}{u}\,\frac{\partial u}{\partial n}\,ds\right)^2}{-\oint_\Gamma \frac{1}{u}\,\frac{\partial u}{\partial n}\,ds}\,; \tag{3'}$$

the additional term on the right is non-negative.

4. *The method of one-dimensional auxiliary problems* [2, 3, 5, 11, 12, 15] *applied to the mixed Stekloff problem.*

4.1. *Leading idea, motivation*

In order to obtain lower bounds for the first eigenvalue λ_1, we have *to apply Rayleigh's principle* not to the given problem itself, but *to some auxiliary problems.* Now we consider as auxiliary problems vibrating strings in G, some parallel to the x-axis, the others to the y-axis, and with end points P_0, P_1 on Γ. Those strings carry no masses in their interior; each one carries two point masses: m_0 at P_0 and m_1 at P_1; *they are elastically supported in their interior.*

As in [5], we shall avoid the resolution of an infinity of auxiliary problems by *choosing* beforehand their first eigenfunction (it must have *constant sign*), and only then determine the one-dimensional problems themselves.

Let us choose two *positive* functions f and g in G: $f(x, y) > 0$ continuous *in* x and twice partially derivable *with respect to* x (but not necessarily continuous in y); and $g(x, y) > 0$ continuous *in* y and twice partially derivable *with respect to* y (but not necessarily continuous in x).

We further suppose that the following condition is fulfilled:

$$\frac{f_{xx}}{f} + \frac{g_{yy}}{g} \leqslant 0 \quad \text{in} \quad \text{all } G\,. \tag{4}$$

For every segment through G, parallel to the x-axis and with extremities P_0 and P_1 on the boundary Γ, $f(x, y)$ is the first eigenfunction of an auxiliary vibrating string with the elastic coefficient $\varkappa(x) = f_{xx}(x, y)/f(x, y)$ and the masses $m_0 = -f_x(P_0)/f(P_0)$, $m_1 = +f_x(P_1)/f(P_1)$ and the corresponding (first) eigenvalue is $\tilde{\lambda} = 1$. Indeed, f has constant sign and satisfies

$$f_{xx} - \varkappa(x)\, f = 0\,; \quad -f_x(P_0) - 1 \cdot m_0\, f(P_0) = 0 \quad \text{and} \quad f_x(P_1) - 1 \cdot m_1\, f(P_1) = 0\,.$$

(One should be careful that m_0 or m_1 may here be negative; but the formal calculation in 4.2 remains valid in this case.)

Similarly, on every segment through G, parallel to the y-axis, $g(x, y)$ is the first eigenfunction of an auxiliary string and corresponds to the eigenvalue $\tilde{\lambda} = 1$.

The following formal calculation is equivalent to applying the one-dimensional Rayleigh principle (more precisely: Picard's method for its proof, see Section 3) to each auxiliary string.

4.2. *Formal calculation*

Let again $u = u_1(x, y)$ be the first eigenfunction of the given Stekloff problem (1).

$$u_x^2 - \left(\frac{f_x}{f}\, u^2\right)_x = u_x^2 - 2\,\frac{f_x}{f}\, u\, u_x + \frac{f_x^2}{f^2}\, u^2 - \frac{f_{xx}}{f}\, u^2 = \left(u_x - \frac{f_x}{f}\, u\right)^2 - \frac{f_{xx}}{f}\, u^2\,;$$

$$u_y^2 - \left(\frac{g_y}{g}\, u^2\right)_y = u_y^2 - 2\,\frac{g_y}{g}\, u\, u_y + \frac{g_y^2}{g^2}\, u^2 - \frac{g_{yy}}{g}\, u^2 = \left(u_y - \frac{g_y}{g}\, u\right)^2 - \frac{g_{yy}}{g}\, u^2\,;$$

now *assuming that the following integrals have a sense,* we obtain

$$D(u) - \oint \left(\frac{f_x}{f}\,\frac{\partial x}{\partial n} + \frac{g_y}{g}\,\frac{\partial y}{\partial n}\right) u^2\, ds = \iint\limits_G \left[\left(u_x - \frac{f_x}{f}\, u\right)^2 + \left(u_y - \frac{g_y}{g}\, u\right)^2 - \left(\frac{f_{xx}}{f} + \frac{g_{yy}}{g}\right) u^2\right] dA \geqslant 0\,;$$

since u is harmonic, $D(u) = \oint_\Gamma u\, \partial u/\partial n\, ds = \oint_\Gamma [\lambda_1 \varrho(s) - k(s)]\, u^2\, ds$, whence

$$\oint_\Gamma \left\{\lambda_1 \varrho(s) - \left[k(s) + \frac{f_x}{f}\frac{\partial x}{\partial n} + \frac{g_y}{g}\frac{\partial y}{\partial n}\right]\right\} u^2\, ds \geqslant 0\,;$$

the integrand cannot be everywhere negative: therefore, if $\varrho(s) \geqslant 0$,

$$\lambda_1 \geqslant \inf_\Gamma \left\{\frac{1}{\varrho(s)}\left[k(s) + \frac{f_x}{f}\frac{\partial x}{\partial n} + \frac{g_y}{g}\frac{\partial y}{\partial n}\right]\right\}. \tag{5}$$

In particular, if we have chosen $f = g$ (positive, continuous and twice differentiable, satisfying $\Delta f \leqslant 0$),

$$\lambda_1 \geqslant \inf_\Gamma \left\{\frac{1}{\varrho(s)}\left[k(s) + \frac{1}{f}\frac{\partial f}{\partial n}\right]\right\}, \tag{5'}$$

which is the analogue here of the BARTA inequality [1] for vibrating membranes.

Furthermore, if by chance we have chosen exactly $f = g = u_1(x, y)$, then

$$k\,(s) + \frac{1}{f}\frac{\partial f}{\partial n} = \lambda_1 \varrho\,(s)\,,$$

so we have equality in (5) and (5'); hence

$$\lambda_1 = \operatorname*{Max}_{\substack{f>0\\ g>0\\ \frac{f_{xx}}{f}+\frac{g_{yy}}{g}\leqslant 0}} \inf_\Gamma \left\{\frac{1}{\varrho(s)}\left[k(s) + \frac{f_x}{f}\frac{\partial x}{\partial n} + \frac{g_y}{g}\frac{\partial y}{\partial n}\right]\right\}. \tag{5''}$$

Remark. – We note the rather trivial inequalities

$$\inf_\Gamma \frac{k(s)}{\varrho(s)} \leqslant \lambda_1 \leqslant \frac{\oint_\Gamma k(s)\, ds}{\oint_\Gamma \varrho(s)\, ds} = \frac{K}{M}\,; \tag{6}$$

the inequality on the right follows immediately from Rayleigh's principle applied to the function $v \equiv 1$; the inequality on the left follows from (5') applied to $f \equiv 1$.

Physical interpretation: the upper bound K/M in (6) becomes actually the first eigenvalue of the problem when one augments to infinity the modulus of elasticity of the membrane; the lower bound on the left, when one reduces the modulus of elasticity to zero. Both inequalities thus express monotony.

In particular, if $k(s) = c\,\varrho(s)$ with a constant c, then $\lambda_1 = c$ and $u_1 =$ const. (Special case: the classical Stekloff problem: $k(s) \equiv 0$, the first eigenvalue is then zero.)

4.3. *Vector formulation*

As was done in [5] for vibrating membranes, we construct, to each pair of admissible functions f, g, the vector field $\boldsymbol{p} = (-f_x/f;\, -g_y/g)$; the condition (4) becomes $\operatorname{div} \boldsymbol{p} - \boldsymbol{p}^2 = -f_{xx}/f - g_{yy}/g \geqslant 0$ and the lower bound (5) becomes

$$\lambda_1 \geqslant \inf_\Gamma \left\{\frac{1}{\varrho(s)}\,[k(s) - \boldsymbol{p}\cdot\boldsymbol{n}]\right\}.$$

Direct proof. – We consider a vector field $\boldsymbol{p}$ in G, of which we assume:

(i) The first component of $\boldsymbol{p}$ must be continuous *in* x and partially derivable *with respect to* x, the second component continuous *in* y and partially derivable *with respect to* y;

(ii) the condition

$$\operatorname{div} \boldsymbol{p} - \boldsymbol{p}^2 \geqslant 0 \tag{4'}$$

must be fulfilled;

(iii) the integrals following below must exist.

Let once more $u = u_1(x, y)$ be the first eigenfunction of the given problem (1). Then

$$\begin{aligned}\operatorname{grad}^2 u + \operatorname{div}(u^2 \boldsymbol{p}) &= \operatorname{grad}^2 u + 2\, u\, \boldsymbol{p} \cdot \operatorname{grad} u + u^2 \operatorname{div} \boldsymbol{p} \\ &= (\operatorname{grad} u + u\, \boldsymbol{p})^2 + u^2 (\operatorname{div} \boldsymbol{p} - \boldsymbol{p}^2) \geqslant 0 ,\end{aligned}$$

whence, if we may integrate,

$$0 \leqslant D(u) + \oint_\Gamma u^2 \boldsymbol{p} \cdot \boldsymbol{n}\, ds = \oint_\Gamma \left\{ u \frac{\partial u}{\partial n} + u^2 \boldsymbol{p} \cdot \boldsymbol{n} \right\} ds = \oint_\Gamma \{\lambda_1 \varrho(s) - [k(s) - \boldsymbol{p} \cdot \boldsymbol{n}]\}\, u^2\, ds ;$$

the integrand cannot be everywhere negative, whence, if $\varrho(s) \geqslant 0$,

$$\lambda_1 \geqslant \inf_\Gamma \left\{ \frac{1}{\varrho(s)} [k(s) - \boldsymbol{p} \cdot \boldsymbol{n}] \right\}. \tag{5'''}$$

If we introduce the vector field $\boldsymbol{p} = -\operatorname{grad} u/u$, we have equality, whence

$$\lambda_1 = \operatorname{Max}_{div\, \boldsymbol{p} - \boldsymbol{p}^2 \geqslant 0} \inf_\Gamma \left\{ \frac{1}{\varrho(s)} [k(s) - \boldsymbol{p} \cdot \boldsymbol{n}] \right\}. \tag{5^{IV}}$$

5. We shall henceforth restrict our consideration to the case where $k(s) = +\infty$ along a 'fixed' part Γ_0 of the boundary Γ, while $k(s) = 0$ on the 'free' remaining boundary $\Gamma_1 = \Gamma - \Gamma_0$. It is then sufficient to consider only masses $\varrho(s)$ along Γ_1 (masses along Γ_0 cannot vibrate and therefore play no role in the problem).

$$\Delta u = 0 \ \text{ in } \ G, \quad u = 0 \ \text{ along } \ \Gamma_0, \quad \frac{\partial u}{\partial n} - \lambda\, \varrho(s)\, u = 0 \ \text{ along } \ \Gamma_1 .$$

We first consider a '*trilateral*' T, i.e. a Jordan domain with three designated boundary points. Let the Jordan arcs a, b, c be its 'sides'. We assume a mass distribution $\varrho(s)$ along c, but *no masses along a and b*. Total mass: $M_c = \int_c \varrho(s)\, ds$. *Let λ_a be the first eigenvalue of the problem with fixed side a ($k = \infty$), free sides b and c ($k = 0$):*

$$\Delta u = 0 \ \text{ in } \ T, \quad u = 0 \ \text{ along } \ a, \quad \frac{\partial u}{\partial n} = 0 \ \text{ along } \ b, \quad \frac{\partial u}{\partial n} - \lambda_a\, \varrho(s)\, u = 0 \ \text{ along } \ c .$$

Let λ_b be the first eigenvalue when b is fixed, a and c free.

We shall prove the isoperimetric inequality

$$\left(\frac{1}{\lambda_a} + \frac{1}{\lambda_b} \right) \frac{1}{M_c} \geqslant \frac{4}{\pi} . \tag{7}$$

Proof. – Let the given trilateral T be in the complex z-plane; we map it conformally on to the circular sector $\tilde{S}$: $\xi^2 + \eta^2 < 1$, $\xi > 0$, $\eta > 0$ of the $\zeta = \xi + i\,\eta$-plane in such a way that $\zeta(a)$ (the image of a) is the real interval $\tilde{a}$: $0 < \xi < 1$, $\zeta(b)$ the imaginary interval $\tilde{b}$: $0 < \eta < 1$, and $\zeta(c)$ the circular arc $\tilde{c}$: $\zeta = e^{i\alpha}$, $0 < \alpha < \pi/2$. This mapping is possible, since all trilaterals are conformally equivalent.

The corresponding problems in $\tilde{S}$ *with constant mass density* $\tilde{\varrho}(\alpha) \equiv 1$ along $\tilde{c}$, have respectively the first eigenfunctions $\tilde{u}_{\tilde{a}} = \eta$ and $\tilde{u}_{\tilde{b}} = \xi$, and the same corresponding first eigenvalue $\tilde{\lambda} = (1/\xi)\, \partial\xi/\partial r = (1/\xi)\, \xi/1 = 1$; here $M_{\tilde{c}} = \pi/2$, so that for $\tilde{S}$ we have equality in (7). We note that $\tilde{u}_{\tilde{a}}^2 = \tilde{u}_{\tilde{b}}^2 = \eta^2 + \xi^2 = 1$ along $\tilde{c}$, and $D(\tilde{u}_{\tilde{a}}) = D(\tilde{u}_{\tilde{b}}) = \pi/4$.

We now 'transplant' the functions $\tilde{u}_{\tilde{a}}$ and $\tilde{u}_{\tilde{b}}$ on to the given trilateral T: $U(z) = \tilde{u}(\zeta(z))$, i.e.

$$U_a(z) = \eta(z) \quad \text{and} \quad U_b(z) = \xi(z);$$

again $U_a^2 + U_b^2 = 1$ along c; *since Dirichlet's integral remains invariant under conformal transplantation*, $D(U_a) = D(U_b) = \pi/4$. [It follows also this way: $D(U_a) = D(\eta(z)) = \iint_T |d\zeta/dz|^2\, dA_z = \iint_{\tilde{S}} dA_\zeta = \pi/4$.]

Now we apply Rayleigh's principle twice:

$$\lambda_a \leqslant R[U_a] \quad \text{and} \quad \lambda_b \leqslant R[U_b], \quad \frac{1}{\lambda_a} + \frac{1}{\lambda_b} \geqslant \frac{4}{\pi} \int_c \varrho\, (U_a^2 + U_b^2)\, ds = \frac{4}{\pi} M_c;$$

this is (7).

It is readily seen that we have *equality* not only for the particular rectangular sector $\tilde{S}$, but *for all circular sectors* with constant mass density along the circular arc c; we even have equality for *any* trilateral $a\, b\, c$, provided the masses along c are those obtained by the (unique) conformal mapping onto $\tilde{S}$.

Inequality (7) can be compared with the more general one

$$(\lambda_a^{-1} + \lambda_b^{-1} + \lambda_c^{-1})\, M^{-1} \geqslant \frac{3}{\pi},$$

obtained in [7, 8] for all nonhomogeneous membranes on a trilateral $a\, b\, c$, of total mass M, fixed in turn along each one of the three 'sides' a, b, c (extremal membrane: homogeneous membrane on a trirectangular spherical triangle). In our present situation all the masses are on the boundary arc c, $\lambda_c = \infty$, hence $(\lambda_a^{-1} + \lambda_b^{-1})\, M^{-1} \geqslant 3/\pi$, which is less precise than our specific bound (7).

Example. – Consider the rectangle $-p$, O, $i\,\pi/2$, $-p + i\,\pi/2$ (in the complex z-plane) as a trilateral with the designated points $-p$, O, $i\,\pi/2$, the side a being the segment $-p$, O; c the segment O, $i\,\pi/2$; b the rest of the boundary; $\varrho \equiv 1$ along c, $M = \pi/2$. We have the *first* eigenfunctions $u_a = Ch\,(x + p) \sin y$ and $u_b = Sh\,(x + p) \cos y$ with the corresponding first eigenvalues $\lambda_a = Th\, p$ and $\lambda_b = Cth\, p$; $(\lambda_a^{-1} + \lambda_b^{-1})\, M^{-1} = (Cth\, p + Th\, p)\, 2/\pi \geqslant 4/\pi$ in agreement with (7). Asymptotically we have equality when $p \to \infty$; this is not astonishing: a half-strip can be considered as a circular sector with infinite radius and vanishing aperture.

6. We now consider the following mixed Stekloff problem on a *doubly connected domain* D of the z-plane, bounded by the Jordan curves Γ_0 and Γ_1:

$$\Delta u = 0 \;\text{ in }\; D, \quad u = 0 \;\text{ along }\; \Gamma_0, \quad \frac{\partial u}{\partial n} - \lambda\, \varrho(s)\, u = 0 \;\text{ along }\; \Gamma_1$$

(all masses are on Γ_1). Total mass: $M = \oint_{\Gamma_1} \varrho(s)\, ds$.

The *modulus* μ of D can be defined by conformal mapping $\zeta(z)$ on to a circular ring $\tilde{D}$: $1 < |\zeta| < R$, $\mu = (1/2\pi) \ln R$.

The corresponding problem in $\tilde{D}$ with $\tilde{\Gamma}_0$: $|\zeta| = 1$, $\tilde{\Gamma}_1$: $|\zeta| = R$, and constant mass density $\tilde{\varrho} \equiv 1$ along $\tilde{\Gamma}_1$, total mass $M = 2\pi R$, has the first eigenfunction $\tilde{u} = \ln|\zeta|$ and the first eigenvalue

$$\tilde{\lambda} = \frac{1}{R \ln R} = \frac{1}{2\pi R \mu} = \frac{1}{M\mu};$$

we shall prove, for the first eigenvalue $\lambda = \lambda_1$ of our arbitrary D, the inequality

$$\lambda \leqslant \frac{1}{M\mu}. \tag{8}$$

Proof. – We transplant $\tilde{u}$ on to D: $U(z) = \tilde{u}(\zeta(z))$; $U = 0$ along Γ_0 and $U = \ln R$ along Γ_1; since the Dirichlet integral remains invariant under conformal transplantation, $D(U) = D(\tilde{u}) = 2\pi \ln R$; by Rayleigh's principle,

$$\lambda \leqslant \frac{D(U)}{\oint\limits_{\Gamma_1} \varrho\, U^2\, ds} = \frac{2\pi \ln R}{(\ln R)^2 M} = \frac{1}{M\mu}, \quad \text{i.e. } (8).$$

Remark. – Inequality (8) also immediately follows from Rayleigh's principle applied to the harmonic function v in D satisfying $v = 0$ on Γ_0 and $v = 1$ on Γ_1: $D(v) = 1/\mu$, thus $\lambda \leqslant R[v] = D(v)/\oint_{\Gamma_1} \varrho\, v^2\, ds = 1/M\mu$. (In fact, $U = v \ln R$.)

We have *equality* for all circular rings with $\varrho = \text{const}$ along Γ_1, and even *for any doubly connected domain D with an adequate mass distribution* ϱ (obtained by conformal mapping onto the conformally equivalent circular ring $\tilde{D}$) along the free boundary curve Γ_1.

Inequality (8) can be compared with the more general one $(\lambda_{\Gamma_0}^{-1} + \lambda_{\Gamma_1}^{-1}) M^{-1} \geqslant 8\mu/\pi^2$, obtained in [6, 8] for all doubly connected nonhomogeneous membranes of modulus μ and total mass M, fixed alternatively along each of the boundary curves (extremal membrane: straight cylinder with homogeneous mass distribution). In our present situation, all the masses are on Γ_1, $\lambda_{\Gamma_1} = \infty$, hence $\lambda_{\Gamma_0} \leqslant \pi^2/8 M\mu$, which is less precise than our present specific bound (8).

Corollary. – It follows that the modulus μ of a given doubly connected domain D can be characterized as

$$\mu^{-1} = \mathrm{Max}_{\textit{choice of } \varrho(s) \textit{ along } \Gamma_1} (\lambda_1 M), \tag{9}$$

or in other words:

The inverse modulus μ^{-1} of a doubly connected domain with boundary curves Γ_0 and Γ_1, is equal to the largest total mass $M = \oint_{\Gamma_1} \varrho\, ds$ that Γ_1 can carry, while the first eigenvalue λ_1 of the mixed Stekloff problem (with fixed Γ_0) does not become inferior to unity.

This characterization is analogous to that of DE LA VALLÉE POUSSIN and FROSTMAN [4] for the Capacity in space as the least upper bound of the masses the set can carry while its potential does nowhere exceed unity. – *Contrarily to the usual ways of Potential theory,* we here give a characterization in terms of an *eigenvalue* (and not of an energy or a potential), and we consider the *Stekloff problem* (instead of a Dirichlet problem) to obtain theorems on harmonic functions.

It is worth mentioning that the restriction to everywhere positive masses $\varrho(s)$ can be dropped: let h be any positive harmonic function in D, vanishing along Γ_0: it is then the first eigenfunction to the mass density $\varrho = (1/h)\ \partial h/\partial n$ along Γ_1, with corresponding first eigenvalue $\lambda_1 = 1$. The density ϱ may well change sign; we have

$$\frac{1}{\mu} = \mathrm{Max}_{\substack{\Delta h = 0 \text{ in } D \\ h = 0 \text{ along } \Gamma_0 \\ h > 0 \text{ along } \Gamma_1}} \oint_{\Gamma_1} \frac{1}{h} \frac{\partial h}{\partial n}\, ds\,. \tag{9'}$$

This follows immediately from inequality (3), in which we set $u = h$ and introduce for v the harmonic function with boundary values zero on Γ_0 and one on Γ_1: then $D(v) = 1/\mu$. The Maximum is attained for $h = v$. – This variational problem is, of course, invariant under conformal mapping of the domain D.

The same is true in 3-space and can be used as a characterization of Capacity: let D be a three-dimensional domain with boundary surfaces Γ_0 and Γ_1; then

$$4\,\pi\, C = \mathrm{Max}_{\substack{\Delta h = 0 \text{ in } D \\ h = 0 \text{ along } \Gamma_0 \\ h > 0 \text{ along } \Gamma_1}} \iint_{\Gamma_1} \frac{1}{h} \frac{\partial h}{\partial n}\, dS\,. \tag{9''}$$

7. The following problem is closely related to the preceding one. We consider a '*quadrilateral*' Q in the z-plane, i.e. a Jordan domain with four designated boundary points; let a, b, c, d be its 'sides'. We consider the mixed Stekloff problem with fixed side a ($k = \infty$), free b, c, d ($k = 0$) and all masses $\varrho(s)$ along c ($\varrho = 0$ along a, b and d); side c lies opposite to side a.

$\Delta u = 0$ in Q, $u = 0$ on a, $\partial u/\partial n = 0$ along b and d, $\partial u/\partial n - \lambda\, \varrho(s)\, u = 0$ along c.
Total mass: $M_c = \int_c \varrho(s)\, ds$.

The modulus $\mu = \mu_{ac}$ of Q can be defined by conformal mapping onto a rectangle $\tilde{R}$ in the $\zeta = \xi + i\,\eta$ plane: $-p, 0, i\,q, -p + i\,q$; $\zeta(a) = \tilde{a}$ is the segment $-p + i\,q$, $-p$; $\tilde{b}$ the segment $-p, 0$; $\tilde{c}$ the segment $0, i\,q$; $\tilde{d}$ the segment $i\,q, -p + i\,q$; then $\mu = p/q$.

The corresponding mixed Stekloff problem in $\tilde{R}$ with $\tilde{\varrho} = 1$ along $\tilde{c}$, $M_{\tilde{c}} = q$, has first eigenfunction $\tilde{u} = \xi + p$ with the corresponding first eigenvalue $\tilde{\lambda} = 1/p = 1/M_{\tilde{c}}\,\mu$. We shall prove that, for our arbitrary quadrilateral Q and a mass density $\varrho(s)$ along c,

$$\lambda \leqslant \frac{1}{M_c\, \mu_{ac}}\,. \tag{8'}$$

Proof. – We transplant $\tilde{u}$ on to Q: $U(z) = \tilde{u}(\zeta(z)) = \xi(z) + p$; $D(U) = D(\tilde{u}) = p\,q$; by Rayleigh's principle, $\lambda \leqslant R[U] = p\,q/p^2 \int_c \varrho\, ds = q/p\, M_c = 1/M_c\,\mu$.

Remark. – As in Section 6, inequality (8') also immediately follows from Rayleigh's principle applied to the harmonic function v in Q which solves the mixed Dirichlet-Neumann problem: $v = 0$ along a, $v = 1$ along c, $\partial v/\partial n = 0$ along b and d: $D(v) = 1/\mu_{ac}$, thus $\lambda \leqslant R[v] = D(v)/\int_c \varrho\, v^2\, ds = 1/M_c\,\mu$. (In fact, $U = p\, v$.)

As is readily verified, we have *equality* for all rectangles and for all sectors of circular rings (c = circular arc), with ϱ = const along c; we even have equality for every quadrilateral with an appropriate mass density $\varrho(s)$ on c (obtained by conformal mapping onto $\tilde{R}$).

Inequality (8′) can be compared with the more general one $(\lambda_a^{-1} + \lambda_c^{-1})\, M^{-1} \geqslant 8\,\mu/\pi^2$, obtained in [6, 8] for all nonhomogeneous membranes of total mass M on a quadrilateral of modulus $\mu = \mu_{ac}$, fixed in turn along a and along c (extremal membrane: homogeneous rectangle). In our present situation, all the masses are on c, $\lambda_c = \infty$, hence $\lambda = \lambda_a \leqslant \pi^2/8\, M_c\, \mu$, which is less precise than our specific bound (8′).

Corollary. – As in Section 6, it follows that the modulus μ_{ac} of a given quadrilateral $a\,b\,c\,d$ can be characterized as

$$\mu_{ac}^{-1} = \mathrm{Max}_{\text{choice of } \varrho(s) \text{ along } c}\, (\lambda_1\, M_c) : \qquad (9''')$$

The inverse modulus μ_{ac}^{-1} *of a quadrilateral* $a\,b\,c\,d$ *is equal to the largest total mass* $M_c = \int_c \varrho\, ds$ *that the side* c *can carry while the first eigenvalue* λ_a *of the mixed Stekloff problem (with* a *fixed;* b, c *and* d *free) does not become inferior to unity.*

As we did in Section 6, we again note here that the restriction to everywhere positive masses can be dropped: the modulus μ_{ac} can be characterized by

$$\frac{1}{\mu_{ac}} = \mathrm{Max}_{\substack{\Delta h = 0 \text{ in } Q \\ \partial h/\partial n = 0 \text{ along } b \text{ and } d \\ h = 0 \text{ along } a \\ h > 0 \text{ along } c}} \int_c \frac{1}{h}\, \frac{\partial h}{\partial n}\, ds\,. \qquad (9^{IV})$$

This again follows from (3). The Maximum is attained by the functions h which are constant along c.

In particular, if the sides b and d are horizontal segments and a is a segment on the y-axis, we may choose $h = x$; since $\partial x/\partial n = \partial y/\partial s$, we obtain

$$\frac{1}{\mu_{ac}} \geqslant \int_c \frac{dy}{x}\,,$$

which is a well-known superadditivity property of moduli ([9], p. 608).

8. In an arbitrary quadrilateral Q with sides a, b, c, d and given mass distribution $\varrho(s)$ along c, we now consider two mixed Stekloff problems:

(i) $\Delta u = 0$ in Q, $u = 0$ along a and b, $\partial u/\partial n = 0$ along d, $\partial u/\partial n - \lambda\, \varrho(s)\, u = 0$ along c; first eigenvalue λ_{ab}.

(ii) $\Delta u = 0$ in Q, $u = 0$ along a and d, $\partial u/\partial n = 0$ along b, $\partial u/\partial n - \lambda\, \varrho(s)\, u = 0$ along c; first eigenvalue λ_{ad}.

In the conformally equivalent rectangle $\tilde{R}$ (see Section 7) with $q = \pi/2$, $\mu_{ac} = \mu_{\tilde{a}\tilde{c}} = p/q = 2\,p/\pi$ and $\tilde{\varrho} \equiv 1$ along $\tilde{c}$, $M_{\tilde{c}} = \pi/2$, we have the following first eigenfunctions of problems (i) and (ii): $\tilde{u}_{\tilde{a}\tilde{b}} = Sh\,(\xi + p)\,\sin\eta$ and $\tilde{u}_{\tilde{a}\tilde{d}} = Sh\,(\xi + p)\,\cos\eta$; $\tilde{u}_{\tilde{a}\tilde{b}}^2 + \tilde{u}_{\tilde{a}\tilde{d}}^2 = Sh^2\, p$ on $\tilde{c}$; $D(\tilde{u}_{\tilde{a}\tilde{b}}) = D(\tilde{u}_{\tilde{a}\tilde{d}}) = \pi/4 \int_{-p}^{0} [Sh^2\,(\xi + p) + Ch^2\,(\xi + p)]\, d\xi = \pi/4\, Sh\, p\, Ch\, p$; $\tilde{\lambda}_{\tilde{a}\tilde{b}} = \tilde{\lambda}_{\tilde{a}\tilde{d}} = Cth\, p = Cth\, \pi\, \mu_{\tilde{a}\tilde{c}}/2$; $(1/\tilde{\lambda}_{\tilde{a}\tilde{b}} + 1/\tilde{\lambda}_{\tilde{a}\tilde{d}})\, 1/M_{\tilde{c}} = (4/\pi)\, Th\, \pi\, \mu_{\tilde{a}\tilde{c}}/2$; we shall prove that, for our arbitrary quadrilateral Q,

$$\left(\frac{1}{\lambda_{ab}} + \frac{1}{\lambda_{ad}}\right) \frac{1}{M_c} \geqslant \frac{4}{\pi}\, Th\left(\frac{\pi}{2}\, \mu_{ac}\right). \qquad (10)$$

Proof. – We transplant $\tilde{u}_{\tilde{a}\tilde{b}}$ and $\tilde{u}_{\tilde{a}\tilde{d}}$ on to Q: $U_{ab}(z) = \tilde{u}_{\tilde{a}\tilde{b}}(\zeta(z))$ and $U_{ad}(z) = \tilde{u}_{\tilde{a}\tilde{d}}(\zeta(z))$; $U_{ab}(z)$ vanishes on a and b, $U_{ad}(z)$ on a and d; $D(U_{ab}) = D(\tilde{u}_{\tilde{a}\tilde{b}}) = (\pi/4)\, Sh\, p\, Ch\, p = D(\tilde{u}_{\tilde{a}\tilde{d}}) = D(U_{ad})$; $U_{ab}^2 + U_{ad}^2 = Sh^2\, p$ on c. We apply Rayleigh's principle twice:

$$\frac{1}{\lambda_{ab}} + \frac{1}{\lambda_{ad}} \geqslant \frac{1}{R[U_{ab}]} + \frac{1}{R[U_{ad}]} = \frac{M_c\, Sh^2\, p}{\frac{\pi}{4}\, Sh\, p\, Ch\, p} = \frac{4}{\pi} M_c\, Th\, p = \frac{4}{\pi} M_c\, Th\left(\frac{\pi}{2}\mu_{ac}\right).$$

Again, it is readily seen that we have *equality* for all rectangles and for all sectors of circular rings (c = circular arc), with ϱ = const along c; and even for every quadrilateral with appropriate mass distribution $\varrho(s)$ on c (obtained by conformal mapping on to $\tilde{R}$).

Inequality (10) can be compared with the more general one

$$(\lambda_{ab}^{-1} + \lambda_{bc}^{-1} + \lambda_{cd}^{-1} + \lambda_{da}^{-1})\, M^{-1} \geqslant 16\,\pi^{-2}\,(\mu + \mu^{-1})^{-1}\,,$$

obtained in [6, 8] for all quadrilateral nonhomogeneous membranes of modulus μ and total mass M, fixed in turn along each pair of adjacent sides (extremal membrane: homogeneous rectangle again). In our present situation, all the masses are on c, $\lambda_{bc} = \lambda_{cd} = \infty$, hence

$$(\lambda_{ab}^{-1} + \lambda_{ad}^{-1})\, M_c^{-1} \geqslant 16\,\pi^{-2}\,\mu\,(1 + \mu^2)^{-1}\,.$$

Since, whatever μ, there are Stekloff problems giving equality in (10), our specific bound (10) is necessarily sharper for all μ (> 0):

$$Th\left(\frac{\pi}{2}\mu\right) \geqslant \frac{4}{\pi}\,\frac{\mu}{1 + \mu^2}.$$

Limit cases:

(a) $\mu_{ac} \to \infty$: The side a disappears, the quadrilateral is reduced to the *trilateral* $b\,c\,d$ (with masses along c): (10) becomes $(\lambda_b^{-1} + \lambda_d^{-1})\, M_c^{-1} \geqslant 4/\pi$; *this is* (7) *of Section* 5.

Interpretation for the extremal domains: the rectangle becomes a half-strip = sector with angle zero; the sector of a circular ring with a = smaller circular arc, becomes a circular sector (extremal domain of Section 5).

(b) $\mu_{ac} \to 0$: If b and d both tend to disappear, we obtain asymptotically, from (10), the same inequality $2\,\lambda_a^{-1}\, M_c^{-1} \gtrsim 2\,\mu_{ac}$, i.e. $\lambda_a \lesssim 1/M_c\,\mu_{ac}$, as from (8′) in Section 7.

9. We now consider a quadrilateral Q (sides a, b, c, d) and given mass distributions $\varrho_a(s)$ along a and $\varrho_c(s)$ along c; $M_a = \int_a \varrho_a(s)\, ds$; $M_c = \int_c \varrho_c(s)\, ds$; $\varrho = 0$ along b and d.

Let λ_b be the first eigenvalue with $u = 0$ along b (fixed) and $\partial u/\partial n = 0$ along d (a, c, d free); and let λ_d be the first eigenvalue with the same mass distributions along a and c, but with d fixed and a, b, c free.

In the conformally equivalent rectangle $\tilde{R}$ with $q = \pi/2$ (see Sections 7 and 8: $\mu_{ac} = \mu_{\tilde{a}\tilde{c}} = p/q = 2\,p/\pi$) and with $\tilde{\varrho} \equiv \tilde{\alpha}$ = const along $\tilde{a}$ and $\tilde{\varrho} \equiv \tilde{\gamma}$ = const along $\tilde{c}$, we have the following first eigenfunctions of both problems:

$$\tilde{u}_{\tilde{b}} = Ch\,(\xi - \xi_0)\,\sin\eta\,, \quad \tilde{u}_{\tilde{d}} = Ch\,(\xi - \xi_0)\,\cos\eta\,,$$

where ξ_0 is determined by

$$\frac{-Th(-p-\xi_0)}{\tilde{\alpha}} = \frac{Th(-\xi_0)}{\tilde{\gamma}}, \text{ i.e.}$$

$$\frac{Sh(p+2\xi_0)+Sh\,p}{Sh(p+2\xi_0)-Sh\,p} = \frac{Sh(p+\xi_0)\,Ch\,\xi_0}{Sh\,\xi_0\,Ch(p+\xi_0)} = \frac{Th(p+\xi_0)}{Th\,\xi_0} = -\frac{\tilde{\alpha}}{\tilde{\gamma}},$$

whence

$$Sh(p+2\xi_0) = \frac{\tilde{\alpha}-\tilde{\gamma}}{\tilde{\alpha}+\tilde{\gamma}}\,Sh\,p \tag{11}$$

with the corresponding first eigenvalue $\tilde{\lambda} = Th(-\xi_0)/\tilde{\gamma}$.

We choose arbitrarily ξ_0 (i.e. $\tilde{\alpha}:\tilde{\gamma}$), we construct $\tilde{u}_{\tilde{b}}$ and $\tilde{u}_{\tilde{d}}$, and we transplant both those functions on to the given quadrilateral Q: $U_b(z) = \tilde{u}_{\tilde{b}}(\zeta(z))$, $U_d(z) = \tilde{u}_{\tilde{d}}(\zeta(z))$; $D(U_b) = D(\tilde{u}_{\tilde{b}}) = \pi/4\,Sh\,p\,Ch(p+2\xi_0) = D(\tilde{u}_{\tilde{d}}) = D(U_d)$. Along a, we have $U_b^2 + U_d^2 = \mathrm{Ch}^2(-p-\xi_0) = Ch^2(p+\xi_0)$; and along c, $U_b^2 + U_d^2 = Ch^2(-\xi_0) = Ch^2\,\xi_0$; whence, by Rayleigh's principle applied twice,

$$\frac{1}{\lambda_b} + \frac{1}{\lambda_d} \geqslant \frac{\int\limits_a \varrho_a U_b^2\,ds + \int\limits_c \varrho_c U_b^2\,ds}{D(U_b)} + \frac{\int\limits_a \varrho_a U_d^2\,ds + \int\limits_c \varrho_c U_d^2\,ds}{D(U_d)} \tag{12}$$

$$= \frac{M_a\,Ch^2(p+\xi_0) + M_c\,Ch^2\,\xi_0}{\frac{\pi}{4}\,Sh\,p\,Ch(p+2\xi_0)} = \frac{2}{\pi\,Sh\,p}\,\frac{M_a[1+Ch(2p+2\xi_0)] + M_c[1+Ch(2\xi_0)]}{Ch(p+2\xi_0)}.$$

Let us choose ξ_0 such that this bound be best possible; we obtain easily the optimal value $\hat{\xi}_0$:

$$Sh(p+2\hat{\xi}_0) = \frac{M_a - M_c}{M_a + M_c}\,Sh\,p; \tag{11'}$$

this is not astonishing: it corresponds to transplanting the eigenfunctions $\tilde{u}_{\tilde{b}}$ and $\tilde{u}_d$ of the rectangle $\tilde{R}$ *with* $\tilde{\alpha}:\tilde{\gamma} = M_a:M_c$, see (11); this rectangle realizes equality, our bound will therefore be exact.

When we introduce (11′) into (12), we obtain, decomposing

$$2p + 2\xi_0 = (p+2\xi_0) + p \quad\text{and}\quad 2\xi_0 = (p+2\xi_0) - p:$$

$$\frac{1}{\lambda_b} + \frac{1}{\lambda_d} \geqslant \frac{2}{\pi\,Sh\left(\frac{\pi}{2}\mu_{ac}\right)}\left\{(M_a+M_c)\,Ch\left(\frac{\pi}{2}\mu_{ac}\right) + \sqrt{(M_a+M_c)^2 + (M_a-M_c)^2\,Sh^2\left(\frac{\pi}{2}\mu_{ac}\right)}\right\}. \tag{12'}$$

(a) *Special case* $M_a = 0$:

$$\frac{1}{\lambda_b} + \frac{1}{\lambda_d} \geqslant \frac{4}{\pi}\,M_c\,Cth\left(\frac{\pi}{2}\mu_{ac}\right). \tag{12''}$$

Limit case $a \to$ point, i.e. $\mu_{ac} \to \infty$:

$$\frac{1}{\lambda_b} + \frac{1}{\lambda_d} \geqslant \frac{4}{\pi}\,M_c$$

we obtain again (7) of Section 5.

(b) *Special case* $M_a = M_c$:

$$\frac{1}{\lambda_b} + \frac{1}{\lambda_d} \geqslant \frac{4}{\pi}\,M_a\,Cth\left(\frac{\pi}{4}\mu_{ac}\right). \tag{12'''}$$

(c) If M_a and M_c are not known separately, but only the total mass $M_a + M_c = M$ is given, we obtain:

$$\frac{1}{\lambda_b} + \frac{1}{\lambda_d} \geqslant \frac{2}{\pi} M \, Cth \left(\frac{\pi}{4} \mu_{ac}\right); \tag{12^{IV}}$$

equality can here only be realized when $M_a = M_c = M/2$.

10. *Some Examples*

10.1. *An Application of Section 9*

We consider a mixed Stekloff problem in a given square of side s; as indicated in Figure 1 this square is to be considered as a quadrilateral such that all four 'sides' must have the same length.

Since in this case we have $\mu_{ac} = 1$, $M_a = M_c = s$ and $\lambda_b = \lambda_d$, it follows from (12‴) of Section 9 that

$$\lambda_b \leqslant \frac{\pi}{2\,s} \, Th \, \frac{\pi}{4},$$

with equality if the four designated points defining the quadrilateral coincide with the vertices of the square.

10.2. *An Application of Sections 4 and 7*

We consider a mixed Stekloff problem in the square $-1 \leqslant x, y \leqslant 1$ (Fig. 2). This square is to be considered as a quadrilateral *with the four designated points* $1, i, -1, -i$. The modulus μ of this quadrilateral is equal to $1/\mu$, therefore $\mu = 1$. The total mass is $M = M_c = \int_c \varrho \, ds = 2$. By (8′) of Section 7,

$$\lambda_1 \leqslant \frac{1}{M_c \, \mu} = \frac{1}{2}.$$

(*More generally, we have here an extremal property quite similar to the foregoing example 10.1.*)

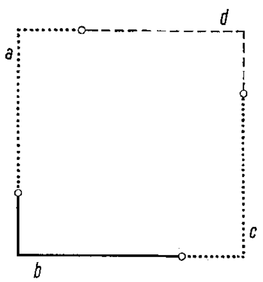

Figure 1

side b —— $k = \infty, \varrho = 0$; side d - - - $k = 0, \varrho = 0$
sides a . . . and c ····· $k = 0, \varrho = 1$

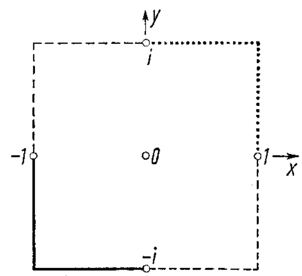

Figure 2

side a —— $k = \infty, \varrho = 0$; sides b and d - - - $k = 0, \varrho = 0$
side c ····· $k = 0, \varrho = 1$

A lower bound for λ_1 can be constructed, as indicated in Section 4, using e.g. the following simple functions $f(x, y)$ and $g(x, y)$:

square $x < 0, y < 0$: $f = A\,(x + 1)$; $g = A\,(y + 1)$;
square $x > 0, y < 0$: $f = B \cos[k\,(x - 1)]$; $g = C\,Ch[k\,(y + 1)]$;
square $x < 0, y > 0$: $f = C\,Ch[k\,(x + 1)]$; $g = B \cos[k\,(y - 1)]$;
square $x > 0, y > 0$: $f = D + E\,x$; $g = D + E\,y$;

for the x-continuity of f (and the y-continuity of g), we must have $A = B \cos k$ and $D = C\, Ch\, k$; for the x-continuity of f_x (and the y-continuity of g_y), we must have $A = B\, k \sin k$ and $E = C\, k\, Sh\, k$.

Since f and g should not vanish inside the square, A, B, C, D must not vanish, thus $k \operatorname{tg} k = 1$, $k \simeq 0.860_3$.

f and g satisfy $f_{xx}/f + g_{yy}/g = 0$, condition (4) is satisfied. We therefore have by (5):

$$\lambda_1 \geqslant \inf_c \left[\frac{f_x}{f}\frac{\partial x}{\partial n} + \frac{g_y}{g}\frac{\partial y}{\partial n}\right] = \frac{E}{D+E} = \frac{1}{1+(k\, Th\, k)^{-1}} \simeq 0.3746\,.$$

We have thus obtained

$$0.3746 \leqslant \lambda_1 \leqslant 0.5\,.$$

10.3. *An Application of Sections 4 and 5*

10.3.1. We consider a mixed Stekloff problem in the rectangle displayed in Figure 3. This rectangle is to be considered as a trilateral (in the sense of Section 5) with the three designated points 0, $2\,i\,q$ and $-p + i\,q$. One 'side' a of this trilateral is fixed, while side b is free and without masses, whereas side c (on the imaginary axis) is free and carries all masses. Total mass: $M = M_c = 2\,q$.

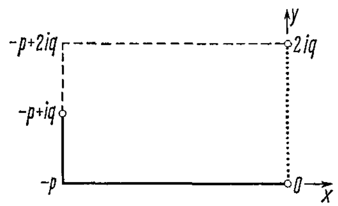

Figure 3

side a —— $k = \infty, \varrho = 0$; side b - - - $k = 0, \varrho = 0$; side c ····· $k = 0, \varrho = 1$

Limit case $p = \infty$ (half strip): First eigenfunction $u_1(x, y) = e^{\pi x/4q} \sin(\pi\, y/4\, q)$; corresponding first eigenvalue

$$\lambda_1 = \frac{1}{u^1}\frac{\partial u_1}{\partial x}\bigg|_{x=0} = \frac{\pi}{4\,q} \simeq \frac{0.7854}{q}\,.$$

10.3.2. *Elementary upper and lower bounds, due to monotony.*

(a) The modified problem: left segment $x = -p$ completely *fixed*, has higher eigenvalues, whence $\lambda_1 < (\pi/4\,q)\, Cth\, (\pi\, p/4\, q)$.

(b) The modified problem: left segment $x = -p$ completely *free*, has lower eigenvalues, whence $\lambda_1 > (\pi/4\,q)\, Th\, (\pi\, p/4\, q)$.

(c) The modified problem: the whole lower half-rectangle $y \leqslant q$ completely *fixed*, has higher eigenvalues, whence $\lambda_1 < (\pi/2\,q)\, Th\, (\pi\, p/2\, q) = \lambda_1^+$.

10.3.3. *Application of Section 5.*

Since we have clearly $\lambda_a = \lambda_b = \lambda_1$ and here $M_c = 2\,q$, our inequality (7) gives simply

$$\lambda_1 \leqslant \frac{\pi}{4\,q} \simeq \frac{0.7854}{q}$$

[a much sharper bound than 10.3.2(a)]. – Therefore we know: λ_1 *is a maximum when* $p = \infty$.

10.3.4. *Application of Section 4* (one-dimensional auxiliary problems).

We choose simple functions $f(x, y)$ and $g(x, y)$ in the following way: In the lower half-rectangle $y \leqslant q: f = Sh[\nu_1 (x + p)]$; $g = C_1 \sin(\nu_1 y)$; in the upper half-rectangle $y > q: f = Ch[\nu_2 (x + p)]$; $g = C_2 \cos[\nu_2 (2q - y)]$.

g and g_y must be continuous in y for $y = q$, whence

$$\nu_1 \operatorname{cotg}(\nu_1 q) = \nu_2 \operatorname{tg}(\nu_2 q).$$

Then, by (5) of Section 4,

$$\lambda_1 \geqslant \lambda_1^- = \min[\nu_1\, Cth(\nu_1 p); \quad \nu_2\, Th(\nu_2 p)];$$

a 'good' choice of ν_1 and ν_2 will realize $\nu_1\, Cth(\nu_1 p) = \nu_2\, Th(\nu_2 p)$: we then have

$$\frac{\nu_1}{\nu_2} = \operatorname{tg}(\nu_1 q)\operatorname{tg}(\nu_2 q) = Th(\nu_1 p)\, Th(\nu_2 p).$$

For given p and q, those two transcendental equations determine ν_1 and ν_2, whence the lower bound for λ_1; but in order to construct the diagram giving, for fixed q, a lower bound for λ_1 as a function of p, we choose the quotient ν_1/ν_2 as a parameter, we calculate ν_1 and then p, each one from a single transcendental equation.

ν_1/ν_2 chosen	p/q calculated	$q\,\lambda_1^-$ calculated
1	∞	$\pi/4 \simeq 0.7854$
1/2	1.5658	0.7284
0	1.3945	0.7171
i	1	0.688
$2i$	0.579	0.636
$3i$	0.380	0.553
$4i$	0.280	0.470
$\to i\infty$	$\to 0$	$(\pi^2/4)(p/q) + O(p^2/q^2) \to 0$

In conjunction with 10.3.2(c), this shows that, when $p/q \to 0$,

$$q\,\lambda_1 = \frac{\pi^2}{4}\,\frac{p}{q} + O\left(\frac{p^2}{q^2}\right);$$

i.e. in our diagram $q\,\lambda_1 = F(p/q)$ we know the exact tangent at the origin (see Fig. 4).

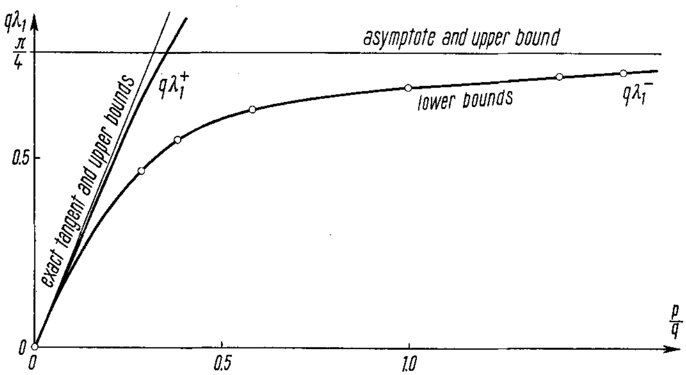

Figure 4

Upper bounds [from section 5 and from 10. 3. 2 (c)] and lower bounds (from section 4, see 10. 3. 4) for the example of Figure 3.

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Zusammenfassung

Das betrachtete Eigenwertproblem kann aufgefasst werden als dasjenige einer schwingenden Membran mit teilweise festem, teilweise freiem Rand, welche aber nicht im Innern, sondern auf dem freien Randteil Massen trägt. *Die Eigenfunktionen sind harmonisch:* das Stekloffsche Problem mit dem zugehörigen Rayleighschen Prinzip liefert für harmonische Funktionen andere Erkenntnisse als das Dirichletsche Problem mit dem Dirichletschen Prinzip [siehe insbesondere die Ungleichung (3)]. Isoperimetrische Ungleichungen werden durch konforme Abbildung auf ein Normalgebiet und Anwendung des Rayleighschen Prinzips auf die «verpflanzten» (siehe PÓLYA-SZEGÖ [14]) Eigenfunktionen des Normalgebietes hergeleitet.

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