Minimum Induced Drag of Wings with Given Lift and Root-Bending Moment

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1. Introduction

It is a well known fact that wings with elliptic spanwise loading have minimum induced drag for given lift and wing span. If the span is taken as a free parameter, different loadings will, however, be optimum with respect to induced drag, their shapes depending on the auxiliary conditions imposed. From the practical point of wing design the span is restricted by structural considerations. Hence the most reasonable circulation-distribution would be that one which yields minimum total drag at cruise if the structural weight of the wing is given besides its lift. This optimization problem is an extremely difficult one. Even the much easier task to determine the minimum not for the total drag but for the induced drag alone, requires additional simplifications. Such a solution is due to Prandtl [1]. He assumed that the weight of the spars is everywhere proportional to the local bending moment, and he derived the optimum spanwise lift-distribution which pertains to this condition. Prandtl showed also that for a certain span the induced drag attains then an absolute minimum which is by about 11% lower than the corresponding value for elliptic loading. As an alternative to Prandtl's problem, a solution is derived in the present paper for wings for which the lift and the root-bending moment that is produced by the spanwise lift-distribution are given. Prandtl's assumptions do viz. not hold good for swept wings. However, for the high-aspect ratio wings with comparatively small sweep of transport aircraft it seems to be the root-bending moment of the lift which to a large extent determines the required structural weight at cruise ([2]). In such a case the present solution may prove to be a better approximation for the actual fundamental problem than Prandtl's.

2. Prandtl's Solution

Assuming a direct proportionality between the weight of the spars and the local bending moment, Prandtl [1] showed that the induced drag of such wings is determined, besides by their lift, by the moment of inertia of their spanwise lift-distributions. The mathematical formulation of Prandtl's problem consisted therefore in making the induced drag

$$D_i = 4\rho_\infty V_\infty^2 s^2 \int_0^1 \gamma(\eta)\, \alpha_i(\eta) \tag{1}$$

of such wings a minimum by prescribing the lift

$$L = 2\rho_\infty V_\infty^2 s^2 \int_0^1 \gamma(\eta)\, d\eta \tag{2}$$

and the moment of inertia

$$Lr^2 = 2\rho_\infty V_\infty^2 s^4 \int_0^1 \gamma(\eta)\, \eta^2\, d\eta . \tag{3}$$

In these equations r is the radius of gyration of the lift-distribution, γ is the circulation Γ referred to the span $2s$ and the undisturbed speed V_∞ of the flow, η is the spanwise coordinate referred to the semi-span s, $\alpha_i(\eta)$ the distribution of the flow angle induced by the circulation $\gamma(\eta)$ far behind the wing, and ρ_∞ is the density of the fluid. The first part of the problem was then to determine, by application of variational principles, the spanwise loading $\gamma(\eta)$ which gives minimum induced drag for any prescribed span. Thus Prandtl had to solve for

$$\delta D_i = 2 \int_0^1 \delta\gamma(\eta)\, \alpha_i(\eta) = 0 \tag{4}$$

with the auxiliary conditions

$$\delta L \quad = \int_0^1 \delta\gamma(\eta)\, d\eta = 0, \tag{5}$$

$$\delta(Lr^2) = \int_0^1 \delta\gamma(\eta)\, \eta^2\, d\eta = 0. \tag{6}$$

The solution reads

$$\gamma(\eta) = \gamma_R \left[1 - \frac{4(\sigma^2-1)}{2\sigma^2-1}\eta^2\right] \sqrt{1-\eta^2}, \tag{7}$$

where γ_R is the loading at the wing root and $\sigma = s/s_e$ is the ratio between the semi-span s of the wing and the semi-span s_e of a wing with the same structural weight having elliptic spanwise loading. This loading induces a velocity-distribution which far behind the wing varies parabolically along the span.

With eq. (7) the induced drag becomes, if referred to the elliptic-loading value,

$$\frac{D_i}{D_{ie}} = \frac{4\sigma^4 - 6\sigma^2 + 3}{\sigma^6}. \tag{8}$$

The relations eqs. (7) and (8) as presented here differ somewhat from those originally derived by Prandtl who did not use the ratio σ explicitely. The form chosen in the present paper is, however, more comprehensive.

The second part of Prandtl's problem consisted in finding the absolute minimum of the induced drags within the solutions obtained by variational principles. This minimum occurs at $\sigma = 1.2247$, for which $D_i/D_{ie} = 0.8889$, where D_{ie} is the induced drag of the wing with elliptic loading having the same structural weight within the approximations made. The corresponding optimum loading is

$$\gamma_{\text{opt}}(\eta) = \gamma_R (1-\eta^2)\sqrt{1-\eta^2}. \tag{9}$$

3. Solution for Prescribed Wing-Root Bending Moment

As an alternative to Prandtl's solution the spanwise loading is derived in the following of wings, for which the root-bending moment M_R, besides the lift, is given. In this case the mathematical statement of the problem as formulated in the previous chapter is retained with the exception of eqs. (3) and (6) which have to be replaced by

$$M_R = 2\rho_\infty V_\infty^2 s^3 \int_0^1 \gamma(\eta)\,\eta\,d\eta \tag{10}$$

and

$$\delta M_R = \int_0^1 \delta\gamma(\eta)\,\eta\,d\eta = 0. \tag{11}$$

From eqs. (4), (5) and (11) it follows that

$$\alpha_i(\eta) = C_1 + C_2\,|\eta|.$$

This means that the required circulation-distribution produces a downwash which varies linearly along the span. This downwash, referred to V_∞, is found as a solution of the integral equation

$$\alpha_i(\eta) = \frac{1}{2\pi}\int_{-1}^{+1} \frac{d\gamma(\eta')}{d\eta'}\,\frac{d\eta'}{\eta-\eta'} = C_1 + C_2\,|\eta|, \tag{12}$$

which is established by application of Biot-Savart's law. This solution reads

$$\frac{d\gamma(\eta)}{d\eta} = \left(2C_1 + \frac{4C_2}{\pi}\right)\frac{1-\eta}{\sqrt{1-\eta^2}} - \frac{2}{\pi}C_2\,\eta\log\frac{1-\sqrt{1-\eta^2}}{1+\sqrt{1-\eta^2}} + \frac{\text{const}}{\sqrt{1-\eta^2}}. \tag{13}$$

The value of the constant is determined from the condition

$$\frac{d\gamma(-\eta)}{d\eta} = -\frac{d\gamma(\eta)}{d\eta}$$

to be $-(2C_1+4C_2/\pi)$. If we now integrate the above expression for $d\gamma(\eta)/d\eta$, we obtain

$$\gamma(\eta)=\frac{1}{\pi}\left[2\sqrt{1-\eta^2}(\pi C_1+C_2)-C_2\eta^2\log\frac{1-\sqrt{1-\eta^2}}{1+\sqrt{1-\eta^2}}\right]. \tag{14}$$

In deriving eq. (13), use has been made of the well-known solution of an integral equation in aerofoil theory, which is similar to ours. The chordwise circulation-distribution of a lifting two-dimensional symmetrical aerofoil of finite thickness can be expressed by the distribution which represents the lifting flat plate and an additive term which depends on the section shape. This additive term produces a chordwise loading $\Delta C_p(x)$. For incompressible flow the normal velocity Δw which the additional circulation induces on the chord-line is obtained from

$$\Delta w(x)=-\frac{1}{4\pi}\int_0^1 -\Delta C_p(x')\frac{dx'}{x-x'}.$$

The solution of this equation is

$$-\Delta C_p(x)=\frac{4}{\pi}\sqrt{\frac{1-x}{x}}\int_0^1 \Delta w(x')\sqrt{\frac{x'}{1-x'}}\frac{dx'}{x-x'},$$

(see Weber [3]).

Similarly the solution of eq. (12) is

$$\frac{d\gamma(\eta)}{d\eta}=\frac{2}{\pi}\sqrt{\frac{1-\eta}{\eta}}\int_{-1}^{+1}(C_1+C_2|\eta|)\sqrt{\frac{\eta'}{1-\eta'}}\frac{d\eta'}{\eta-\eta'}.$$

The integral can be solved in a closed form so that eq. (13) is obtained.

The constants in eq. (14) are determined in the following manner: $\gamma(\eta)$ according to eq. (14) is introduced into eqs. (2) and (10) for the lift and the wing-root bending moment, and these relations are equated to those of the corresponding wing with elliptic loading of semispan s_e in a flow with the same free-stream dynamic pressure. The integrals occurring in the two resulting equations turn out to have the following values:

$$\int_0^1 \sqrt{1-\eta^2}\,d\eta=\frac{\pi}{4};\qquad \int_0^1 \eta\sqrt{1-\eta^2}\,d\eta=\tfrac{1}{3},$$

$$\int_0^1 \eta^2\log\frac{1-\sqrt{1-\eta^2}}{1+\sqrt{1-\eta^2}}\,d\eta=-\frac{\pi}{6},\qquad \int_0^1 \eta^3\log\frac{1-\sqrt{1-\eta^2}}{1+\sqrt{1-\eta^2}}\,d\eta=-\tfrac{1}{3}.$$

Solving for the two constants gives

$$C_1=\frac{\gamma_{Re}}{2\sigma^3}(9\sigma-8), \tag{15a}$$

$$C_2=-\frac{3\pi\gamma_{Re}}{\sigma^3}(\sigma-1). \tag{15b}$$

The ratio between the circulations γ_{Re} and γ_R at the roots of the wings with elliptic loading (i.e. for $\sigma=1$) and with the loading specified by eq. (14) is then obtained as

$$\frac{\gamma_{Re}}{\gamma_R}=\frac{\sigma^3}{3\sigma-2}. \tag{16}$$

Eq. (14) can then be finally expressed as

$$\gamma(\eta)=\gamma_R\left[\sqrt{1-\eta^2}+\frac{3(\sigma-1)}{3\sigma-2}\eta^2\log\frac{1-\sqrt{1-\eta^2}}{1+\sqrt{1-\eta^2}}\right]. \tag{17}$$

For $\sigma=1$, this relation reduces to that for elliptic loading. All the circulations according to eq. (17) induce an angle of attack which varies linearly along the span. It is found from eqs. (12), (15a), (15b), and (16) to be

$$\alpha_i(\eta)=\frac{\gamma_R}{2(3\sigma-2)}\left[9\sigma-8-3\pi(\sigma-1)\eta\right]. \tag{18}$$

For elliptic loading, $\sigma=1$, eq. (18) reduces to $\alpha_{ie}=\gamma_{Re}/2$.

The second part of the problem is to determine, among the loadings specified by eq. (17), the particular loading which provides the absolute minimum for the induced drag and to find the corresponding value of σ. This is easily accomplished by introducing $\gamma(\eta)$ and $\alpha_i(\eta)$ from eqs. (17) and (18) into eq. (1) and dividing by the value for the wing with elliptic loading. The result is

$$\frac{D_i}{D_{ie}}=\frac{\sigma^2+8(1-\sigma)^2}{\sigma^4}. \tag{19}$$

This function has its minimum $(D_i/D_{ie})_{\text{opt}}=\frac{27}{32}=0.844$ at $\sigma_{\text{opt}}=\frac{4}{3}=1.333$. The respective values of Prandtl's [1] optimum loading are $(D_i/D_{ie})_{\text{opt}}=0.889$ at $\sigma_{\text{opt}}=1.225$. The relation for $\gamma(\eta)$ at $\sigma=\frac{4}{3}$ is

$$\gamma_{\text{opt}}(\eta)=\gamma_R\left[\sqrt{1-\eta^2}+\frac{\eta^2}{2}\log\frac{1-\sqrt{1-\eta^2}}{1+\sqrt{1-\eta^2}}\right]. \tag{20}$$

It is shown in Figure 1 together with Prandtl's optimum loading [1] and with elliptic loading. In Figure 2 the relation eq. (19) is plotted. We see that the "minimum" of the induced drag at $\sigma_{\text{opt}}=\frac{4}{3}$ is actually a point of inflexion. However, for values $\sigma>\frac{4}{3}$ the lift at the wing tip becomes negative and our solution is hence not valid any more. In this respect the same considerations apply that Prandtl discussed in [1]. His induced-drag relationship according to eq. (8), which is included in Figure 2, has viz. also a point of inflexion instead of a genuine minimum.

Comparison of the curves in Figures 1 and 2 shows that the spanwise loading derived in this paper departs even more from elliptic loading than Prandtl's. This results in even smaller induced drags. The absolute minimum occurs at a larger span-ratio σ than in Prandtl's case.

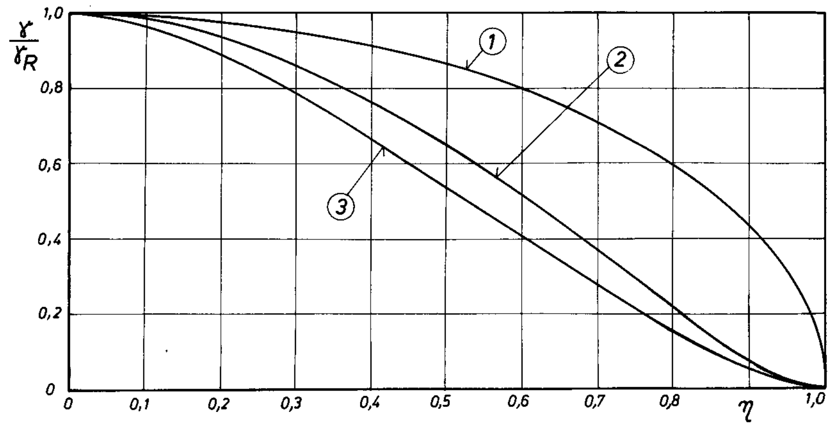

Figure 1
Comparison of three different optimum spanwise loadings. ① Elliptic loading. ② Prandtl's optimum loading, eq. (9). ③ Optimum loading of present investigation, eq. (20).

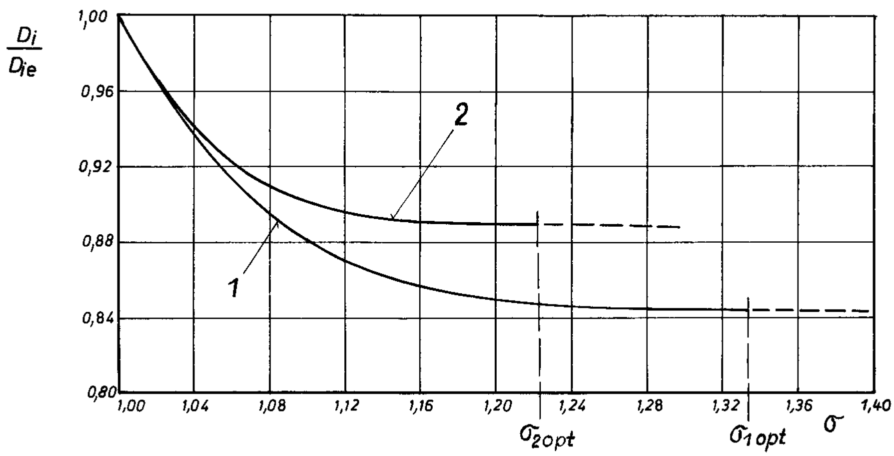

Figure 2
Induced drag versus span ratio $\sigma = s/s_e$. 1 Prandtl's solution, eq. (8). 2 Solution of present investigation, eq. (19).

References

[1] L. PRANDTL, *Über Tragflügel kleinsten induzierten Widerstandes*, Z. Flugtechn. Motorluftschiffahrt *24*, Nr. 11, 305–306 (1933).

[2] D. KÜCHEMANN and J. WEBER, *An Analysis of Some Performance Aspects of Various Types of Aircraft Designed to Fly over Different Ranges at Different Speeds*, Progr. Aeron. Sci. *9*, 329–456, Pergamon Press 1968.

[3] J. WEBER, *The Calculation of the Pressure Distribution over the Surface of Two-dimensional and Swept Wings with Symmetrical Aerofoil Sections*, ARC R&M 2918 (1953).

Summary

The spanwise aerodynamic loading of wings having minimum induced drag is derived for prescribed lift and root-bending moment. This problem is an alternative to Prandtl's solution for the case that the lift and its moment of inertia about the longitudinal axis of the aircraft are given.

Zusammenfassung

Es wird die Zirkulationsverteilung über der Spannweite hergeleitet, die für Tragflügel mit gegebenem Auftrieb und Biegemoment an der Flügelwurzel den minimalen induzierten Widerstand ergibt. Dieses Problem stellt eine Alternative zu der Lösung dar, die L. Prandtl für den Fall angegeben hat, dass der Auftrieb und das von ihm um die Flugzeuglängsachse erzeugte Trägheitsmoment gegeben sind.

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