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On the Equations of Fully Fluidized Granular Materials

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1. Introduction

The analysis of the dynamical behaviour of granular materials has been one of the most interesting subjects in the fields of civil engineering and chemical engineering. However, granular materials are very complicated in structure and composition, and a consistent treatment of the dynamical problem is extremely difficult. To characterize the motion of granular materials, the rotational motion of particles [1, 2], the micromotion [3, 4] and the motion of void[5] have been introduced as the internal variables.

The aim of this paper is to describe the characteristic motion of granular materials in fully fluidized state. We observe that the dissipation processes for energy of fully fluidized granular materials differ from those of usual continua such as gas or liquid. In the case of gas or liquid the momentum transfer is produced by the thermal motion of molecules and there is the close relation between the coefficient of viscosity and the temperature. On the other hand, in the case of granular materials the momentum transfer occurs by particle collision, the scale of which is far larger than those of molecules, hence there is no direct relation between the momentum transfer process and the temperature. The energy corresponding to the random motion of particles will be converted into the thermal energy ultimately, and the relaxation time of this process will not be negligibly short, particularly in loose granular materials. In [6], we have introduced a new internal variable corresponding to the random motion of granular particles (quasi-thermal motion). In this paper we deduce the constitutive equations of random motion by using a simple kinematical model of collision of granular particles. Our constitutive equations have similar properties to those of fully developed turbulence and are coincident with Bagnold's relation [7, 8] when the assumption of local equilibrium for quasi-thermal motion is made. To investigate the characteristics of equations, steady and one-dimensional gravity flow is solved. Velocity profiles similar to those of experiments [9] are predicted by the theory.

2. Field Equations

To make the problem simple, neither the rotational motion of particles nor the motion of voids, etc., are considered in this paper. We introduce a new internal variable corresponding to the random motion of granular particles. Basic field equations are obtained by using the usual procedures [12].

The mass conservation equation is

$$\partial_t \rho + \partial_j (v^j \rho) = 0, \tag{1}$$

where ρ is the mass density and v^i the mean velocity, and abbreviations have been made use of:

$$\partial_t = \partial/\partial t \quad \text{and} \quad \partial_j = \partial/\partial x^j. \tag{2}$$

The equation of balance of momentum is

$$\rho(\partial_t v^i + v^j \partial_j v^i) = \partial_j \sigma^{ji} + \rho f^i, \tag{3}$$

where σ^{ij} and f^i are the stress tensor and the body force, respectively. The equation of balance of random motion energy is [6]

$$\rho(\partial_t e_\lambda + v^j \partial_j e_\lambda) = \sigma_\lambda^{ij} v_{(j,i)} - \partial_j q_\lambda^j - \gamma, \tag{4}$$

where e_λ is the specific energy of random motion and σ_λ^{ij} is the stress allotted to the production of random motion. q_λ^i and γ are the flux vector of random energy and the body sink of random energy, respectively. In statistical description, the specific energy of random motion e_λ will be given by $\langle \tilde{v}_\lambda \cdot \tilde{v}_\lambda \rangle/2$, where $\tilde{v}_\lambda$ is the random velocity or fluctuating velocity and $\langle\ \rangle$ denotes the ensemble average.

The equation of balance of internal energy is written in the similar form to Eqn. (3),

$$\rho(\partial_t e + v^j \partial_j e) = \sigma_h^{ij} v_{(j,i)} - \partial_j q_h^j + \gamma, \tag{5}$$

where e is the specific internal energy and σ_h^{ij} the dissipative stress and q_h^i the usual heat flow. Since the total mechanical work is $\sigma^{ij} v_{(j,i)}$, $\sigma^{ij} = \sigma_\lambda^{ij} + \sigma_h^{ij}$ is satisfied. In the above the energy of random motion is introduced as the transient energy state in the dissipation process of fully fluidized granular materials. The dissipation process of our material model is illustrated in Figure 1.

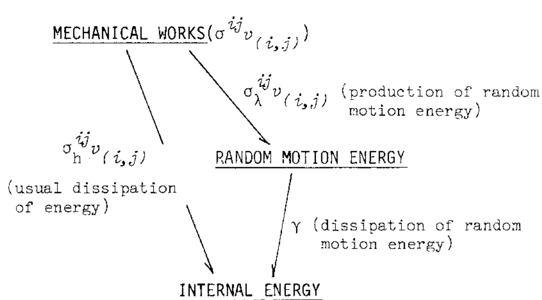

Figure 1
Energy dissipation process of fully fluidized granular materials.

3. Constitutive Equations

Let us obtain the constitutive equations by using a simple kinematical model of the microscopic behaviour of granular particles. The energy dissipation processes of granular materials are caused by the frictional force and the collisions of particles. The motion of particles will be very complicated and it is impossible to describe, hence the model shown in Figure 2 is used. A particle moving with the velocity $\tilde{v}_\lambda$ is considered and it is assumed to collide with its neighbouring particles, which are replaced by a spherical wall. The gain of kinetic energy of the particle considered is calculated and, by comparing with Eqn. (3), the constitutive equations are obtained.

Now, let us deduce the relation of velocities of the particle before and after the collision with the surface. Considering the usual granular materials contain adhesive materials as water films, a fraction $(1 - \alpha)$ of the granular particles is assumed to be reflected from the surface, while the fraction α adhere to the surface (similar parameters to α are used as the simplest theoretical models of interaction molecules with a surface in the kinetic theory of gases [15]). Then the velocity of the particle after the collision will be given as

$$\tilde{v}'_\lambda = \alpha\tilde{v}_\omega + (1 - \alpha)\tilde{v}''_\lambda, \tag{6}$$

where $\tilde{v}_\omega$ is the velocity of the surface and $\tilde{v}''_\lambda$ the velocity of reflected particles, which is determined as follows. Let $\tilde{v}_\lambda$ be the velocity of the particle and $\tilde{n}$ be the unit vector in the direction of the velocity (i.e., $\tilde{v}_\lambda = v_\lambda\tilde{n}$). The unit normal vector of the wall surface is denoted by $\tilde{\nu}$. In the reflection particles are assumed to slip in direction of wall surface. From the momentum conservation relation, the following will be satisfied. In $\tilde{\nu}$ direction

$$v_\lambda(\tilde{n}\cdot\tilde{\nu})\tilde{\nu} - \tilde{\nu}\int_0^{t_0} f(t)/m^*\,dt = (\tilde{v}''_\lambda\cdot\tilde{\nu})\tilde{\nu}, \tag{7}$$

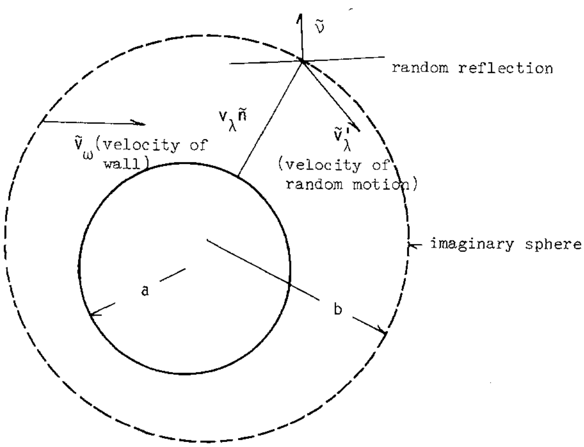

Figure 2
Kinematical model of collision of granular particles.

where a force $-\tilde{v}f(t)$ is assumed to act during impact $(0 < t < t_0)$. The component of $\tilde{v}_\lambda$ perpendicular to $\tilde{v}$ is given by $v_\lambda \tilde{n}\cdot(\boldsymbol{\delta} - \tilde{v}\tilde{v})$ and in $\tilde{n}\cdot(\boldsymbol{\delta} - \tilde{v}\tilde{v})$ direction the following equation will be satisfied

$$v\tilde{n}\cdot(\boldsymbol{\delta} - \tilde{v}\tilde{v}) - \tilde{n}\cdot(\boldsymbol{\delta} - \tilde{v}\tilde{v}) \int_0^{t_0} \mu f(t)/m^*\, dt = \tilde{v}''\cdot(\boldsymbol{\delta} - \tilde{v}\tilde{v}), \tag{8}$$

where the frictional force is assumed to be linear in $f(t)$ (Coulomb's friction law) and μ is the coefficient of friction. Since the function $f(t)$ is not determined without more microscopic consideration, we use the following empirical relation of collision

$$[(\tilde{v}_\lambda'' - \tilde{v}_\omega)\cdot\tilde{v}]/[(\tilde{v}_\lambda - \tilde{v}_\omega)\cdot\tilde{v}] = -\varepsilon, \tag{9}$$

where ε is the coefficient of restitution. Substituting the above relation into Eqn. (7), we obtain

$$\int_0^{t_0} f(t)/m^*\, dt = (1 + \varepsilon)(\tilde{v}_\lambda - \tilde{v}_\omega)\cdot\tilde{v}. \tag{10}$$

Summarizing Eqns. (7), (8) and (10), the velocity of the particle after a collision with the surface is given in the following form,

$$\tilde{v}_\lambda' = \alpha\tilde{v}_\omega + (1 - \alpha)\{\tilde{v}_\lambda - (1 + \varepsilon)[\tilde{v} + \mu\tilde{n}(\boldsymbol{\delta} - \tilde{v}\tilde{v})](v_\lambda\tilde{n}\cdot\tilde{v} - \tilde{v}_\omega\cdot\tilde{v})\}. \tag{11}$$

The random motion of granular particles will be generated only in shear field, hence the particle is assumed to be located in shear field. Let D_{ij} be the velocity gradient tensor of the field, then the relative velocity of the spherical wall surface to the center will be given by

$$\tilde{v}_\omega = \mathbf{D}\cdot(b\tilde{n}), \tag{12}$$

where b is the radius of an imaginary sphere. In the calculation, only the relative velocity is available, whence we may consider that the velocity of particle is $\tilde{v}_\lambda$ (random velocity) and the velocity of the spherical wall is given by Eqn. (12).

Now, let us calculate the gain of kinetic energy during a collision with the spherical wall. The direction of random velocity is assumed to be isotropic in the meaning of first approximation (the random velocity will have a preferred direction in the direction of flow, however the calculation including the effect of anisotropy is too difficult with the present level), and since the collision of particles will be the random reflection, the orientation of the wall $\tilde{v}$ is also assumed to be isotropic. Then, the gain of kinetic energy of one particle during a collision is calculated by

$$\Delta E_s = (1/2)m^*(\langle\langle\tilde{v}_\lambda'\cdot\tilde{v}_\lambda'\rangle_{\tilde{v}}\rangle_{\tilde{n}} - v_\lambda^2), \tag{13}$$

where m^* is the mass of particle (i.e., in this paper we consider the identical granular spheres), and $\langle\ \rangle_{\tilde{v}}$ denotes the statistical average with respect to $\tilde{v}$ which varies in the space of $\tilde{n}\cdot\tilde{v} \geq 0$, and $\langle\ \rangle_{\tilde{n}}$ is the statistical average with respect to $\tilde{n}$ which varies in whole space. The calculation of $\langle\langle\tilde{v}_\lambda'\cdot\tilde{v}_\lambda'\rangle_{\tilde{v}}\rangle_{\tilde{n}}$ is done as follows. Let ξ, ψ_1, ψ_2 and ψ_3 be the scalars defined by

$$\xi \equiv \tilde{n}\cdot\tilde{v}, \qquad \psi_1 \equiv \tilde{v}\cdot\mathbf{D}\cdot\tilde{n}, \qquad \psi_2 \equiv \tilde{n}\cdot\mathbf{D}\cdot\tilde{n}, \qquad \psi_3 \equiv \tilde{n}\cdot\mathbf{D}^{\tau}\cdot\mathbf{D}\cdot\tilde{n}, \tag{14}$$

then the velocity of the particles after collision is written as

$$\tilde{v}'_\lambda = \alpha b\mathbf{D}\cdot\tilde{n} + (1-\alpha)\{v_\lambda\tilde{n} - (1+\varepsilon)(v_\lambda\xi - b\psi_1)[\tilde{v} + \mu(\tilde{n} - \xi\tilde{v})]\}, \tag{15}$$

and the inner product $\tilde{v}'_\lambda\cdot\tilde{v}'_\lambda$ is

$$\begin{aligned}\tilde{v}_\lambda\cdot\tilde{v}_\lambda = {} & (\alpha b)^2\cdot\psi_3 + 2\alpha(1-\alpha)b \\ & \times\{v_\lambda\psi_2 - (1-\varepsilon)[(\xi\psi_1 + \mu\xi\psi_2 - \mu\xi^2\psi_1)v_\lambda - (\psi_1 + \mu\psi_2 - \mu\xi\psi_1)b\psi_1]\} \\ & + (1-\alpha)^2\{v_\lambda^2 - 2(1+\varepsilon)v_\lambda[\xi + \mu(1-\xi^2)](v_\lambda\xi - b\psi_1) \\ & + (1+\varepsilon)^2[1 + \mu^2(1-\xi^2)](v_\lambda^2\xi^2 - 2bv_\lambda\xi\psi_1 + b^2\psi_1^2)\}.\end{aligned} \tag{16}$$

In calculating the statistical average with respect to $\tilde{v}$, the following fundamental relations are used:

$$\begin{aligned}&\langle\xi^k\rangle_{\tilde{v}} = (1/2\pi)\cdot\int\xi^k\cdot d\omega = \int_0^{\pi/2}\cos^k\theta\cdot\sin\theta\,d\theta = 1/(k+1), \\ &\langle\xi^k\tilde{v}\rangle_{\tilde{v}} = 1/(k+2)\cdot\tilde{n}, \\ &\langle\xi^k\tilde{v}\tilde{v}\rangle_{\tilde{v}} = (\boldsymbol{\delta} + k\tilde{n}\tilde{n})/[(k+1)(k+3)],\end{aligned} \tag{17}$$

and

$$\begin{aligned}&\langle\xi^k\psi_1\rangle_{\tilde{v}} = \langle\xi^k\tilde{v}\rangle_{\tilde{v}}\cdot\mathbf{D}\cdot\tilde{n} = \psi_2/(k+1), \\ &\langle\xi^k\psi_1^2\rangle_{\tilde{v}} = \xi^k\tilde{v}\tilde{v}\rangle_{\tilde{v}}:(\mathbf{D}\cdot\tilde{n})(\mathbf{D}\cdot\tilde{n}) = (\psi_3 + k\psi_2^2)/[(k+1)(k+3)].\end{aligned}$$

Using these relations, the statistical average of $\tilde{v}'_\lambda\cdot\tilde{v}'_\lambda$ is calculated as

$$\begin{aligned}&\langle\tilde{v}'_\lambda\cdot\tilde{v}'_\lambda\rangle_{\tilde{v}} = (\alpha b)^2\psi_3 + 2\alpha(1-\alpha)b\cdot C_1(\tilde{n}) + (1-\alpha)^2\cdot C_2(\tilde{n}), \\ &C_1(\tilde{n}) \equiv v_\lambda\psi_2 - (1+\varepsilon)v_\lambda\psi_2(1/3 + \mu/4) + (1+e)b[\psi_3/3 + (3\psi_2^2 - \psi_3)/8], \\ &C_2(\tilde{n}) \equiv v_\lambda^2 - 2(1+\varepsilon)v_\lambda(1/3 + \mu/4)(v_\lambda - b\psi_2) \\ &\qquad + (1+\varepsilon)^2[v_\lambda(1/3 + 2\mu^2/15)(v_\lambda - 2b\psi_2) \\ &\qquad\qquad - (2/15)(\mu b\psi_2)^2 + (1/3 + 4\mu^2/15)b^2\psi_3].\end{aligned} \tag{18}$$

In calculating the statistical average with respect to $\tilde{n}$, the following fundamental relations are used:

$$\begin{aligned}&\langle 1\rangle_{\tilde{n}} = 1, \\ &\langle\tilde{n}\tilde{n}\rangle_{\tilde{n}} = \boldsymbol{\delta}/3, \\ &\langle\psi_2\rangle_{\tilde{n}} = D_1/3, \qquad D_1 \equiv D_{ii}, \\ &\langle\psi_3\rangle_{\tilde{n}} = D_2/3, \qquad D_2 \equiv D_{ij}D_{ij}, \\ &\langle\psi_3^2\rangle_{\tilde{n}} = (D_1^2 + 2D_2)/15,\end{aligned} \tag{19}$$

where $\mathbf{D}$ is assumed to be a symmetric tensor. Using the above relations, we have

$$\langle\langle \tilde{v}'_\lambda \cdot \tilde{v}'_\lambda \rangle_{\tilde{v}} \rangle_{\tilde{n}} = (\alpha b)^2 D_2 + 2\alpha(1-\alpha) b \bar{C}_1 + (1-\alpha)^2 \bar{C}_2,$$

$$\bar{C}_1 \equiv [1 - (1+\varepsilon)(1/3 + \mu/4)] v_\lambda D_1 + (1+\varepsilon)(1/9 + \mu/120) b D_2 + \mu b D_1^2/40,$$

$$\begin{aligned}\bar{C}_2 \equiv {} & [1 - 2(1+\varepsilon)(1/3+\mu/4) + (1+\varepsilon)^2(1/3 + 2\mu^2/15)] v_\lambda^2 \\ & + (2/3)(1+\varepsilon)[1/3 + \mu/4 - (1+\varepsilon)(1/3 + 2\mu^2/15)] b v_\lambda D_1 \\ & + (1/9)(1+\varepsilon)^2(1 + 16\mu^2/25) b^2 D_2 + \mu b^2 D_1/40.\end{aligned} \quad (20)$$

Now, let us consider the collision frequency of granular particles. How should we take the radius of the imaginary sphere? In usual cell model methods [10, 11], if N is the number density of particles, then the space allotted to one particle will be given by $1/N$, the radius of which will be $b = (3/4\pi N)^{1/3}$. However in case of the collisions of granular particles, we think that the radius determined by the above is meaningless because even in packed state, $b > a$ holds, where a is the particle radius. Therefore, we determine the radius of imaginary sphere by another method. It is natural to consider that the mean distance between the centers of neighbouring particles is proportionate to $(1/N)^{1/3}$, where N is the number density of particles. Since N is proportionate to the mass density ρ, the mean distance L can be written as

$$L = C_{sp}(1/\rho)^{1/3}, \quad (21)$$

where C_{sp} is the proportional constant determined by the statistical properties of the distribution of particles (e.g., in cubic array systems, $C_{sp} = (4\pi m^*/3)^{1/3} a$, where m^* and a are the mass and the radius of particle). In the flow of fully fluidized granular materials, a certain statistical distribution of particles will be realized, and we assume that the statistical distribution is not altered by the change of mass density ρ. Let ρ^* be the mass density of the packed state of granular materials, then the condition

$$2a \simeq C_{sp}(1/\rho^*)^{1/3} \quad (22)$$

will be satisfied as the particles touch with each other in packed state. Eliminating C_{sp} from (21), we obtain the formula

$$L \simeq 2a(\rho^*/\rho)^{1/3}. \quad (23)$$

We therefore take the radius of the imaginary sphere to be $L/2$. This selection satisfies the condition that as $\rho \to \rho^*$, then $b \to a$. Since the particle velocity is $\tilde{v}_\lambda$, the collision frequency of the particle is approximately given by $v_\lambda/2(b-a)$.

We can now calculate the total gain of energy of random motion. In a unit volume, N particles are contained and the absolute values of random velocities are assumed to be uniformly given by v_λ. Multiplying ΔE_s by the number density N and the collision frequency, the total gain of energy of random motion per unit time per unit mass is

$$\Delta E = (1/4)\rho v_\lambda/(b-a)[\langle\langle \tilde{v}_\lambda \cdot \tilde{v}_\lambda \rangle_{\tilde{v}} \rangle_{\tilde{n}} - v_\lambda^2], \quad (24)$$

where $\rho = m^*N$ was used. Substituting Eqn. (20), ΔE is obtained as

$$\Delta E = (1/4)\rho v_\lambda/(b - a)[K_0 v^2 + K_1 v D_{jj} + K_2 b^2 D_{ij} D_{ij} + K_3 b^2 (D_{jj})^2], \tag{25}$$

where K_0, K_1, K_2 and K_3 are the constants defined by

$$\begin{aligned} K_0 &\equiv (1 - \alpha)[1 - 2(1 + \varepsilon)(1/3 + \mu/4) + (1 + \varepsilon)^2(1/3 + 2\mu^2/15)] - 1, \\ K_1 &\equiv (2/3)(1 - \alpha)\{\alpha[1 - (1 - \varepsilon)(1/3 + \mu/4)] + (1 - \alpha) \\ &\qquad \times [(1 - \varepsilon)(1/3 + \mu/4) - (1 + \varepsilon)^2(1/3 + 2\mu^2/15)]\}, \\ K_2 &\equiv \alpha^2/3 + \alpha(1 - \alpha)(1 + \varepsilon)(2/9 + \mu/60) + (1 - \alpha)^2(1 + \varepsilon)^2(1/9 + 16\mu^2/225), \\ K_3 &\equiv (1/5)\mu(1 - \alpha)[\alpha/4 - (2/45)\mu(1 - \alpha)(1 + \varepsilon)^2]. \end{aligned} \tag{26}$$

The total gain of energy of random motion per unit time per unit mass may correspond to the right-hand side of Eqn. (4). Comparing the right-hand side of Eqn. (4) with ΔE, we obtain the following constitutive equations:

$$\sigma_{\lambda ij} = \Phi(\rho)[K_1 v_\lambda^2 \delta_{ij} + b(\rho) v_\lambda (K_2 v_{(i,j)} + K_3 v^s{}_{,s} \delta_{ij})], \tag{27}$$

$$\gamma = -K_0 \Phi(\rho) v_\lambda^3 / b(\rho), \tag{28}$$

where Φ and b are the functions of ρ and are defined by

$$\Phi(\rho) \equiv (1/4)\rho/[1 - (\rho/\rho^*)^{1/3}], \qquad b(\rho) \equiv a(\rho^*/\rho)^{1/3}. \tag{29}$$

We note that, in order to deduce the constitutive expression for flux vector of random energy, q_λ^i, we must consider another kinematical model, one which has the effects of random velocity gradient $\partial_j v_\lambda^i$ included.

4. Equations of Fully Fluidized Granular Materials

In the previous section we obtained approximate forms of the constitutive equations for σ_λ^{ij} and γ, which relate these quantities to the random motion of granular particles. In order to describe the motion of granular materials, we must investigate the form of dissipative stress σ_h^{ij}. In the case of packed granular materials we obtained [13], using a simple model in which the particles have been approximated by a series of parallel layers, the dissipative stress caused by the frictional force between the particles. In the case of fully fluidized granular materials the packing is loose and a model similar to that of dense packing model cannot be used. There are still some ambiguities at the present stage in modeling the complicated motion of granular materials, however, we think that the effects of the dissipative stress in fully fluidized state will be less than in the packed state. In this paper we neglect the dissipative stress, which corresponds to assuming that the dissipation process of energy occurs as follows:

Mechanical Work → Energy of Random Motion → Thermal Energy.

Substituting (27) and (28) into the conservation equations of Section 2, we obtain the following a set of governing equations for fully fluidized granular materials. The mass conservation equation is (1). The equation of balance of momentum is

$$\rho(\partial_t v^i + v^j \partial_j v^i) = -\partial_i p_\lambda + \partial_j \bar{\sigma}_\lambda^{ji} + \rho f^i, \tag{30}$$

where p_λ and $\bar{\sigma}_\lambda^{ij}$ are defined by

$$p_\lambda \equiv -K_1 \Phi v_\lambda^2, \tag{31}$$

$$\bar{\sigma}_\lambda^{ij} \equiv b\Phi v_\lambda (K_2 v_{(i,j)} + K_3 v^j_{,j} \delta^{ij}). \tag{32}$$

In Eqn. (4), the specific energy of random motion e_λ must be expressed in terms of the internal variable v_λ. Since the uniformity of random motion of particles has been assumed in calculating the constitutive equations, it will be natural to assume

$$e_\lambda = (1/2) v_\lambda^2, \tag{33}$$

then the equation of balance of random motion energy (4) is written as

$$\rho(\partial_t v_\lambda + v^j \partial_j v_\lambda) = \Phi[K_1 v_\lambda v^s_{,s} + b(K_2 v_{(m,n)} v_{(m,n)} + K_3 (v^s_{,s})^2 + K_0 v_\lambda^2 / b]. \tag{34}$$

In most problems the change of internal energy has negligible effects on the motion of materials, hence we neglect Eqn. (5) in our considerations.

We notice that these equations have the close resemblance with those of turbulence. In the turbulent theory (see, for example, [14]) the following facts are well known:

a) the mean dissipation of the turbulent energy per unit time per unit mass is $\varepsilon \sim (\Delta u)^3 / L$,
b) the turbulent viscosity is $\mu \sim \rho L \, \Delta u$,
c) the variation pressure is $\Delta p \sim \rho(\Delta u)^2$.

It may be easily shown that, if $b/(1 - \rho/\rho^*)^{1/3}$ and v_λ are replaced with L and Δu in our constitutive equations, similar relations hold in the present theory.

In the case of the local equilibrium of random motion ($\partial_t v_\lambda = \partial_i v_\lambda = 0$) and a simple shearing motion (i.e., $\partial_2 v^1 \neq 0$, all other components are zero), we obtain from Eqn. (34) that $v_\lambda \sim b(\partial_2 v^1)$.

Substituting this relation into (31) and (32), the pressure p_λ and the stress $\bar{\sigma}_\lambda^{ij}$ are

$$p_\lambda \sim b^2 (\partial_2 v^1)^2, \qquad \bar{\sigma}_\lambda^{ij} \sim b^2 (\partial_2 v^1)^2.$$

These relations suggest the same functional dependence as the semi-empirical constitutive equations obtained by Bagnold [7, 8].

5. Steady One-Dimensional Gravity Flow of Granular Materials

Let us consider the gravity flow of granular materials bounded below by the fixed granular bed inclined at an angle θ to the horizontal (Figure 3). Since the equations are

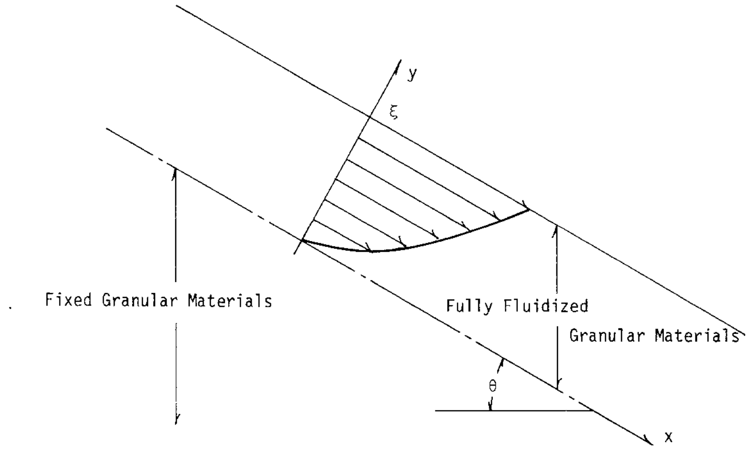

Figure 3
Steady one-dimensional gravity flow of granular materials.

complicated nonlinear partial differential equations, the solutions will not be easily obtained with the exception of some special cases. When the flow is assumed to be steady and parallel to the fixed plane, we can solve the equations.

In case of the steady and one-dimensional flow, $\partial_t = 0$ holds and the variables are the functions of y only. The conservation equation of mass (1) is satisfied automatically. The equation of balance of momentum is rewritten in the following forms:

$$(1/2)K_2\, d(\Phi b v_\lambda v_{x,y})/dy + \rho g \sin\theta = 0 \quad \text{(in the } x \text{ direction)}, \tag{35}$$

where $v_{x,y} \equiv dv_x/dy$, and g is the constant of the gravity force, and

$$K_1 d(\Phi v_\lambda^2)/dy - \rho g \cos\theta = 0 \quad \text{(in the } y \text{ direction)}. \tag{36}$$

The equation of balance of random motion is, from Eqn. (34),

$$(1/4)K_2 \Phi b (v_{x,y})^2 + K_0 \Phi v_\lambda^2/b = 0. \tag{37}$$

On substituting $v_{x,y}$ given by Eqn. (37) into Eqn. (35), the following equation is obtained:

$$d(\Phi v_\lambda^2)/dy = -\rho g \sin\theta/(|K_0|K_2)^{1/2}, \tag{38}$$

where $|K_0|$ is the absolute value of K_0, which always takes a negative value. In case of cohesionless granular particles ($\alpha \sim 0$), K_1 is also negative. Comparing Eqn. (38) with Eqn. (36), we see the solution of steady one-dimensional flow problem exists if and only if the angle θ takes the value defined by

$$\tan\theta^* = (|K_0|K_2)^{1/2}/|K_1|. \tag{39}$$

Let us obtain the solutions in case of $\theta = \theta^*$. The thickness of granular materials ξ is assumed to be constant. The boundary conditions are given at $y = 0$ and $y = \xi$. At

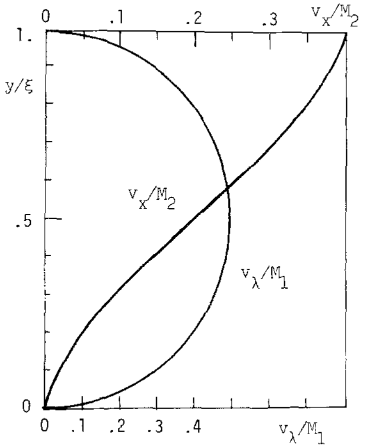

Figure 4
Profiles of the velocity and the random motion.

the free boundary $y = \xi$, we may put, neglecting the effects of air,

$$v_{x,y} = 0 \quad \text{at } y = \xi. \tag{40}$$

At the boundary $y = 0$, though there are some ambiguities, we put

$$v_x = 0 \quad \text{at } y = 0. \tag{41}$$

Since the profile of mass density $\rho(y)$ is not determined by the equations, we assume ρ is the linear function of y,

$$\rho = \rho^*[1 - (\Delta\rho/\Delta y)y], \tag{42}$$

where, at $y = 0$, $\rho = \rho^*$ is assumed and $-(\Delta\rho/\Delta y)$ is the density gradient in the direction of y. Then v_λ and v_x are obtained by Eqn. (36) and Eqn. (37):

$$v_\lambda = M_1[\bar{y}(1 - \bar{y})]^{1/2}, \tag{43}$$

$$v_x = M_2 \int_0^{\bar{y}} [\bar{y}(1 - \bar{y})]^{1/2}\, d\bar{y}, \tag{44}$$

where $\bar{y} \equiv y/\xi$, and M_1, M_2 are constants defined by

$$M_1 \equiv [g \cos \theta^*(\Delta\rho/\Delta y)/(3|K_1|)]^{1/2} \cdot \xi, \qquad M_2 \equiv (2/a)(|K_0|/K_2)^{1/2}\xi M_1. \tag{45}$$

In Figure 4, the solution of steady flow (Eqn. (43), (44)) is illustrated. The velocity profile has the close resemblance of those of experimental works [9].

Conclusions

We have so far investigated the equations of fully fluidized granular materials. To describe the characteristic motion of granular particles, we have introduced a new internal variable corresponding to the random motion. The constitutive equations relating to the random motion have been obtained by using a simple kinematical model of the microscopic behaviour of granular particles. Our constitutive equations have the similar properties of those of turbulence, and in the case of local equilibrium for the random motion, our constitutive equations coincide with Bagnold's relations. To investigate the characteristics of equations, steady and one-dimensional gravity flow has been solved and similar velocity profiles to those of experimental works have been obtained. We think that some aspects of complicated behaviour of granular materials are described by our equations.

We note that the analysis of the dynamical behaviour of granular materials has just started and many interesting developments can be expected in the future.

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Abstract

Equations for fully fluidized granular materials are proposed and are solved in a simple case. In fully fluidized granular materials, the granular particles slip or collide with each other and energy is dissipated. In describing the energy dissipation process characteristic to granular materials, a measure of random motion of granular particles is introduced as a new internal variable. We derive the constitutive equations by using a simple kinematical model of the collision of particles. The set of equations for fully fluidized granular materials obtained has properties similar to the equations that describe turbulence. For reasonable assumptions, these equations predict the results of Bagnold, namely that the shear and normal stress depend upon the square of the velocity gradient. In case of steady one-dimensional gravity flow the calculated flow profiles resemble experimental ones.

Résumé

Des équations pour des matériaux granulés entièrement fluides sont proposées et résolues dans un cas simple. Par le frottement et les collisions des particules entre elles, de l'énergie est dissipée. Pour décrire l'énergie de dissipation, on introduit une mesure du mouvement aléatoire des particules comme nouvelle variable intense. Un module cinématique simple de la collision des particules permet d'établir les équations. Ces équations ont des propriétés semblables aux équations de la turbulence. Sous des hypothèses raisonnables, elles prédisent les résultats de Bagnold, à savoir que l'abrasion et la tension normale dépendent du gradient de la vitesse. Pour un flux de granité stable, les profils de flux calculés ressemblent à ceux obtenus expérimentalement.

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