Journal of Applied Mathematics and Physics (ZAMP)
Vol. 31, 1980 Birkhäuser Verlag Basel

Applications of Variational Principles in Finite Elasticity

By Sang J. Lee[1]) and Richard T. Shield, Dept of Theoretical and Applied Mechanics, University of Illinois at Urbana-Champaign, Urbana, Illinois, USA

1. Introduction

In [1], a variational principle of the complementary energy type was developed for finitely deformed elastic bodies. Under certain restrictions, the principle can be used in conjunction with the potential energy principle to provide bounds on the potential energy, and for some deformations the bounds can lead to estimates for overall quantities of direct physical interest.

The problem of the all-around extension of a plane sheet with a circular hole has been treated by Rivlin and Thomas [2] who solved the problem numerically and compared their solution to experimental results. The problem is one-dimensional and most efficiently treated by a direct numerical approach, but it is considered here in order to provide experience for more complex problems. In Section 2, the variational principles are used to obtain bounds for the total strain energy of the deformed sheet for various values of the overall extension ratio for particular strain energy functions. The approximate solution of Wong and Shield [3] is used to generate trial functions, and accurate estimates for the stress resultant at the outer edge of the sheet are obtained.

Section 3 considers the large extension and torsion of a long elastic cylinder which is bonded at the ends to rigid plates. An approach is described for estimating the resultant end loads through the use of variational principles. As an illustration, a neo-Hookean cylinder with an elliptical cross section is considered. Green and Shield [4] have determined second-order effects in the torsion of a finitely extended cylinder for an incompressible Mooney material which includes the neo-Hookean material as a special case. The results in [4] suggest forms to be used for the trial functions. Accurate estimates for the twisting moment and the axial force are obtained for elliptical cylinders with axes in the ratios of 2:1 and 4:1 for a wide range of extension and twist.

2. All-Around Extension of a Plane Sheet with a Circular Hole

We suppose that a plane sheet of uniform thickness h_0 occupies the annular region $a \leq r \leq A$ in the reference state, where r, θ are polar coordinates. We assume that the material is incompressible, homogeneous, and isotropic. With the major surfaces and

[1]) Now at Mead Office Systems, Richardson, Texas, 75081, USA.

the edge $r = a$ of the hole traction free, the sheet is subject to a uniform radial displacement at $r = A$, so that a particle initially at (A, θ) is displaced to $(\mu A, \theta)$, where μ is a constant ($\mu > 1$). Under these conditions, a particle initially at (r, θ) in the middle plane is displaced to the point (ρ, θ), where ρ is a function of r only.

We use the membrane approximation and ignore variations in the deformation gradients throughout the thickness. From symmetry considerations, the principal directions of strain are radial, circumferential, and normal to the middle plane. Denoting the principal extension ratios in these directions by λ_1, λ_2, and λ_3, respectively, we have

$$\lambda_1 = \frac{d\rho}{dr}, \qquad \lambda_2 = \frac{\rho}{r}, \qquad \lambda = \lambda_3 = \frac{1}{\lambda_1 \lambda_2}. \tag{2.1}$$

and λ_1, λ_2 and λ_3 are functions of r only. The principal strain invariants I_1, I_2 are

$$I_1 = \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2 \lambda_2^2}, \qquad I_2 = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \lambda_1^2 \lambda_2^2.$$

We first assume the neo-Hookean form for the strain energy function. The total strain energy $U(\mu)$ is given by the integral

$$F = 2\pi h_0 C_1 \int_a^A (I_1 - 3) r \, dr,$$

where C_1 is a material constant, and the Euler differential equation associated with the functional F provides the equilibrium equation

$$\frac{d}{dr}\left[r\left(\lambda_1 - \frac{1}{\lambda_1^3 \lambda_2^2}\right)\right] - \lambda_2 + \frac{1}{\lambda_1^2 \lambda_2^3} = 0 \quad \text{in } a \le r \le A. \tag{2.2}$$

The boundary conditions are

$$T = 2h_0 C_1\left(\lambda_1 - \frac{1}{\lambda_1^3 \lambda_2^2}\right) = 0 \quad \text{at } r = a, \qquad \rho = \mu A \quad \text{at } r = A, \tag{2.3}$$

where T is the radial stress resultant measured per unit length of the undeformed middle plane.

For the present case, the functional Φ in $(3.14)^{\dagger}$ [2]) becomes

$$\Phi = 2\pi h_0 C_1 \int_a^A \left\{\lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2 \lambda_2^2} - 3\right\} r \, dr, \tag{2.4}$$

in which λ_1, λ_2 are quantities obtained through (2.1) from a trial function ρ satisfying the second condition in (2.3). This gives an upper bound for $U(\mu)$ provided the inequality in $(3.16)^{\dagger}$ holds. When the terms $1/\lambda_1^3 \lambda_2^2$, $1/\lambda_1^2 \lambda_2^3$ in (2.2) and (2.3) are much smaller than unity, the solution is very close to [3]

$$\rho = \mu r(1 + a^2/r^2)/(1 + a^2/A^2). \tag{2.5}$$

[2]) Numbers marked with a dagger (†) refer to equations and inequalities in [1].

This suggests that we choose the trial function to be

$$\rho = \mu r\left[1 + c\left(\frac{1}{r^2} - \frac{1}{A^2}\right)\right],$$

where c is a parameter and c is required to satisfy

$$0 < c < A^2/(1 + A^2/a^2)$$

in order to have $\lambda = 1/\lambda_1\lambda_2$ finite in $a \leq r \leq A$. Substituting in (2.4) and evaluating the integral, we obtain

$$\begin{aligned}\frac{1}{2\pi h_0 A^2 C_1}\Phi = \mu^2\left(1 - \frac{1}{\alpha^2}\right)\left[\left(1 - \frac{C}{\alpha^2}\right)^2 + \frac{C^2}{\alpha^2}\right] - \frac{3}{2}\left(1 - \frac{1}{\alpha^2}\right) + \frac{\alpha^6}{2\mu^4(\alpha^2 - C)^4}\\ \times\left[(\alpha^2 - 1) - \frac{C^2}{2(\alpha^2 - 2C)} - \frac{\alpha^4 C^2}{2[(\alpha^4 - 1)C - \alpha^2(\alpha^2 - 2C)]}\right.\\ \left. + \frac{3\alpha^2 C}{4(\alpha^2 - C)}\ln\left\{\left(1 - \frac{2C}{\alpha^2}\right)\left[\frac{(\alpha^2 - 1)C + \alpha^2}{-(\alpha^2 + 1)C + \alpha^2}\right]\right\}\right],\end{aligned} \tag{2.6}$$

where we have set $\alpha = A/a$, $C = c/a^2$. The parameter C is to be chosen so that the functional Φ is minimized.

With the change of variable $\zeta = r/a$, the functional Ψ in $(3.17)^\dagger$ becomes

$$\frac{1}{2\pi h_0 A^2 C_1}\Psi = \frac{1}{\alpha^2}\int_1^\alpha \left\{\lambda_1^2 + \lambda_2^2 - \frac{5}{\lambda_1^2\lambda_2^2} + 3\right\}\zeta\, d\zeta - 2\mu\left(\lambda_1 - \frac{1}{\lambda_1^3\lambda_2^2}\right)_{\zeta=\alpha}, \tag{2.7}$$

in which λ_1, λ_2 are required to satisfy (2.2) and the boundary condition

$$\lambda_1^2\lambda_2 = 1 \quad \text{at } \zeta = 1.$$

This leads to a lower bound for $U(\mu)$ provided the inequality in $(3.19)^\dagger$ holds. Note that λ_1, λ_2 are not necessarily quantities derived from a function ρ through (2.1). The approximate solution (2.5) suggests that we take

$$\lambda_1 - \frac{1}{\lambda_1^3\lambda_2^2} = K\left(1 - \frac{1}{\zeta^2}\right), \qquad \lambda_2 - \frac{1}{\lambda_1^2\lambda_2^3} = K\left(1 + \frac{1}{\zeta^2}\right), \tag{2.8}$$

where $K > 0$ is a parameter, and the equilibrium equation and the boundary condition are then satisfied. With the forms (2.8), the integral in (2.7) can be transformed into one using $\lambda = 1/\lambda_1\lambda_2$ as the variable of integration. The details can be found in [5].

It can be shown [5] that the second variations of Φ and Ψ, given by $(3.16)^\dagger$ and $(3.19)^\dagger$, are positive definite for radial deformations of the neo-Hookean sheet. Hence the functions Φ and Ψ have local minima at the actual extension ratios λ_1, λ_2 and Φ and $-\Psi$ provide upper and lower bounds on the total strain energy $U(\mu)$, as indicated in $(3.20)^\dagger$. In a small change $\delta\mu$, the change in the total strain energy $U(\mu)$ is equal to the work of the stress resultant ar $r = A$,

$$\delta U = 2\pi\mu A^2 S\,\delta\mu,$$

where S is the stress resultant measured per unit length of the deformed middle plane. It follows that

$$S = \frac{1}{2\pi\mu A^2}\frac{dU}{d\mu}. \tag{2.9}$$

For $I_1 < 12$ and $4 < I_2 < 100$, the strain energy function

$$W = C_1\{(I_1 - 3) + k(I_2^N - 3^N)/N\},$$

where $C_1 = 0.181 MPa$, $k = 0.077$, and $N = 0.5$, approximates the empirical strain energy function derived by Shield [6] from experimental results of Treloar [7]. For this strain energy function, the functional Φ becomes

$$\frac{1}{2\pi h_0 A^2 C_1}\Phi = \frac{1}{\alpha^2}\int_1^\alpha \left\{\lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2} - 3 + \frac{k}{N}\left[\left(\lambda_1^2\lambda_2^2 + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_2^2}\right)^N - 3^N\right]\right\}\zeta\, d\zeta,$$

where λ_1 and λ_2 are derived from a trial function ρ such that $\rho = \mu\alpha$ at $\zeta = \alpha$. We choose the same trial function as that for the neo-Hookean sheet and after some calculation, we obtain

$$\begin{aligned}
\frac{1}{2\pi h_0 A^2 C_1}\Phi = {} & \mu\left(1 - \frac{1}{\alpha^2}\right)\left[\left(1 - \frac{C}{\alpha^2}\right)^2 + \frac{C^2}{\alpha^2}\right] - \frac{3}{2}\left(1 - \frac{1}{\alpha^2}\right) + \frac{\alpha^6}{2\mu^4(\alpha^2 - C)^4} \\
& \times \left[(\alpha^2 - 1) - \frac{C^2}{2(\alpha^2 - 2C)} - \frac{\alpha^4 C^2}{2[(\alpha^4 - 1)C^2 - \alpha^2(\alpha^2 - 2C)]}\right. \\
& \left. + \frac{3\alpha^2 C}{4(\alpha^2 - C)}\ln\left\{\left(1 - \frac{2C}{\alpha^2}\right)\left[\frac{(\alpha^2 - 1)C + \alpha^2}{-(1 + \alpha^2)C + \alpha^2}\right]\right\}\right] - \frac{k3^N}{2\alpha^2 N}(\alpha^2 - 1) \\
& + \frac{k}{\alpha^2 N}\int_1^\alpha \left\{\mu^4\left(\frac{\alpha^2 - C}{\alpha^2}\right)^4\left[1 - \frac{\alpha^4 C^2}{(\alpha^2 - C)^2\zeta^4}\right]^2 + \frac{\alpha^4}{\mu^2(\alpha^2 - C)^2}\right. \\
& \left. \times \left[\left(1 - \frac{\alpha^2 C}{(\alpha^2 - C)\zeta^2}\right)^{-2} + \left(1 + \frac{\alpha^2 C}{(\alpha^2 - C)\zeta^2}\right)^{-2}\right]\right\}^N \zeta\, d\zeta.
\end{aligned} \tag{2.10}$$

The functional Ψ becomes

$$\begin{aligned}
\frac{1}{2\pi h_0 A^2 C_1}\Psi = {} & \frac{1}{\alpha^2}\int_1^\alpha \left\{\lambda_1^2 + \lambda_2^2 - \frac{5}{\lambda_1^2\lambda_2^2} + 3 + 2k\left(\lambda_1^2\lambda_2^2 + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right)^{N-1}\right. \\
& \times \left(2\lambda_1^2\lambda_2^2 - \frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}\right) \\
& \left. - \frac{k}{N}\left[\left(\lambda_1^2\lambda_2^2 + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right)^N - 3^N\right]\right\}\zeta\, d\zeta \\
& - 2\mu\left[\lambda_1 - \frac{1}{\lambda_1^3\lambda_2^2} + k\left(\lambda_1^2\lambda_2^2 + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right)^{N-1}\left(\lambda_1\lambda_2^2 - \frac{1}{\lambda_1^3}\right)\right]_{\zeta=\alpha}
\end{aligned} \tag{2.11}$$

and if we choose λ_1, λ_2 so that

$$\begin{aligned}&\left(\lambda_1 - \frac{1}{\lambda_1^3\lambda_2^2}\right)\left\{1 + k\lambda_2^2\left(\lambda_1^2\lambda_2^2 + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right)^{N-1}\right\} = K\left(1 - \frac{1}{\zeta^2}\right),\\&\left(\lambda_2 - \frac{1}{\lambda_1^2\lambda_2^3}\right)\left\{1 + k\lambda_1^2\left(\lambda_1^2\lambda_2^2 + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right)^{N-1}\right\} = K\left(1 + \frac{1}{\zeta^2}\right),\end{aligned} \tag{2.12}$$

where K is a parameter, then the equilibrium equation and the condition $\lambda_1^2\lambda_2 = 1$ at $\zeta = \alpha$ are satisfied.

Numerical calculations were performed on the University of Illinois CYBER 175. Upper bounds on $U(\mu)$ for the neo-Hookean plane sheet and the sheet with the empirical strain energy were obtained by minimizing numerically the expressions (2.6) and (2.10), respectively, with respect to C. Numerical integration was used to evaluate the integral in (2.10). Lower bounds on $U(\mu)$ for the neo-Hookean sheet were obtained essentially by evaluating the functional (2.7) through numerical integration and varying the value of K to maximize the lower bound, but the transformed integral of [5] was used. In order to obtain lower bounds for the sheet with the empirical strain energy, the equations in (2.12) were solved numerically for λ_1, λ_2 for a particular value of K and the functional (2.11) was evaluated through numerical integration. The value of K was varied to maximize the lower bound. For both of the sheets, values of $\alpha(=A/a)$ from 2 to 8 were used and calculations were performed at intervals of 0.02 for μ for the range $1.2 \leq \mu \leq 4$. The values of the bounds obtained determined $U(\mu)$ to within 0.5% over the range of μ considered. In general, the bounds become closer as μ is increased. The mean of the upper and lower bounds was chosen for the value of $U(\mu)$ and numerical differentiation was used to give S through the relation (2.9). Figure 1 shows the variations of S/h_0C_1 with μ. If the upper bounds alone or the lower bounds alone were used to estimate S, the estimates would differ by not more than 0.6% from the values indicated in Figure 1.

3. Large Extension and Torsion of a Cylinder

We consider a cylinder of homogeneous elastic material with length l and cross section R_0 with ends bonded to rigid plates which are parallel to each other and perpendicular to the axis of the cylinder. We choose the rectangular Cartesian coordinate system x_i so that the x_3-axis is the line of centroids of the cross sections of the cylinder in the reference state and the ends are in the planes $x_3 = 0$, $x_3 = l$.

The end plate at $x_3 = 0$ is held fixed and the end plate at $x_3 = l$ is displaced $(\lambda - 1)l$ axially and rotated through an angle ψl about the x_3-axis. We then have the end displacement conditions

$$\begin{aligned}&\hat{y}_i = x_i \quad \text{on } x_3 = 0,\\&\hat{y}_i = (x_1 \cos\psi l - x_2 \sin\psi l,\ x_1 \sin\psi l + x_2 \cos\psi l,\ \lambda l) \quad \text{on } x_3 = l,\end{aligned} \tag{3.1}$$

where x_i and y_i are the coordinates of a particle in the reference and deformed states,

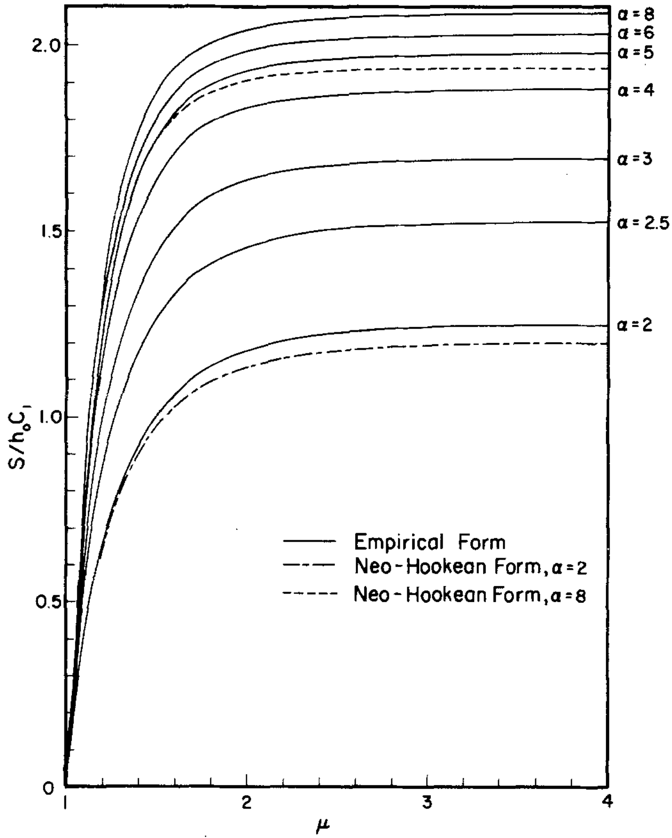

Figure 1
Stress resultant at the outer edge for extended plane sheets with a circular hole.

respectively. The lateral surface is free from traction and we denote the axial force and the twisting moment on the end plates by L and M, respectively. In small changes $\delta\psi$, $\delta\lambda$, the change in the total strain energy $U(\psi, \lambda)$ is equal to the work of the end loads,

$$\delta U = Ml\,\delta\psi + Ll\,\delta\lambda,$$

and it follows that

$$M = \frac{1}{l}\frac{\partial U}{\partial \psi}, \qquad L = \frac{1}{l}\frac{\partial U}{\partial \lambda}. \tag{3.2}$$

When the cross section has two axes of symmetry, the resultant load on the ends will be the axial force L and the twisting moment M. For other sections, the location of the 'natural' axis of torsion will depend on the values of λ and ψ and on the material properties, and forces and moments other than L, M will be needed on the ends in order to maintain the end displacements (3.1).

The principles in Section 3 of [1] can be used to obtain bounds for the total strain energy of the cylinder, provided the conditions in (3.4)† and (3.12)† are satisfied for the actual deformation y_i. We assume that the length of the cylinder is long enough for the end effects to be negligible. Then we may take 'two-dimensional' trial fields for the functional $P\{Y_i\}$ except in end regions $0 \le x_3 \le c$ and $l - c \le x_3 \le l$ in which the fields are adjusted in order to satisfy the boundary conditions (3.1). Here c is of order of the maximum diameter d of the cross section. Except in the end regions, we take

$$Y_i = [(x_1/\sqrt{\lambda} + u)\cos\theta - (x_2/\sqrt{\lambda} + v)\sin\theta,$$
$$(x_1/\sqrt{\lambda} + u)\sin\theta + (x_2/\sqrt{\lambda} + v)\cos\theta,\ \lambda x_3 + w], \tag{3.3}$$

where $\theta = \psi(\lambda x_3 + w)/\lambda$ and u, v, w are the functions of x_1, x_2 only. The Cauchy strains $C_{ik} = Y_{r,i}Y_{r,k}$ derived from (3.3) are independent of x_3 and we have

$$\frac{1}{l}P\{Y_i\} = \int_{R_0} W\,dA, \tag{3.4}$$

with neglect of terms of $0(d/l)$. This gives an upper bound for $U(\psi, \lambda)/l$. As an alternative to (3.3) we can take the equivalent forms

$$\mathrm{Y}_i = [(\mathrm{x}_1/\sqrt{\lambda} + \mathrm{u})\cos\phi - (x_2/\sqrt{\lambda} + v)\sin\phi,$$
$$(x_1/\sqrt{\lambda} + u)\sin\phi + (x_2/\sqrt{\lambda} + v)\cos\phi,\ \lambda x_3 + w], \tag{3.5}$$

where $\phi = \psi x_3$, and u, v are different functions of x_1, x_2 to those in (3.3). These forms suggest that for the functional $Q\{Y_{ik}\}$, we can take the 'two-dimensional' trial fields

$$Y_{ik} = \begin{bmatrix} A\cos\phi - B\sin\phi, & C\cos\phi - D\sin\phi, & -\psi(Y_1\sin\phi + Y_2\cos\phi) \\ A\sin\phi + B\cos\phi & C\sin\phi + D\cos\phi & \psi(Y_1\cos\phi - Y_2\sin\phi) \\ W_1 & W_2 & \lambda \end{bmatrix}, \tag{3.6}$$

where $\phi = \psi x_3$. The quantities $A, B, C, D, Y_1, Y_2, W_1, W_2$ are functions of x_1, x_2 only and they are to be chosen so that the trial functions Y_{ik} satisfy the equilibrium equations throughout the cylinder and the condition of zero traction on the lateral surface. The functional $Q\{Y_{ik}\}$ becomes

$$Q\{Y_{ik}\} = l\int_{R_0}\left\{\frac{\partial W'}{\partial Y_{ik}}Y_{ik} - W'\right\}dA - \left[\int_{R_0}\frac{\partial W'}{\partial Y_{i3}}\hat{y}_i\,dA\right]_{x_3=0}^{x_3=l}.$$

Then we have

$$\frac{1}{l}Q\{Y_{ik}\} = \int_{R_0}\left\{\frac{\partial W'}{\partial Y_{ik}}Y_{ik} - W'\right\}dA - \lambda\int_{R_0}\frac{\partial W'}{\partial Y_{33}}\,dA, \tag{3.7}$$

if we again neglect terms of $0(d/l)$, and this gives a lower bound for $U(\psi, \lambda)/l$. The integrands in (3.7) are functions of x_1, x_2 only.

For an incompressible material, we take the same trial fields as above and use the functionals of Section 4 of [1]. The trial fields Y_i for the functional $P\{Y_i\}$ are required to satisfy the condition of incompressibility, and with the forms (3.3) the condition is

$$\lambda\left[\left(\frac{1}{\sqrt{\lambda}}+\frac{\partial u}{\partial x_1}\right)\left(\frac{1}{\sqrt{\lambda}}+\frac{\partial v}{\partial x_2}\right)-\frac{\partial v}{\partial x_1}\frac{\partial u}{\partial x_2}\right]-1=0. \tag{3.8}$$

With neglect of end effects, the functional $Q_1\{Y_{ik}, P\}$ in (4.21)† leads to

$$\begin{aligned}\frac{1}{l}Q_1\{Y_{ik}, P\} &= \int_{R_0}\left\{\frac{\partial W'}{\partial Y_{ik}}Y_{ik}-W'+P[3-\ln(|Y_{rs}|)]\right\}dA\\ &\quad -\lambda\int_{R_0}\left\{\frac{\partial W'}{\partial Y_{33}}+PX_{33}\right\}dA,\end{aligned} \tag{3.9}$$

where the trial pressure P is a function of x_1, x_2 only.

For a homogeneous cylinder we can alternatively consider an equilibrium state of extension and twist in which we have the same state of strain at each section. The deformations y_i are then characterized by (3.3) or by (3.5) throughout the cylinder. Considering the work done by tractions in small changes $\delta\psi$, $\delta\lambda$ on the portion of the cylinder initially between $x_3 = 0$ and $x_3 = l$ and the change in the total strain $U(\psi, \lambda)$ for that portion, it can be shown [8] that the relations (3.2) still apply. The Cauchy strains C_{ik} are independent of x_3 and they can be expressed in terms of $y_{i,\beta}$ and ψy_α evaluated on $x_3 = 0$, the explicit dependence on ψ and y_α occurring only through the product ψy_α. (Greek indices range over 1, 2.) Thus the strain energy per unit initial volume can be written as

$$W = \bar{W}(\eta_{i,\beta}, \eta_\alpha) = \hat{W}(\eta_{i,\beta}, \psi\eta_\alpha),$$

where we have written η_i for $(y_i)_{x_3=0}$. The total strain energy per unit initial length of the cylinder is given by the integral

$$F\{\eta_i\} = \int_{R_0}\bar{W}\,dA. \tag{3.10}$$

The Euler differential equations and the natural boundary conditions associated with the functional $F\{\eta_i\}$ are equivalent to the equilibrium equations and the boundary conditions for the deformed cylinder and they are

$$\frac{\partial}{\partial x_\beta}\left(\frac{\partial\bar{W}}{\partial\eta_{i,\beta}}\right)-\frac{\partial\bar{W}}{\partial\eta_i}=0 \quad \text{in } R_0, \qquad \frac{\partial\bar{W}}{\partial\eta_{i,\beta}}n_\beta = 0 \quad \text{on } C_0, \tag{3.11}$$

where C_0 is the bounding curve of the region R_0 and n_β is the unit outward normal on C_0. In (3.11), $\partial\bar{W}/\partial\eta_3$ is zero. With the approach leading to (3.14)† and (3.17)†, we can form the functionals P and Q corresponding to the system (3.11) and the resulting integrals will be of the same forms as (3.4) and (3.7).

We note that if we differentiate the integral in (3.10) with respect to ψ and use (3.11), we can show that

$$M = \int_{R_0} \left(\frac{\partial \bar{W}}{\partial \psi}\right)_{\exp} dA = \frac{1}{\psi} \int_{R_0} \frac{\partial \bar{W}}{\partial \eta_\alpha} \eta_\alpha \, dA, \tag{3.12}$$

where $(\partial \bar{W}/\partial \psi)_{\exp}$ indicates an explicit partial differentiation of $\bar{W}$ with respect to ψ (holding $\eta_{i,\beta}$ and η_α constant). If we set $\zeta_i = \psi \eta_i$ and $\xi_\alpha = \psi x_\alpha$ and transform the integral in (3.10), by a similar process we can show that

$$M = \frac{1}{\psi} \int_{R_0} \frac{\partial \bar{W}}{\partial x_\alpha} x_\alpha \, dA. \tag{3.13}$$

This formula was obtained previously by Green [9] by a different approach.

When the strains are the same at each section, the Jacobian $|y_{i,k}|$ is equal to

$$\Delta = \begin{vmatrix} \eta_{1,1} & \eta_{1,2} & -\psi\eta_2 \\ \eta_{2,1} & \eta_{2,2} & \psi\eta_1 \\ \eta_{3,1} & \eta_{3,2} & \lambda \end{vmatrix},$$

and incompressibility requires Δ to be unity. For an incompressible material, we can associate a multiplier p with the condition $\ln \Delta = 0$ and arrive at equilibrium equations and boundary conditions in the form

$$\begin{aligned} &\frac{\partial}{\partial x_\beta}\left(\frac{\partial \bar{W}}{\partial \eta_{i,\beta}} + p \frac{\partial \ln \Delta}{\partial \eta_{i,\beta}}\right) - \left(\frac{\partial \bar{W}}{\partial \eta_i} + p \frac{\partial \ln \Delta}{\partial \eta_i}\right) = 0 \quad \text{in } R_0, \\ &\left(\frac{\partial \bar{W}}{\partial \eta_{i,\beta}} + p \frac{\partial \ln \Delta}{\partial \eta_{i,\beta}}\right) n_\beta = 0 \quad \text{on } C_0, \end{aligned} \tag{3.14}$$

where $\partial \bar{W}/\partial \eta_3 = \partial \Delta/\partial \eta_3 = 0$. With the approach of Section 4 of [1], we can form the functionals P and Q_1 corresponding to the system (3.14) and the resulting integrals will be of the same forms as (3.4) and (3.9). The formula corresponding to (3.12) is given by

$$M = \frac{1}{\psi} \int_{R0} \left(\frac{\partial \bar{W}}{\partial \eta_\alpha} + p \frac{\partial \Delta}{\partial \eta_\alpha}\right) \eta_\alpha \, dA$$

but (3.13) is unchanged in form.

In the following, we consider a cylinder of an (incompressible) neo-Hookean material with an elliptical cross section

$$R_0 \colon \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \leq 1$$

with semi-axes a and b. The condition $(3.4)^\dagger$, or the condition $(4.16)^\dagger$ for an incompressible material, under which the functional $P\{Y_i\}$ has a relative minimum implies that the equilibrium state would be stable under dead loading as mentioned earlier. The stability of an extended and twisted circular cylinder of neo-Hookean material has been examined by Green and Spencer [10], and they found that when $\lambda = 1$ an infinite cylinder is unstable during twisting for all values of the twist ψ. For $\lambda = 1.2$, the cylinder became unstable at $\psi a/\lambda = 1.07$ where a is the radius of the cross section. Instability on twisting for $\lambda = 1$ is expected because the axial force L is given by [11]

$$\frac{L}{2C_1 \pi a^2} = \lambda - \frac{1}{\lambda^2} - \frac{\psi^2 a^2}{4\lambda^2}$$

for a neo-Hookean circular cylinder and is compressive for $\lambda = 1$. For other sections, the axial force is [12]

$$\frac{L}{C_1} = 2A_0\left(\lambda - \frac{1}{\lambda^2}\right) + \frac{\psi^2}{\lambda^3}\left[\frac{4}{\lambda^2}(I_0 - S_0) - \lambda I_0\right]$$

with neglect of terms of $0(\psi^4)$. Here A_0 is the area of the cross section R_0, I_0 is the moment of inertia of the cross section about the centroid and S_0 is the classical geometrical torsional rigidity of the unstrained cylinder. For sections with S_0/I_0 less than 0.75, twisting with $\lambda = 1$ induces a tensile force initially. For the elliptical cross section, we have

$$S_0 = \frac{\pi a^3 b^3}{a^2 + b^2}, \qquad I_0 = \frac{\pi}{4} ab(a^2 + b^2), \qquad \frac{S_0}{I_0} = \frac{4\gamma^2}{(1 + \gamma^2)^2},$$

where $\gamma = b/a$, and S_0/I_0 is less than 0.75 if γ is less than 0.577. Thus for sections not too close to a circle, we expect that the functional $P\{Y_i\}$ will have a relative minimum even for $\lambda = 1$ for a range of ψ.

The functional $P\{Y_i\}$ is given by

$$\frac{1}{l} P\{Y_i\} = \int_{R_0} C_1(Y_{i,k}Y_{i,k} - 3)\, dA, \tag{3.15}$$

with neglect of terms of $0(d/l)$. We take (3.3) and choose

$$u = K_1 x_1, \qquad v = -\frac{K_1}{1 + \sqrt{\lambda} K_1} x_2, \qquad w = -K_2 x_1 x_2 + K_3 x_1^3 x_2 + K_4 x_1 x_2^3, \tag{3.16}$$

where K_1, K_2, K_3 and K_4 are parameters to be chosen so that the functional $P\{Y_i\}$ is minimized. The incompressibility condition (3.8) is satisfied. (The form of w in (3.16) is suggested by the expression (3.21) for the warping function correct to $0(\psi^3)$.) Substituting the trial functions into (3.15) and evaluating the integral, we obtain

$$\begin{aligned}\frac{1}{C_1 l A_0} P\{Y_i\} &= \left(\frac{1}{\sqrt{\lambda}} + K_1\right)^2 \left(1 + \frac{\psi'^2}{4}\right) + \left(\frac{1}{\sqrt{\lambda}} - \frac{K_1}{1 + \sqrt{\lambda} K_1}\right)^2 \\ &\quad \times \left(1 + \frac{\psi'^2}{4}\gamma^2\right) + \frac{\psi'^2 \gamma^2}{8\lambda^2}\left(\frac{1}{\sqrt{\lambda}} - \frac{K_1}{1 + \sqrt{\lambda} K_1}\right)^2\end{aligned}$$

$$\times\left[(1+3\gamma^2)k_2\left(\frac{k_2}{3}-\frac{k_3}{4}\right)+\frac{k_3^2}{80}(5+27\gamma^2)\right.$$
$$\left.+\frac{\gamma^4}{16}k_4^2(9+7\gamma^2)-\frac{\gamma^2}{4}(3+5\gamma^2)k_4(k_2-\tfrac{3}{10}k_3)\right]$$
$$+\frac{\psi'}{2\lambda}(1-\gamma^2)\left(\frac{1}{\sqrt{\lambda}}+K_1\right)\left(\frac{1}{\sqrt{\lambda}}-\frac{K_1}{1+\sqrt{\lambda}K_1}\right)$$
$$\times\left(-k_2+\frac{k_3}{2}+\frac{\gamma^2}{2}k_4\right)+\frac{\psi'^2}{8\lambda^2}\left(\frac{1}{\sqrt{\lambda}}+K_1\right)^2$$
$$\times\left[(3+\gamma^2)k_2\left(\frac{k_2}{3}-\frac{\gamma^2}{4}k_4\right)+\frac{k_3^2}{16}(7+9\gamma^2)\right.$$
$$\left.+\frac{\gamma^4}{80}k_4^2(27+5\gamma^2)-\frac{k_3}{4}(5+3\gamma^2)\left(k_2-\tfrac{3}{10}\gamma^2k_4\right)\right]$$
$$+\tfrac{1}{4}(1+\gamma^2)(k_2^2-k_2k_3-\gamma^2k_2k_4+\tfrac{3}{8}\gamma^2k_3k_4)$$
$$+\frac{k_3^2}{64}(5+9\gamma^2)+\frac{\gamma^4}{64}k_4^2(9+5\gamma^2)+\lambda^2-3, \qquad (3.17)$$

where $A_0=\pi ab$, $\psi'=\psi a$, $k_2=K_2a$, $k_3=K_3a^3$, $k_4=K_4a^3$.

The trial functions Y_{ik}, P for the functional $Q_1\{Y_{ik}, P\}$ are required to satisfy equilibrium and the boundary conditions on the lateral surface. These require for a neo-Hookean material

$$\frac{\partial}{\partial x_k}(2C_1Y_{ik}+PX_{ki})=0 \quad \text{throughout the cylinder,}$$

$$(2C_1Y_{ik}+PX_{ki})n_k=0 \quad \text{on the lateral surface,}$$

where n_k is the unit outward normal on the lateral surface. When (3.6) is used, the equations reduce to

$$\left.\begin{aligned}&\frac{\partial}{\partial x_1}[A+\hat{P}(D\lambda-\psi Y_1W_2)/\Delta]+\frac{\partial}{\partial x_2}[C-\hat{P}(B\lambda-\psi Y_1W_1)/\Delta]\\&\qquad=\psi^2Y_1-\hat{P}\psi(AW_2-CW_1)/\Delta\\&\frac{\partial}{\partial x_1}[B-\hat{P}(C\lambda+\psi Y_2W_2)/\Delta]+\frac{\partial}{\partial x_2}[D+\hat{P}(A\lambda+\psi Y_2W_1)/\Delta]\\&\qquad=\psi^2Y_2-\hat{P}\psi(BW_2-DW_1)/\Delta\\&\frac{\partial}{\partial x_1}[W_1+\hat{P}\psi(CY_1+DY_2)/\Delta]+\frac{\partial}{\partial x_2}[W_2-\hat{P}\psi(AY_1+BY_2)/\Delta]=0,\end{aligned}\right\}\ \text{in}\, R_0,$$

(3.18)

$$\left.\begin{aligned}&[A + \hat{P}(D\lambda - \psi Y_1 W_2)/\Delta]n_1 + [C - \hat{P}(B\lambda - \psi Y_1 W_1)/\Delta]n_2 = 0\\&[B - \hat{P}(C\lambda + \psi Y_2 W_2)/\Delta]n_1 + [D + \hat{P}(A\lambda + \psi Y_2 W_1)/\Delta]n_2 = 0\\&[W_1 + \hat{P}\psi(CY_1 + DY_2)/\Delta]n_1 + [W_2 - \hat{P}\psi(AY_1 + BY_2)/\Delta]n_2 = 0\end{aligned}\right\} \text{ on } C_0,$$

where $\hat{P} = P/2C_1$, n_α is the unit outward normal on C_0, and Δ is the determinant $|Y_{ik}|$, i.e.,

$$\Delta = \psi W_1(CY_1 + DY_2) - \psi W_2(AY_1 + BY_2) + \lambda(AD - BC).$$

The functional $Q_1\{Y_{ik}, P\}/l$ in (3.9) becomes

$$\frac{1}{C_1 l} Q_1\{Y_{ik}, \hat{P}\} = \int_{R_0} \{A^2 + B^2 + C^2 + D^2 + \psi^2(Y_1^2 + Y_2^2) + W_1^2 + W_2^2 - \lambda^2 + 3 + 2\hat{P}(3 - \ln \Delta) - 2\lambda\hat{P}(AD - BC)/\Delta\}\, dA. \quad (3.19)$$

Green and Shield [4] determined the displacements u, v and the warping function w in (3.5) correct to $0(\psi^2)$. Because of symmetry considerations, u, v are even functions of ψ and w is an odd function of ψ for fixed λ, so that w has no second-order terms. The results of [4] suggest that we choose the quantities $A, B, C, D, Y_1, Y_2, W_1, W_2$ and $\hat{P}$ so that

$$\begin{aligned}&A + \hat{P}(D\lambda - \psi Y_1 W_2)/\Delta = A_0 + A_1 x_1^2 + A_2 x_2^2,\\&C - \hat{P}(B\lambda - \psi Y_1 W_1)/\Delta = B_1 x_1 x_2,\\&B - \hat{P}(C\lambda + \psi Y_2 W_2)/\Delta = B_2 x_1 x_2,\\&D + \hat{P}(A\lambda + \psi Y_2 W_1)/\Delta = D_0 + D_1 x_1^2 + D_2 x_2^2,\\&W_1 + \hat{P}\psi(CY_1 + DY_2)/\Delta = E_1 x_2 + E_2 x_1^2 x_2 + E_3 x_2^3,\\&W_2 - \hat{P}\psi(AY_1 + BY_2)/\Delta = F_1 x_1 + F_2 x_1 x_2^2 + F_3 x_1^3,\\&\psi^2 Y_1 - \hat{P}\psi(AW_2 - CW_1)/\Delta = (2A_1 + B_1)x_1,\\&\psi^2 Y_2 - \hat{P}\psi(BW_2 - DW_1)/\Delta = (B_2 + 2D_2)x_2,\end{aligned} \quad (3.20)$$

where $A_0, A_1, \ldots, F_3$ are parameters at our disposal. If we let the parameters satisfy

$$A_1 = -\frac{A_0}{a^2}, \quad \frac{A_2}{a^2} + \frac{B_1}{b^2} = -\frac{A_0}{a^2 b^2}, \quad \frac{B_2}{a^2} + \frac{D_1}{b^2} = -\frac{D_0}{a^2 b^2}, \quad D_2 = -\frac{D_0}{b^2},$$

$$E_2 = -F_2, \quad \frac{E_2}{a^2} + \frac{F_3}{b^2} = -\frac{1}{a^2}\left(\frac{E_1}{a^2} + \frac{F_1}{b^2}\right), \quad \frac{E_3}{a^2} - \frac{E_2}{b^2} = -\frac{1}{b^2}\left(\frac{E_1}{a^2} + \frac{F_1}{b^2}\right),$$

then the equations and the boundary conditions in (3.18) are satisfied. It remains to choose the trial pressure $\hat{P}$ and the parameters $A_0, B_1, B_2, D_0, E_1, E_2, F_1$ so that the functional $Q_1\{Y_{ik}, \hat{P}\}/C_1 l$ is minimized. With the second-order results of [4], the third-order warping function can be found from the third and sixth equations in (3.18), and we obtain

$$w = [\psi'\alpha/(a\lambda^2) + \psi'^3\rho]x_1 x_2 + \psi'^3\sigma x_1^3 x_2 + \psi'^3\tau x_1 x_2^3, \quad (3.21)$$

where

$$\rho = \frac{\gamma^2}{(1+\gamma^2)a}\left\{-\frac{\alpha\gamma^2}{6(\alpha^2+2)\lambda^5}\left[3(\alpha-4)\lambda^3 + 8(5-\alpha) - 2(\alpha^2-2) + \frac{16\alpha}{\alpha^2-2}\right]\right.$$

$$-\frac{\alpha}{2(\alpha^2+2)\lambda^5}\left[3\alpha\lambda^3 - 8(\alpha+1) - 2(\alpha^2-2) + \frac{16\alpha}{\alpha^2-2}\right]$$

$$\left.+\frac{1}{2\lambda^3}(1-\gamma^2) + \frac{\alpha}{\lambda^3}(1+\gamma^2)\right\},$$

$$\sigma = -\frac{\alpha}{6(\alpha^2+2)\lambda^5 a^3}\left[(\alpha+2)\lambda^3 + 2(\alpha^2-2) + \frac{16\alpha}{\alpha^2-2}\right],$$

$$\tau = \frac{\alpha}{6(\alpha^2+2)\lambda^5 a^3}\left[(\alpha^2-2)\lambda^3 - 2(\alpha^2-2) + \frac{16\alpha}{\alpha^2-2}\right].$$

Here ψ' is ψa as before and we have set

$$\alpha = -\frac{a^2-b^2}{a^2+b^2}.$$

The second-order solutions in [4] and the warping function (3.21) suggest that as an approximation for the optimum values of the parameters and the trial pressure we may take

$$A_0 = -\frac{(\psi'k)^2}{\sqrt{\lambda}\lambda^3(\alpha^2+2)}[\lambda^3 + \alpha(\alpha^2+2\alpha+2)],$$

$$B_1 = \frac{(\psi'k)^2\alpha}{\sqrt{\lambda}\lambda^3(\alpha^2+2)a^2}[\alpha\lambda^3 - \alpha^2 - 4\alpha - 2],$$

$$B_2 = \frac{(\psi'k)^2\alpha}{\sqrt{\lambda}\lambda^3(\alpha^2+2)a^2}[\alpha\lambda^3 + \alpha^2 - 4\alpha + 2],$$

$$D_0 = -\frac{(\psi'k)^2\gamma^2}{\sqrt{\lambda}\lambda^3(\alpha^2+2)}[\lambda^3 + \alpha(-\alpha^2+2\alpha-2)],$$

$$E_1 = (\psi'k)^3\rho + \frac{\psi'k}{\lambda^2 a}(\alpha-1) - \frac{(\psi'k)^3}{(\alpha^2+2)\lambda^5 a}(\alpha^2-2+\lambda^3) - \frac{(\psi'k)^3}{\lambda^5 a^3}(\alpha+1), \quad (3.22)$$

$$E_2 = 3(\psi'k)^3\sigma + \frac{(\psi'k)^3\alpha}{(\alpha^2+2)\lambda^5 a^3}(4-\lambda^3)(\alpha-1) + \frac{(\psi'k)^3}{2\lambda^5 a^3}(\lambda^3+2\alpha),$$

$$F_1 = (\psi'k)^3\rho - \frac{\psi'k\gamma^2}{\lambda^2 a}(\alpha-1) + \frac{(\psi'k)^3\gamma^2}{(\alpha^2+2)\lambda^5 a}(\alpha^2-2+\lambda^3) - \frac{(\psi'k)^3\gamma^2}{\lambda^5 a}(\alpha-1),$$

$$\hat{P} = -\frac{1}{\lambda} + \frac{(\psi' k)^2}{2\lambda^2 a^2}\left\{\lambda(x_1^2 + x_2^2) + \frac{2\alpha}{\lambda}\left[\frac{1}{\lambda} - \frac{1}{\alpha^2+2}\left(\frac{4}{\lambda} - \lambda^2\right)\right](x_1^2 - x_2^2)\right.$$
$$\left. + \frac{1}{\lambda}\left[\frac{\alpha^2-1}{\lambda} - \frac{1}{\alpha^2+2}\left(\frac{\alpha^2-2}{\lambda} + \lambda^2\right)\right](a^2+b^2)\right\},$$

where k is a parameter to be chosen so that the functional $Q_1\{Y_{ik}, \hat{P}\}/C_1 l$ is minimized.

Numerical calculations were performed on the University of Illinois CYBER 175 (using the routines of IMSL and MSL). Upper bounds on $U(\psi, \lambda)/l$ were obtained by minimizing expression (3.17) numerically with respect to K_1, k_2, k_3 and k_4. In order to obtain lower bounds on $U(\psi, \lambda)/l$, the equations in (3.20) were solved numerically for $A, B, \ldots, W_2$ for a particular value of k in (3.22). The initial guesses were taken from the second-order solution. The functional (3.19) was evaluated by numerical integration [13] and the value of k was varied to improve the lower bounds. For $b/a = 0.25$ and 0.5, values of λ from 1 to 5 were used and calculations were performed at intervals of 0.05 for $\psi a/\lambda$. It was found that the range of ψ for which lower bounds

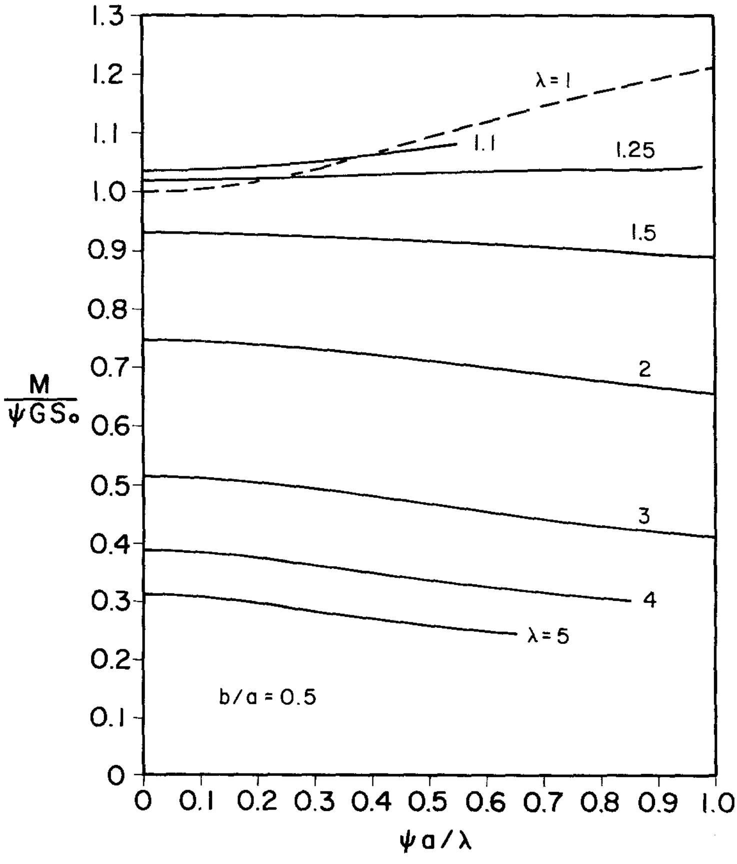

Figure 2
Twisting moment for torsion of an extended elliptical cylinder of neo-Hookean material with $b/a = 0.5$.

could be obtained by this approach varied with the value of λ. The limiting factor was the convergence of the numerical procedure used to solve equations (3.20), routine NONLIQ of MSL.

The range of ψ for which lower bounds could be obtained was extended by reducing the nonlinear system (3.20) of eight equations to four equations. The equations are of the form

$$Y_{ik} + \hat{P}X_{ki} = S_{ik} \tag{3.23}$$

where now Y_{ik} have the values (3.6) with ϕ set equal to zero and X_{ik} is the inverse of Y_{ik}. Here S_{ik} are determined by the right-hand sides of (3.20) with the addition of

$$S_{33} = \lambda + \hat{P}(AD - BC)/\Delta.$$

It follows that

$$Y_{im}Y_{km} + \hat{P}\,\delta_{ik} = S_{im}Y_{km}, \qquad Y_{mi}Y_{mk} + \hat{P}\,\delta_{ik} = S_{mi}Y_{mk}. \tag{3.24}$$

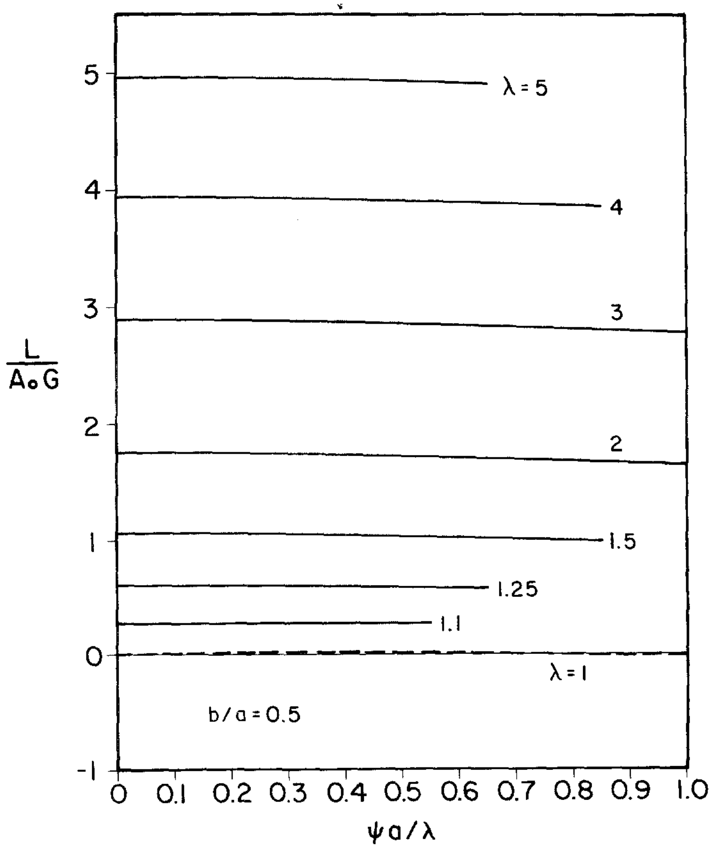

Figure 3
Axial force for torsion of an extended elliptical cylinder of neo-Hookean material with $b/a = 0.5$.

By taking (i, k) to be (1, 3) and (2, 3) in (3.24) in turn, we obtain four equations which are linear in W_1, W_2, Y_1, Y_2. Solving for these quantities in terms of the other unknowns A, B, C, D and substituting the results in the first four equations of (3.20), we arrive at four non-linear equations for A, B, C, D. By this approach, lower bounds for $\lambda = 1.1$, $b/a = 0.25$ were obtained, and lower bounds for $\lambda = 1.25$ and 1.5 for $b/a = 0.25$ and 0.5 were obtained beyond the range of ψ for which lower bounds could be obtained using the eighth-order system (3.20).

The values of the bounds obtained determined $U(\psi, \lambda)/l$ to within $\pm 0.38\%$ and sample numerical results are shown in Table 1. The mean of the upper and lower bounds was chosen for the value of $U(\psi, \lambda)/l$ and numerical differentiation was then used to give M and L through the relations (3.2). In order to determine L, estimates for $U(\psi, \lambda)/l$ were obtained by the approach described above for values of λ differing by 0.01 from the chosen values of λ. Figures 2–5 show the variations of $M/\psi GS_0$ and L/A_0G with $\psi a/\lambda$. Here G is the shear modulus for small strains, $G = 2C_1$. Solid lines indicate the variations of M and L for the ranges of ψ for which upper and lower bounds for $U(\psi, \lambda)/l$ were used to estimate M and L. If the upper or lower bounds alone were used to estimate M, the estimates would differ by not more than 2.1% and

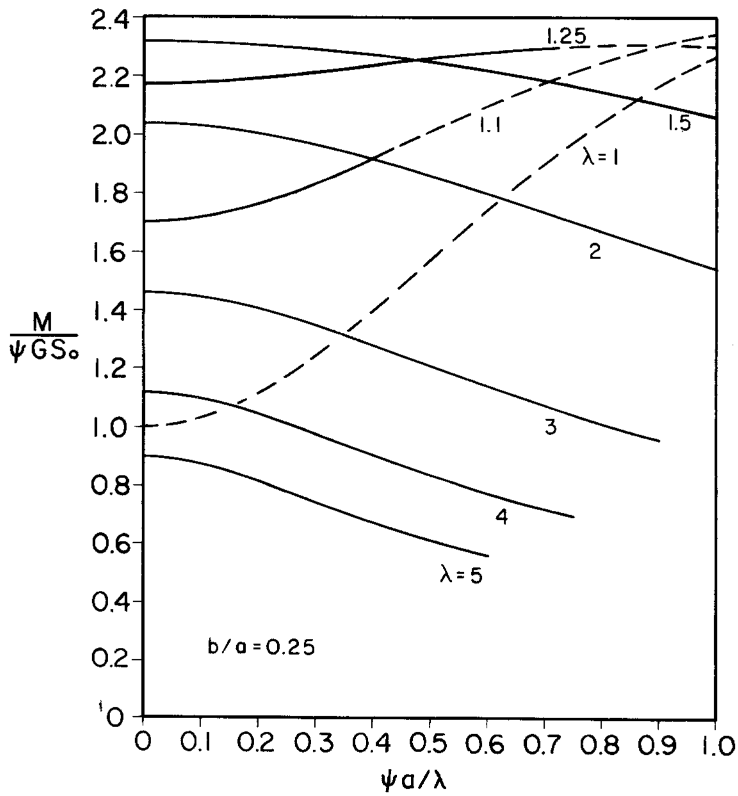

Figure 4
Twisting moment for torsion of an extended elliptical cylinder of neo-Hookean material with $b/a = 0.25$.

Table 1
$\lambda = 2$, $b/a = 0.5$

$\psi a/\lambda$	$P/C_1 l A_0$	$-Q_1/C_1 l A_0$	$U(\psi, \lambda)/C_1 l A_0$
0.0	2.0000	2.0000	2.0000
0.2	2.0238	2.0238	$2.0238 \pm 0.18 \times 10^{-3}\%$
0.4	2.0940	2.0939	$2.0940 \pm 0.33 \times 10^{-2}\%$
0.6	2.2082	2.2073	$2.2077 \pm 0.20 \times 10^{-1}\%$
0.8	2.3634	2.3601	$2.3618 \pm 0.70 \times 10^{-1}\%$
1.0	2.5578	2.5481	$2.5530 \pm 0.19\%$

7.2% for $\lambda \leq 2$ and $\lambda \geq 3$, respectively, from the values indicated in Figure 2 for $b/a = 0.5$. Likewise if the upper or lower bounds alone were used to estimate L, the estimates would differ by not more than 1.4% for all values of λ from the values indicated in Figure 3 for $b/a = 0.5$. Similar remarks with somewhat smaller percentage differences apply for the case $b/a = 0.25$. The broken lines in Figures 2–5 indicate values of M and L calculated from the upper bounds alone. For λ less than 2, the use of

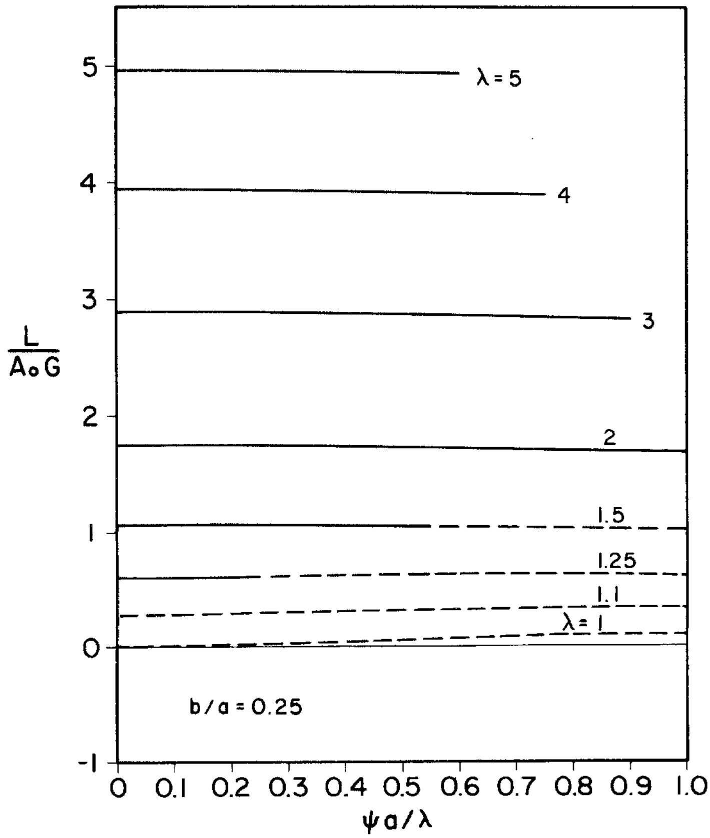

Figure 5
Axial force for torsion of an extended elliptical cylinder of neo-Hookean material with $b/a = 0.25$.

the upper bounds alone is believed to provide reliable results. For $\lambda = 1$ for both values of b/a, the process for solving the nonlinear system of equations converged only for small values of ψ. Shield [14] has examined the extension and torsion of a thin elastic strip of neo-Hookean material, and the variations of M and L for the elongated rectangular section show strong similarities to those for the elliptical cylinder with $b/a = 0.25$.

We remark that by taking 'two-dimensional' trial fields for the functionals P and Q, we have been in fact dealing with the functionals corresponding to the two-dimensional system (3.11) or, for an incompressible material, the system (3.14). We emphasize that these two-dimensional functionals may have relative minima even when the three-dimensional functionals are merely stationary. The results obtained by the 'two-dimensional' approach will be of direct value only for the ranges of ψ and λ for which the cylinder remains stable, with no restriction on the class of instability modes allowed.

References

[1] S. J. LEE, and R. T. SHIELD, *Variational Principles in Finite Elastostatics*, Z. angew. Math. Phys. *31*, 437–453 (1980).
[2] R. S. RIVLIN, and A. G. THOMAS, *Large Elastic Deformations of Isotropic Materials, Part VIII*, Phil. Trans. Roy. Soc., Ser. A, *243*, 289–298 (1950).
[3] F. S. WONG, and R. T. SHIELD, *Large Plane Deformations of Thin Elastic Sheets of Neo-Hookean Material*, Z. angew. Math. Phys., *20*, 175–199 (1969).
[4] A. E. GREEN, and R. T. SHIELD, *Finite Extension and Torsion of Cylinders*, Phil. Trans. Roy. Soc., Ser. A, *244*, 47–86 (1951).
[5] S. J. LEE, *Variational Principles in Finite Elasticity with Applications*, Ph.D. Thesis, University of Illinois at Urbana-Champaign, December, 1979. T. & A.M. Report No. 437, December 1979.
[6] R. T. SHIELD, *On the Stability of Finitely Deformed Elastic Membranes, Part I*, Z. angew. Math. Phys., *22*, 1016–1028 (1971).
[7] L. R. G. TRELOAR, *Stress-Strain Data for Vulcanized Rubber under Various Types of Deformation*, Trans. Faraday Soc., *40*, 59–70 (1944).
[8] R. T. SHIELD, *An Energy Method for Certain Second-Order Effects with Application to Torsion of Elastic Bars under Tension*, J. Appl. Mech., *47*, 75–81 (1980).
[9] A. E. GREEN, *On Some General Formulae in Finite Elastostatics*, Arch. Rat. Mech. Anal., *50*, 73–80 (1973).
[10] A. E. GREEN, and A. J. M. SPENCER, *The Stability of a Circular Cylinder under Finite Extension and Torsion*, J. Math. and Phys., *37*, 316–338 (1958).
[11] R. S. RIVLIN, *Large Elastic Deformations of Isotropic Materials VI. Further Results in the Theory of Torsion, Shear and Flexure*, Phil. Trans. Roy. Soc. A, *242*, 173–195 (1949); reprinted in *Problems of Non-linear Elasticity*, Gordon and Breach, New York (1965).
[12] A. E. GREEN, *A Note on Second-Order Effects in the Torsion of Incompressible Cylinders*, Proc. Camb. Phil. Soc., *50*, 488–490 (1954).
[13] W. H. PIERCE, *Numerical Integration over the Planar Annulus*, J. Soc. Indust. Appl. Math., *5*, No. 2, 66–73 (1957).
[14] R. T. SHIELD, *Finite Extension and Torsion of Thin Elastic Strips*, in preparation.

Summary

For some equilibrium states of a finitely deformed elastic body, variational principles can be used to provide bounds on overall quantities of physical interest. The principles are applied to the problem of the all-around finite extension of a plane sheet with a circular hole, and accurate estimates for the stress resultant at the outer edge are obtained for various extensions. The finite extension and torsion of an elastic cylinder is considered and bounds on the strain energy per unit length are obtained for an elliptical cylinder of neo-Hookean material with axes in the ratios of 2:1 and 4:1. The bounds lead to reliable estimates for the twisting moment and axial force.

Zusammenfassung

Für gewisse Gleichgewichtszustände eines endlich deformierten elastischen Körpers können Variationsprinzipien verwendet werden, um Schranken für globale Grössen von physikalischem Interesse zu erhalten. Die Prinzipien werden auf das Problem der allseitigen endlichen Extension einer ebenen Scheibe mit kreisförmigem Loch angewendet, und es werden für verschiedene Extensionen gute Abschätzungen der Resultierenden am Aussenrand gewonnen. Ferner wird die endliche Zug- und Torsionsverformung eines elastischen Zylinders betrachtet, und es werden Schranken für die Verformungsenergie je Längeneinheit für einen elliptischen Zylinder aus Neo-Hookeschem Material für die Achsenverhältnisse 2:1 und 4:1 erhalten. Die Schranken liefern zuverlässige Schätzungen für das Torsionsmoment und die Axialkraft.

(Received: February 14, 1980)