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Plastic Buckling of Cylindrical Shells Subjected to External Fluid Pressure

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1. Introduction

It is well known that the instability in elastic solids occurs at a point of bifurcation or discrete branching point of the equilibrium path. In plastic solids, however, a point of bifurcation may be reached before an actual loss of stability. This difference arises from the fact that the elastic solid responds identically to loading and unloading, while the plastic solid has separate loading and unloading responses. As a result of the bilinear stress-strain law, it has only been possible to establish sufficient (but not necessary) conditions for uniqueness of the deformation mode in plastic solids. The criterion for uniqueness will therefore give the minimum load for which bifurcation is possible under prescribed boundary conditions. A familiar example is the buckling of straight columns where the bifurcation may occur when the load attains the tangent modulus value, although the loss of stability under dead loading cannot be expected below the reduced modulus value. Owing to geometrical imperfections and other uncertainties, an engineering structure will usually buckle at the minimum load for which the bifurcation is theoretically possible. As the deformation continues in the post-buckling range, the load will initially increase with increasing distortion.

The basic principles of uniqueness and stability in elastic/plastic solids are due to Hill [1], who derived sufficient conditions for these phenomena under dead loading. In a variety of practical problems, however, the loading is of pressure-type which has been treated in detail by Chakrabarty [2] for sufficiently general boundary conditions. The present paper mainly deals with the plastic buckling of thin cylindrical shells of arbitrary length when a gradually increasing external fluid pressure is applied to its cylindrical surface. The Ramberg-Osgood equation for the stress-strain curve is employed to present the results in a form that will be useful for practical purposes. In order to make the paper sufficiently self-contained, the relevant criterion for uniqueness for the conventional elastic/plastic solid is briefly discussed.

2. The Uniqueness Criterion

Consider a metallic body which is subjected to given traction and traction-rate on a part S_F and given velocity on a part S_v of the boundary, the remaining part S_f

being under uniform prescribed fluid pressure. Since the future position of an unconstrained boundary is not known in advance, it is convenient to formulate the boundary value problem in terms of the rate of change of the surface traction based on the initial configuration. The current state of the body is assumed to be completely known and this is taken to be the initial reference state in the field equations below.

Suppose that there are two possible solutions to the problem for given nominal traction-rate $\dot{F}_j$ on S_F, velocity v_j on S_v and fluid pressure rate $\dot{p}$ on S_f. If the material rate of change of the nominal stress is denoted by $\dot{s}_{ij}$, the condition of equilibrium may be expressed as

$$\frac{\partial}{\partial x_i}(\Delta \dot{s}_{ij})=0, \quad \Delta \dot{F}_j = l_i \Delta \dot{s}_{ij}, \tag{1}$$

where l_i is the unit normal to the initial surface element and x_i the initial co-ordinates. The prefix Δ denotes the difference between the corresponding quantities in the two possible solutions at bifurcation. In view of the given boundary conditions, $\Delta \dot{F}_j = 0$ on S_F and $\Delta v_j = 0$ on S_v. The application of Green's theorem for surface and volume integrals gives

$$\int \Delta \dot{F}_j \Delta v_j \, dS = \int \Delta \dot{s}_{ij} \frac{\partial}{\partial x_i}(\Delta v_j) \, dV. \tag{2}$$

The integrand on the left-hand side of the above equation vanishes on S_F and S_v, while on S_f we have [2]

$$\Delta \dot{F}_j = p \left[l_k \frac{\partial}{\partial x_j}(\Delta v_k) - l_j \frac{\partial}{\partial x_k}(\Delta v_k) \right] \tag{3}$$

since $\Delta \dot{p} = 0$ in view of the boundary condition. The condition for having two possible solutions may therefore be expressed as

$$\int \Delta \dot{s}_{ij} \frac{\partial}{\partial x_i}(\Delta v_j) \, dV - p \int \Delta v_j \left[l_k \frac{\partial}{\partial x_j}(\Delta v_k) - l_j \frac{\partial}{\partial x_k}(\Delta v_k) \right] dS_f = 0. \tag{4}$$

In the constitutive equation for the conventional elastic/plastic solid, the stress-rate must be expressed in Jaumann's sense [3]. It corresponds to the material rate of change of the true stress σ_{ij} with respect to axes which take part in the instantaneous rotation of the element. The Jaumann stress-rate $\mathring{\sigma}_{ij}$ is related to the nominal stress-rate $\dot{s}_{ij}$ by the equation

$$\dot{s}_{ij} = \mathring{\sigma}_{ij} + \sigma_{ij} \varepsilon_{kk} + \sigma_{ik} \omega_{jk} - \sigma_{jk} \varepsilon_{ik} \tag{5}$$

where ε_{ij} is the true strain-rate and ω_{ij} the anti-symmetric part of the velocity gradient tensor. It follows from the symmetry of σ_{ij} and ε_{ij} and anti-symmetry of ω_{ij} that

$$\begin{aligned} \dot{s}_{ij} \frac{\partial v_j}{\partial x_i} &= \dot{s}_{ij}(\varepsilon_{ji} + \omega_{ji}) \\ &= \mathring{\sigma}_{ij} \varepsilon_{ij} + \sigma_{ij}(\varepsilon_{kk} \varepsilon_{ij} + 2 \varepsilon_{jk} \omega_{ki} - \varepsilon_{jk} \varepsilon_{ki} - \omega_{jk} \omega_{ki}). \end{aligned} \tag{6}$$

From (3) and (6), a sufficient condition for uniqueness may be written as

$$\begin{aligned}&\int [\Delta\dot{\sigma}_{ij}\,\Delta\varepsilon_{ij}+\sigma_{ij}(\Delta\varepsilon_{kk}\,\Delta\varepsilon_{ij}+2\,\Delta\varepsilon_{jk}\,\Delta\omega_{ki}-\Delta\varepsilon_{jk}\,\Delta\varepsilon_{ki}-\Delta\omega_{jk}\,\Delta\omega_{ki})]\,dV\\&-p\int \Delta v_j[l_k(\Delta\varepsilon_{kj}+\Delta\omega_{kj})-l_j\,\Delta\varepsilon_{kk}]\,dS_f>0\end{aligned}\tag{7}$$

for all continuous differentiable fields Δv_j vanishing on S_v.

The constitutive law for the conventional plastic solid is such that the strain-rate is related to the stress-rate by separate linear equations for loading and unloading. Thus for isotropic solids, we have

$$\varepsilon_{ij}=\begin{cases}\dfrac{1}{2G}\left(\dot{\sigma}_{ij}-\dfrac{\nu}{1+\nu}\,\dot{\sigma}_{kk}\,\delta_{ij}\right)+\dfrac{3}{2H}\,\dot{\sigma}_{kl}\,n_{kl}\,n_{ij}, & \text{when } \dot{\sigma}_{kl}\,n_{kl}\geq 0, \quad (8\,\mathrm{a})\\[2ex] \dfrac{1}{2G}\left(\dot{\sigma}_{ij}-\dfrac{\nu}{1+\nu}\,\dot{\sigma}_{kk}\,\delta_{ij}\right), & \text{when } \dot{\sigma}_{kl}\,n_{kl}\leq 0 \quad (8\,\mathrm{b})\end{cases}$$

where H is the current slope of the true stress-plastic strain curve in uniaxial tension, G is the shear modulus and ν the Poisson ratio; n_{ij} is the outward drawn unit normal to the yield surface assumed to be regular. It is easily shown [2] that $\varepsilon_{ij}\,n_{ij}\gtrless 0$ for $\dot{\sigma}_{ij}\,n_{ij}\gtrless 0$. The scalar product of (8) with $\dot{\varepsilon}_{ij}$ gives

$$\dot{\sigma}_{ij}\,\varepsilon_{ij}=\begin{cases}2G\left[\varepsilon_{ij}\,\varepsilon_{ij}-\dfrac{3G}{3G+H}(\varepsilon_{ij}\,n_{ij})^2+\dfrac{\nu}{1-2\nu}\,\varepsilon_{kk}^2\right], & \varepsilon_{ij}\,n_{ij}\geq 0\\[2ex] 2G\left[\varepsilon_{ij}\,\varepsilon_{ij}+\dfrac{\nu}{1-2\nu}\,\varepsilon_{kk}^2\right], & \varepsilon_{ij}\,n_{ij}\leq 0.\end{cases}\tag{9}$$

Consider now a fictitious solid [4] in which the strain-rate is a linear function of the stress-rate as given by the first equation of (8) whatever the sign of $\dot{\sigma}_{kl}\,n_{kl}$. If $\dot{\tau}_{ij}$ denotes the stress-rate corresponding to the strain-rate ε_{ij} for the linearized elastic/plastic solid, it follows from (9) that

$$\dot{\sigma}_{ij}\,\varepsilon_{ij}\geq\dot{\tau}_{ij}\,\varepsilon_{ij}=2G\left[\varepsilon_{ij}\,\varepsilon_{ij}-\frac{3G}{3G+H}(\varepsilon_{ij}\,n_{ij})^2+\frac{\nu}{1-2\nu}\,\varepsilon_{kk}^2\right]\tag{10}$$

where the equality holds only in the loading part of the current plastic region.

Let $(\dot{\sigma}_{ij},\varepsilon_{ij})$ and $(\dot{\sigma}_{ij}^*,\varepsilon_{ij}^*)$ represent the two possible states at bifurcation in the actual elastic/plastic solid. If both these states correspond to unloading, $\Delta\varepsilon_{ij}$ is related to $\Delta\dot{\sigma}_{ij}$ by an equation of type (8 b) while $\Delta\varepsilon_{ij}$ is always related to $\Delta\dot{\tau}_{ij}$ by an equation of type (8 a). It follows from (9) that in this case $\Delta\dot{\sigma}_{ij}\,\Delta\varepsilon_{ij}>\Delta\dot{\tau}_{ij}\,\Delta\varepsilon_{ij}$. For all other possible combinations of loading and unloading, it can be shown by forming the scalar product of ε_{ij}^* with the relevant equation in (8) that

$$\dot{\sigma}_{ij}\,\varepsilon_{ij}^*\leq 2G\left[\varepsilon_{ij}\,\varepsilon_{ij}^*-\frac{3G}{3G+H}\,\varepsilon_{ij}\,n_{ij}\,\varepsilon_{kl}^*\,n_{kl}+\frac{\nu}{1-2\nu}\,\varepsilon_{kk}\,\varepsilon_{ii}^*\right].$$

This inequality, together with that in (10), leads to[1])

$$\Delta\dot{\sigma}_{ij}\,\Delta\varepsilon_{ij} \geq \Delta\dot{\tau}_{ij}\,\Delta\varepsilon_{ij} = 2\,G\left[\Delta\varepsilon_{ij}\,\Delta\varepsilon_{ij} - \frac{3\,G}{3\,G+H}(n_{ij}\,\Delta\varepsilon_{ij})^2 + \frac{\nu}{1-2\,\nu}(\Delta\varepsilon_{kk})^2\right] \tag{11}$$

where the equality holds when both states correspond to loading.

The sufficient condition for uniqueness of the linearized solid is obviously obtained by replacing $\Delta\dot{\sigma}_{ij}$ by $\Delta\dot{\tau}_{ij}$ in (7). The inequality (11) then shows that uniqueness of the linearized solid also ensures uniqueness of the non-linear elastic/plastic solid. If the constraints are rigid, so that $v_j = 0$ there, every difference field Δv_j is a member of the admissible field v_j. The uniqueness criterion for the linearized solid under these conditions becomes

$$\begin{aligned} &\int [\dot{\tau}_{ij}\,\varepsilon_{ij} + \sigma_{ij}(\varepsilon_{kk}\,\varepsilon_{ij} + 2\,\varepsilon_{jk}\,\omega_{ki} - \varepsilon_{jk}\,\varepsilon_{ki} - \omega_{jk}\,\omega_{ki})]\,dV \\ &- p\int [l_k(\varepsilon_{kj} + \omega_{kj}) - l_j\,\varepsilon_{kk}]\,v_j\,dS_f > 0 \end{aligned} \tag{12}$$

for all continuous differentiable fields vanishing on S_v. If the functional in (12) vanishes for some non-zero field v_j, bifurcation in the linearized solid may occur for any value of the traction-rate on S_F and $\dot{p}$ on S_f. In the actual elastic/plastic solid, however, bifurcation will only occur for those traction-rates for which there is no unloading of the current plastic region. In the buckling type of problems, the plastic modulus H is large in comparison with the components of σ_{ij} and the terms $\sigma_{ij}\,\varepsilon_{kk}\,\varepsilon_{ij}$ and $\sigma_{ij}\,\varepsilon_{jk}\,\varepsilon_{ki}$ may therefore be neglected. The bifurcation of the linearized elastic/plastic solid in this case corresponds to an eigenstate and the eigenfield makes the functional in (12) an absolute minimum. When curvilinear co-ordinates are employed, it is only necessary to interpret ε_{ij} etc. in (12) as the curvilinear components.

3. Cylindrical Shell under External Pressure

Consider a circular cylindrical shell of uniform thickness h subjected to uniform external pressure p on the cylindrical surface. If the length l of the shell is not sufficiently large in comparison with its radius a, the end conditions will have a significant effect on the critical pressure for buckling. At a generic point on the middle surface of the cylinder, consider a right-handed system of axes (x, θ, r) in which the x-axis is along the generator, θ-axis along the tangent to the circumference and r-axis along the outward normal to the surface. Let P be a point on the positive r-axis at a distance z from the middle surface. According to the customary thin shell theory, the components of the velocity vector $\boldsymbol{v}$ at P may be expressed as

$$v_x = u + z\,\omega_\theta, \qquad v_\theta = v - z\,\omega_x, \qquad v_r = w \tag{13}$$

[1]) An inequality identical to [11] has been discussed by the author for the special case of the rigid/plastic solid [5], although this has unfortunately been omitted in reference [2]. The author's attention to this omission has been subsequently drawn by R. Hill in a private communication.

where (u, v, w) are the velocities at the middle surface and $(\omega_x, \omega_\theta)$ the rates of rotation of the r-axis about the x and θ axes respectively measured in the positive sense. Within the order of approximation of the shell theory, $(\omega_x, \omega_\theta)$ are also the appropriate components of the spin-vector $\boldsymbol{\omega}=\frac{1}{2}\operatorname{curl}\boldsymbol{v}$. The components of $\boldsymbol{\omega}$ are expressible in terms of (u, v, w) as

$$\omega_x=\frac{1}{a}\left(\frac{\partial w}{\partial\theta}-v\right), \quad \omega_\theta=-\frac{\partial w}{\partial x}, \quad \omega_r=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{1}{a}\frac{\partial u}{\partial\theta}\right). \tag{14}$$

The velocity field (13) is adequate for calculating all strain-rates except the trough-thickness one which would follow from the stress-strain equations. Thus we have

$$\begin{aligned}
\varepsilon_{xx}&=\frac{\partial u}{\partial x}-z\frac{\partial^2 w}{\partial x^2}\\
\varepsilon_{\theta\theta}&=\frac{1}{a}\left(\frac{\partial v}{\partial\theta}+w\right)-\frac{z}{a^2}\frac{\partial}{\partial\theta}\left(\frac{\partial w}{\partial\theta}-v\right)\\
\varepsilon_{x\theta}&=\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{1}{a}\frac{\partial u}{\partial\theta}\right)-\frac{z}{a}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial\theta}-v\right).
\end{aligned} \tag{15}$$

The remaining shear strain-rates $\varepsilon_{r\theta}$ and ε_{rx} are identically zero, while the non-zero components of ω_{ij} are related to those of the vector $\boldsymbol{\omega}$ by the equations

$$\omega_{\theta x}=-\omega_{x\theta}=\omega_r, \quad \omega_{xr}=-\omega_{rx}=\omega_\theta, \quad \omega_{r\theta}=-\omega_{\theta r}=\omega_x.$$

The current state of stress is a uniaxial compression $\sigma_{\theta\theta}=-pa/h$ and the non-zero components of the unit normal to the yield surface are

$$n_{xx}=n_{rr}=\frac{1}{\sqrt{6}}, \quad n_{\theta\theta}=-\sqrt{\tfrac{2}{3}}.$$

The constitutive law for the linearized solid gives

$$\begin{aligned}
\varepsilon_{xx}&=\left(\frac{1}{E}+\frac{1}{4H}\right)\dot{\tau}_{xx}-\left(\frac{\nu}{E}+\frac{1}{2H}\right)\dot{\tau}_{\theta\theta},\\
\varepsilon_{\theta\theta}&=-\left(\frac{\nu}{E}+\frac{1}{2H}\right)\dot{\tau}_{xx}+\left(\frac{1}{E}+\frac{1}{H}\right)\dot{\tau}_{\theta\theta},\\
\varepsilon_{rr}&=\left(-\frac{\nu}{E}+\frac{1}{4H}\right)\dot{\tau}_{xx}-\left(\frac{\nu}{E}+\frac{1}{2H}\right)\dot{\tau}_{\theta\theta},\\
\varepsilon_{x\theta}&=\frac{1+\nu}{E}\dot{\tau}_{x\theta}
\end{aligned}$$

where E is Young's modulus. If we introduce the tangent modulus $T=EH/(E+H)$, a short calculation yields

$$\begin{aligned}\dot{\tau}_{ij}\,\varepsilon_{ij}&=\dot{\tau}_{xx}\,\varepsilon_{xx}+\dot{\tau}_{\theta\theta}\,\varepsilon_{\theta\theta}+2\,\dot{\tau}_{x\theta}\,\varepsilon_{x\theta}\\&=\frac{E}{1+\nu}(\alpha\,\varepsilon_{xx}^2+2\,\beta\,\varepsilon_{xx}\,\varepsilon_{\theta\theta}+\gamma\,\varepsilon_{\theta\theta}^2+2\,\varepsilon_{x\theta}^2)\end{aligned}\tag{16}$$

where the coefficients (α, β, γ) are

$$\begin{aligned}\alpha&=\frac{4(1+\nu)}{(5-4\,\nu)-(1-2\,\nu)^2\,T/E}\\\beta&=\frac{2(1+\nu)[1-(1-2\,\nu)\,T/E]}{(5-4\,\nu)-(1-2\,\nu)^2\,T/E}\\\gamma&=\frac{(1+\nu)(1+3\,T/E)}{(5-4\,\nu)-(1-2\,\nu)^2\,T/E}.\end{aligned}\tag{17}$$

Neglecting the small terms in the uniqueness criterion (12), as mentioned in the preceding section, the condition for uniqueness of the deformation mode of the cylinder may be written as

$$\iiint\left(1+\frac{z}{a}\right)\dot{\tau}_{ij}\,\varepsilon_{ij}\,dx\,d\theta\,dz-p\,a\iint(\omega_r^2+\omega_x^2-2\,\omega_r\,\varepsilon_{x\theta})\,dx\,d\theta$$
$$+p\iint[(\varepsilon_{xx}+\varepsilon_{\theta\theta})\,w+u\,\omega_\theta-v\,\omega_x]\,dx\,d\theta>0$$

to a sufficient accuracy (since p is small in comparison with E), the strain-rates appearing in the last two integrals being those for the middle surface. Substituting from (14), (15) and (16) into the above integrals and retaining terms that are consistent with the basic approximations [6], we have

$$\begin{aligned}&\iint\left[\alpha\left(\frac{\partial u}{\partial\zeta}\right)^2+2\,\beta\,\frac{\partial u}{\partial\zeta}\left(\frac{\partial v}{\partial\theta}+w\right)+\gamma\left(\frac{\partial v}{\partial\theta}+w\right)^2+\frac{1}{2}\left(\frac{\partial v}{\partial\zeta}+\frac{\partial u}{\partial\theta}\right)^2\right]d\zeta\,d\theta\\&+\frac{h^2}{12\,a^2}\iint\left[\alpha\left(\frac{\partial^2 w}{\partial\zeta^2}\right)^2+2\,\beta\,\frac{\partial^2 w}{\partial\zeta^2}\left(\frac{\partial^2 w}{\partial\theta^2}-\frac{\partial v}{\partial\theta}\right)+\gamma\left(\frac{\partial^2 w}{\partial\theta^2}-\frac{\partial v}{\partial\theta}\right)^2\right.\\&\left.+2\left(\frac{\partial^2 w}{\partial\zeta\,\partial\theta}-\frac{\partial v}{\partial\zeta}\right)^2\right]d\zeta\,d\theta-(1+\nu)\,\frac{p\,a}{E\,h}\iint\left[\left(\frac{\partial u}{\partial\theta}\right)^2+\left(\frac{\partial w}{\partial\theta}\right)^2\right.\\&\left.-w\left(\frac{\partial u}{\partial\zeta}+w\right)+u\,\frac{\partial w}{\partial\zeta}\right]d\zeta\,d\theta>0\end{aligned}\tag{18}$$

where $\zeta=x/a$. In the last integral, use has been made of the fact that the velocities are periodic functions of θ. The occurrence of bifurcation is marked by the vanishing of the above functional which is also a minimum with respect to variations of u, v and w. The Euler-Lagrange differential equations associated with this variational problem are easily derived as

$$\alpha \frac{\partial^2 u}{\partial \zeta^2} + \frac{1}{2} \frac{\partial^2 u}{\partial \theta^2} + (\beta + \tfrac{1}{2}) \frac{\partial^2 v}{\partial \zeta \, \partial \theta} + \beta \frac{\partial w}{\partial \zeta} + q \left(\frac{\partial w}{\partial \zeta} - \frac{\partial^2 u}{\partial \theta^2} \right) = 0, \tag{19}$$

$$\begin{aligned} &(\beta + \tfrac{1}{2}) \frac{\partial^2 u}{\partial \zeta \, \partial \theta} + \frac{1}{2} \frac{\partial^2 v}{\partial \zeta^2} + \gamma \left(\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial w}{\partial \theta} \right) \\ &+ k \left[2 \frac{\partial^2 v}{\partial \zeta^2} + \gamma \left(\frac{\partial^2 v}{\partial \theta^2} - \frac{\partial^3 w}{\partial \theta^3} \right) - (\beta + 2) \frac{\partial^3 w}{\partial \zeta^2 \, \partial \theta} \right] = 0, \end{aligned} \tag{20}$$

$$\begin{aligned} &\beta \frac{\partial u}{\partial \zeta} + \gamma \left(\frac{\partial v}{\partial \theta} + w \right) + q \left(\frac{\partial u}{\partial \zeta} + \frac{\partial^2 w}{\partial \theta^2} + w \right) \\ &+ k \left[-(\beta + 2) \frac{\partial^3 v}{\partial \zeta^2 \, \partial \theta} - \gamma \frac{\partial^3 v}{\partial \theta^3} + \alpha \frac{\partial^4 w}{\partial \zeta^4} + 2(\beta + 1) \frac{\partial^4 w}{\partial \zeta^2 \, \partial \theta^2} + \gamma \frac{\partial^4 w}{\partial \theta^4} \right] = 0 \end{aligned} \tag{21}$$

where we have introduced dimensionless parameters

$$q = (1 + v) \frac{p a}{E h}, \qquad k = \frac{h^2}{12 a^2}.$$

In the case of an elastic cylinder, $T = E$, $\alpha = \gamma = 1/(1 - v)$ and $\beta = v/(1 - v)$. Equations (19) to (21) then reduce to those given by Timoshenko [7] except for certain small order terms, the effects of which are insignificant in the final result. The above equations provide a generalization of the eigenvalue problem when buckling occurs in the plastic range.

4. Solution for a Simply Supported Shell

Consider now the special case when the ends of a shell are supported in such a way that $v = w = 0$ at $x = 0$ and $x = l$. The condition $\partial^2 w / \partial x^2 = 0$ at $x = 0$ and $x = l$ must also be satisfied for the linearized solid when the ends are simply supported. In actual practice, these conditions will prevent uniform deformation near the ends of the shell throughout the application of the pressure. Unless the shell is too short, the bending at the edges will be of local character, having no significant effect on the critical pressure. The virtual velocity field for bifurcation, satisfying these boundary conditions, may be taken in the form

$$\begin{aligned} u &= U \cos \lambda \zeta \cos m \theta \\ v &= V \sin \lambda \zeta \sin m \theta \\ w &= W \sin \lambda \zeta \cos m \theta \end{aligned} \tag{22}$$

where m is an integer and λ an integral multiple of $\pi a / l$; U, V and W are arbitrary constant velocities. Using this velocity field, we find that equations (19) to (21) are satisfied for all values of ζ and θ provided the following conditions are fulfilled:

$$\left[\left(\alpha\lambda^2+\frac{m^2}{2}\right)-q m^2\right] U-(\beta+\tfrac{1}{2})\lambda m V-(\beta+q)\lambda W=0 \qquad \text{(a)}$$

$$-(\beta+\tfrac{1}{2})\lambda m U+\left[\left(\frac{\lambda^2}{2}+\gamma m^2\right)+k(2\lambda^2+\gamma m^2)\right] V$$
$$+[\gamma m+k m\{(2+\beta)\lambda^2+\gamma m^2\}] W=0 \qquad \text{(b)} \qquad (23)$$

$$-(\beta+q)\lambda U+[\gamma m+k m\{(2+\beta)\lambda^2+\gamma m^2\}] V$$
$$+[\gamma-q(m^2-1)+k\{\alpha\lambda^4+2(1+\beta)\lambda^2 m^2+\gamma m^4\}] W=0. \qquad \text{(c)}$$

This system of homogeneous equations will have non-trivial solutions for U, V and W only if the determinant of their coefficient vanishes. It is interesting to note that the matrix of the determinant is symmetric. The determinantal equation is simplified by observing that q and k are small in comparison with unity so that the terms involving q^2, k^2 and qk may be neglected. The equation may therefore be written in the form

$$A+Bk=cq \qquad (24)$$

where

$$A=\delta\lambda^4, \quad \delta=\alpha\gamma-\beta^2=4(1+\nu)^2(T/E)/[(5-4\nu)-(1-2\nu)^2(T/E)], \qquad (25)$$

$$B=[\alpha\lambda^4+2(\delta-\beta)\lambda^2 m^2+\gamma m^4][\alpha\lambda^4+2(1+\beta)\lambda^2 m^2+\gamma m^4]$$
$$-2m^2[(2+\beta)\lambda^2+\gamma m^2][(2\delta-\beta)\lambda^2+\gamma m^2]+(2\lambda^2+\gamma m^2)(2\delta\lambda^2+\gamma m^2) \qquad (26)$$

$$C=(m^2-1)[\alpha\lambda^4+2(\delta-\beta)\lambda^2 m^2+\gamma m^4]+\lambda^2(2\beta\lambda^2-\gamma m^2). \qquad (26)'$$

Since the critical pressure given by (24) increases with λ the least pressure will correspond to $\lambda=\pi a/l$. If the length of the tube is greater than twice its diameter, the ratio λ^2/m^2 will be a small fraction and we may omit all terms containing λ^2 and λ^4 in the expressions for B and C. The equation for the critical pressure therefore becomes

$$q=\frac{\delta\lambda^4}{\gamma m^4(m^2-1)}+k\gamma(m^2-1)$$

or (27)

$$\frac{pa}{Eh}=\frac{4T/E}{1+3T/E}\,\frac{\pi^4 a^4}{m^4(m^2-1)l^4}+\frac{(m^2-1)(1+3T/E)}{(5-4\nu)-(1-2\nu)^2 T/E}\,\frac{h^2}{12a^2}$$

in view of (17) and (25). Equation (27) may be regarded as the true tangent modulus formula for the buckling of thin tubes under uniform external pressure. When the tube is very long, the first term on the right-hand side of the above equation may be neglected and the least value of the critical pressure then corresponds to $m=2$. The critical pressure in this particular case has been found earlier by Chakrabarty [2] and Dubey [8]. For shorter tubes, $m=3$ may give a lower value of the critical pressure than that given by $m=2$ and the condition for this to happen is

$$\frac{\pi^4 a^4}{l^4} > \frac{27E}{5T} \frac{(1+3T/E)^2}{(5-4\nu)-(1-2\nu)^2 T/E} \frac{h^2}{a^2}. \tag{28}$$

As the length of the tube further decreases, a value will be reached such that the critical pressure for still shorter tubes is lower for $m=4$ than for $m=3$. For exceptionally short tubes, however, the validity of equation (27) becomes doubtful.

It is interesting to note that the left-hand side of equation (27) is $1/E$ times the critical compressive stress σ. For a given stress-strain curve, T is a known function of σ and equation (27) can be solved by trial and error to obtain the critical stress and hence the critical pressure for bifurcation.

Substituting for (u, v, w) from (22) into the uniqueness criterion (18) and using the critical value of q given by (27), we find that the eigenfield that makes the functional vanish corresponds to

$$U=\lambda a \rho, \qquad V=m a \rho, \qquad W=\left(\frac{\beta}{\gamma}\lambda^2-m^2\right) a\rho \simeq -m^2 a \rho \tag{29}$$

where ρ is an arbitrary constant. It is evident that for a very long tube, U is negligible in comparison with V and W and the eigenmode corresponds very closely to plane strain [2].

As mentioned before the critical pressure for the actual elastic/plastic tube (with no imperfections) cannot be lower than that given by equation (27) derived for the linearized solid. However, the pressure at bifurcation in the case of the actual non-linear solid must continue to increase in such a way that there is no incipient unloading of the tube. This condition may be stated as $\varepsilon_{\theta\theta}<0$ to a close approximation, the velocity field being any linear combination of the above eigenfield and that corresponding to a uniform radial contraction. In view of (15), (22) and (29), the loading condition becomes

$$\frac{\dot{p}a}{Th}+\frac{z}{a}\rho m^2(m^2-1)\sin\frac{\pi x}{l}\cos m\theta>0.$$

This will be satisfied everywhere in the tube if

$$\frac{\dot{p}a}{Th} \geq m^2(m^2-1)\frac{\rho h}{2a} \tag{30}$$

where ρ is taken to be positive. The tube may therefore buckle in a range of possible ways when the pressure reaches the value given by (27) and the rate of change of pressure at the bifurcation is always positive. Similar results have been obtained by Batterman [9] for a thin spherical shell under external pressure, without the use of the uniqueness criterion.

5. Discussion of Results

Equation (27) is based on the assumption that the critical compressive stress exceeds the yield stress of the material. For thinner tubes, buckling will occur in the elastic range where $T=E$. The critical pressure for all possible tube geometrics can, however, be calculated from a single formula if the stress-strain curve is represented by the Ramberg-Osgood equation

$$\varepsilon=\frac{\sigma}{E}\left\{1+\frac{3}{7}\left(\frac{\sigma}{\sigma_0}\right)^{n-1}\right\} \tag{31}$$

where σ_0 and n are empirical constants. The curve predicted by this equation has an initial slope E and the secant modulus decreases to 70 per cent of its initial value when $\sigma=\sigma_0$, for all values of $n>1$. The tangent modulus according to this equation is given by

$$\frac{T}{E}=\left\{1+\frac{3n}{7}\left(\frac{\sigma}{\sigma_0}\right)^{n-1}\right\}^{-1}. \tag{32}$$

Thus for a given value of σ_0, the strain-hardening increases as the value of n decreases; the non-hardening material corresponds to $n=\infty$.

Inserting the above expression for T/E in equation (27) and noting that the left-hand side of this equation is σ/E, a relationship between the critical stress and the shell parameters is obtained. The resulting equation is most conveniently solved by assuming a value of σ/E for a given value of l/a and calculating the corresponding

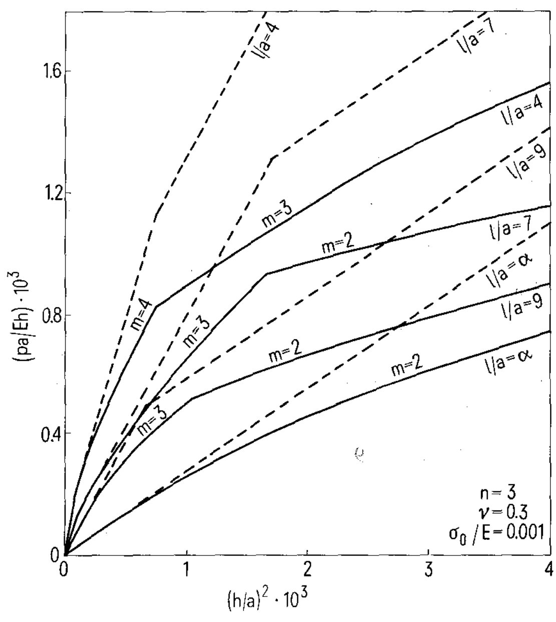

Figure 1
Variation of the critical stress with the shell thickness.

value of h^2/a^2. The calculation may be continued with an appropriate integral value of m, but a stage will be reached when m must be increased for lower values of σ/E to get the largest value of h^2/a^2. For very long tubes, of course, the value of m is always 2. Figure 1 shows the results of the calculation for several values of l/a, taking $n=3$, $\nu=0.3$ and $\sigma_0/E=0.001$. These curves appear as solid lines, while the broken lines correspond to the elastic solution. It is to be noted that for thicker tubes the solid curves are appreciably below the broken ones, indicating the occurrence of plastic buckling which is often experimentally observed.

A formula for the plastic buckling of thin tubes has been obtained earlier by Bijlaard [10] and Gerard [11], using the total strain theory of plasticity. Such calculations are of doubtful validity and no attempt has been made here to compare the present results with those of the above authors.

References

[1] R. Hill, *A General Theory of Uniqueness of Stability in Elastic/Plastic Solids*, J. Mech. Phys. Solids, *6*, 236 (1958).
[2] J. Chakrabarty, *On the Problem of Uniqueness under Pressure Loading*, Z. angew. Math. Phys., *20*, 696 (1969).
[3] W. Prager, *An Elementary Discussion of Definition of Stress Rate*, Quart. Appl. Math., *18*, 403 (1960).
[4] R. Hill, *Some Basic Principles in the Mechanics of Solids Without a Natural Time*, J. Mech. Phys. Solids, *1*, 209 (1959).
[5] J. Chakrabarty, *On Uniqueness and Stability in Rigid/Plastic Solids*, Int. J. Mech. Sci., *11*, 723 (1969).
[6] V. V. Novozhilov, *Thin Shell Theory*, p. 46 (Noordhoff, Groningen, 1964).
[7] S. Timoshenko, *Theory of Elastic Stability*, p. 476 (McGraw Hill, New York 1961).
[8] R. N. Dubey, *Instabilities in Thin Elastic/Plastic Tubes*, Int. J. Solids Struct., *5*, 699 (1969).
[9] P. Batterman, *Plastic Stability of Spherical Shells*, Trans. ASCE, Eng. Mech. Div., *95*, 433 (1969).
[10] P. P. Bijlaard, *Theory and Tests on the Plastic Stability of Plates and Shells*, J. Aero. Sci., *16*, 529 (1949).
[11] G. Gerard, *Plastic Stability Theory of Thin Shells*, J. Aero. Sci., *24*, 269 (1957).

Summary

The problem considered here is that of a thin-walled circular cylindrical shell whose external surface is submitted to uniform fluid pressure. The condition under which bifurcation will occur in the shell beyond the elastic limit is examined and the true tangent modulus formula for the plastic buckling is established. Numerical results are presented for the critical pressure, covering both elastic and plastic ranges of buckling.

Zusammenfassung

Einige Betrachtungen über das Verzweigungsproblem einer dünnen Kreiszylinderschale mit homogen verteiltem Aussendruck führen zu einem exakten Ausdruck für die kritische Last im elastisch-plastischen Bereich. Verschiedene numerische Resultate für die kritische Last im elastischen sowie im plastischen Bereich werden angegeben.

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