On the Problem of Uniqueness under Pressure Loading

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1. Introduction

Consider a typical boundary value problem for an arbitrary solid in which traction and traction-rate are prescribed on a part S_F of the surface and velocity prescribed on the remainder S_v. If changes in geometry are taken into account, the problem does not necessarily possess a unique solution and a bifurcation therefore results at a certain stage of the deformation. It is, however, possible to have a stable bifurcation with the load increasing with continuing deformation. This was first indicated by Shanley [1] in relation to the failure of inelastic columns.

For a wide class of non-linear solids, a sufficient condition for uniqueness may be obtained by linearizing the constitutive law connecting the strain and stress-rates [2]. Under certain boundary conditions, bifurcation in the linearized solid may occur for any value of the traction-rate when the varying parameter (load or modulus) attains a critical value. In the actual non-linear solid, bifurcation may still occur at the same value of the varying parameter, though only for a certain range of values of the traction-rate. The so-called convected derivative of the Kirchoff stress previously employed [2] in this context to describe the constitutive law is, however, unsatisfactory for a general class of solids.

Although load-type sensitivity in relation to bifurcation has long been recognized for both elastic and plastic solids, very little useful general consideration has been given in the literature to load-types other than dead-loading. In the present investigation, the practically important case of pressure-type loading is considered for solids of arbitrary constitutive law under sufficiently general boundary conditions. A sufficient condition for uniqueness is discussed for the important special case of conventional plastic solids.

The theory is illustrated with examples and the tangent modulus formula for the critical external pressure of long thin tubes is corrected.

2. Boundary Conditions

Since the future position of the surface is not known in advance when geometry changes are taken into account, it is convenient to formulate the boundary conditions in terms of the rate of change of nominal traction (i.e. rate of change of traction based on the initial configuration).

Let x_i be the instantaneous co-ordinates of a typical particle with respect to a fixed rectangular Cartesian frame of reference and a_i the co-ordinates at some initial state. If l_i is the unit vector in the direction of the outward normal to the initial surface element dS^0, the vector surface element at the initial state is

$$dS_i^0 = l_i \, dS^0 \, . \tag{1}$$

If this vector becomes dS_i at the instant under consideration, then [3]

$$dS_j = \frac{\varrho^0}{\varrho} \frac{\partial a_i}{\partial x_j} \, dS_i^0 \tag{2}$$

where ϱ^0 and ϱ are the densities at the particle in the initial and instantaneous states respectively.

If the boundary surface is subjected to a uniform fluid pressure p, the load currently acting on the surface element is

$$dP_j = -p \, dS_j = -p \left(\frac{\varrho^0}{\varrho} \frac{\partial a_i}{\partial x_j} \right) l_i \, dS^0$$

in view of equations (1) and (2).

Denoting by F_j the load per unit initial surface area (i.e. the nominal traction), the boundary condition becomes

$$F_j = -p \left(\frac{\varrho^0}{\varrho} \frac{\partial a_i}{\partial x_j} \right) l_i \, . \tag{3}$$

The nominal traction-rate is obtained by taking the material rate of change given by the operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_k \frac{\partial}{\partial x_k}$$

where v_k is the instantaneous velocity of the particle and t the time scale.

Now,

$$\frac{D}{Dt} \left(\frac{\varrho^0}{\varrho} \right) = - \frac{\varrho^0}{\varrho^2} \frac{D\varrho}{Dt} = \frac{\varrho^0}{\varrho} \frac{\partial v_k}{\partial x_k} \tag{4}$$

in view of the equation of continuity

$$\frac{D\varrho}{Dt} + \varrho \frac{\partial v_k}{\partial x_k} = 0 \, .$$

Also,

$$\frac{D}{Dt} \left(\frac{\partial a_i}{\partial x_j} \right) = \frac{\partial}{\partial x_j} \left(\frac{D a_i}{Dt} \right) - \frac{\partial v_k}{\partial x_j} \frac{\partial a_i}{\partial x_k} \, .$$

Since the initial co-ordinates do not change during the motion, $Da_i/Dt = 0$ and

$$\frac{D}{Dt} \left(\frac{\partial a_i}{\partial x_j} \right) = - \frac{\partial v_k}{\partial x_j} \frac{\partial a_i}{\partial x_k} \, . \tag{5}$$

Taking the material derivative of (3) and using (4) and (5), we have

$$\frac{DF_j}{Dt} = - \frac{\varrho^0}{\varrho} \frac{\partial a_i}{\partial x_j} \left(\frac{Dp}{Dt} + p \frac{\partial v_k}{\partial x_k} \right) l_i + \frac{\varrho^0}{\varrho} \frac{\partial a_i}{\partial x_k} \frac{\partial v_k}{\partial x_j} \, p \, l_i \, .$$

If the initial state is taken as that at the instant considered, $\varrho^0 = \varrho$ and $a_i = x_i$. Denoting the rate of change by a dot, we finally obtain the nominal traction-rate as

$$\dot{F}_j = - \dot{p} \, l_j + p \left(l_k \frac{\partial v_k}{\partial x_j} - l_j \frac{\partial v_k}{\partial x_k} \right) . \tag{6}$$

It is to be noted that the part of the nominal traction-rate given by only the first term can be prescribed[1]).

3. General Consideration of Bifurcation

Consider a body (of arbitrary constitutive law) subjected to prescribed nominal traction-rate on S_F and velocity on S_v, the remaining part of the boundary S_f being submitted to a uniform fluid pressure p. The current state of the body is assumed to be completely known and this is taken as the initial reference state in the field equations below.

If $\dot{s}_{ij}$ denotes the material rate of change of the nominal stress and $\dot{g}_j$ the body force rate per unit initial volume, the equation of equilibrium and the boundary condition are

$$\frac{\partial \dot{s}_{ij}}{\partial x_i} + \dot{g}_j = 0 \, , \tag{7}$$

$$\dot{F}_j = \dot{s}_{ij} \, l_i \tag{8}$$

where $\dot{F}_j$ is the rate of increase in nominal traction and l_i the outward drawn unit normal to the surface at the initial configuration.

The material rate of change of the true (Cauchy) stress, denoted by $\dot{\sigma}'_{ij}$, is related to $\dot{s}_{ij}$ by [3]

$$\dot{s}_{ij} = \dot{\sigma}'_{ij} - \sigma_{jk} \frac{\partial v_i}{\partial x_k} + \sigma_{ij} \frac{\partial v_k}{\partial x_k} \tag{9}$$

where σ_{ij} is the initial stress and v_i the initial velocity.

If there is more than one solution of the problem for given boundary conditions and body-force rate, equations (7)–(9) provide

$$\frac{\partial}{\partial x_i} (\Delta \dot{s}_{ij}) = 0 \, , \tag{10}$$

$$\Delta \dot{F}_j = l_i \, \Delta \dot{s}_{ij} \, , \tag{11}$$

$$\Delta \dot{s}_{ij} = \Delta \dot{\sigma}'_{ij} - \sigma_{jk} \frac{\partial}{\partial x_k} (\Delta v_i) + \sigma_{ij} \frac{\partial}{\partial x_k} (\Delta v_k) \tag{12}$$

where the prefix Δ denotes the difference of the corresponding quantities in the two solutions.

Application of Green's theorem to integrals for surface S and Volume V yields

$$\left. \begin{aligned} \int \Delta \dot{F}_j \, \Delta v_j \, dS &= \int \Delta \dot{s}_{ij} \frac{\partial}{\partial x_i} (\Delta v_j) \, dV \, , \\ &= \int \left[\Delta \dot{\sigma}'_{ij} - \sigma_{jk} \frac{\partial}{\partial x_k} (\Delta v_i) + \sigma_{ij} \frac{\partial}{\partial x_k} (\Delta v_k) \right] \frac{\partial}{\partial x_i} (\Delta v_j) \, dV \end{aligned} \right\} \tag{13}$$

in view of equations (10)–(12).

On the part S_f of the boundary, the nominal traction-rate is given by (6), so that

$$\Delta \dot{F}_j = p \left[l_k \frac{\partial}{\partial x_j} (\Delta v_k) - l_j \frac{\partial}{\partial x_k} (\Delta v_k) \right] \tag{14}$$

[1]) An incorrect expression for $\dot{F}_j$ was given by HILL [11].

on S_f, and it follows that

$$\int \Delta \dot{F}_j \, \Delta v_j \, dS = p \int \left[l_k \frac{\partial}{\partial x_j} (\Delta v_k) - l_j \frac{\partial}{\partial x_k} (\Delta v_k) \right] \Delta v_j \, dS_f \tag{15}$$

since the integrand on the left hand side vanishes on S_F (where $\Delta \dot{F}_j = 0$) and S_v (where $\Delta v_j = 0$).

Introducing the true strain-rate

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right),$$

we obtain from equations (13) and (15),

$$\left. \begin{aligned} \int \left[\Delta \dot{\sigma}'_{ij} \, \Delta \varepsilon_{ij} - \sigma_{ij} \frac{\partial}{\partial x_k} (\Delta v_i) \frac{\partial}{\partial x_j} (\Delta v_k) + \sigma_{ij} \frac{\partial}{\partial x_k} (\Delta v_k) \frac{\partial}{\partial x_i} (\Delta v_j) \right] dV \\ - p \int \left[l_k \frac{\partial}{\partial x_j} (\Delta v_k) - l_j \frac{\partial}{\partial x_k} (\Delta v_k) \right] \Delta v_j \, dS_f = 0 \, , \end{aligned} \right\} \tag{16}$$

the boundary condition outside S_f being arbitrary.

If the part of the boundary not submitted to fluid pressure is fully constrained, $\Delta v_j = 0$ on this part of the boundary and the surface integral in (16) can be formally extended over the entire surface of the body. Transformation of the surface into volume integral by Green's theorem then furnishes the result

$$\left. \begin{aligned} \int \Big[\Delta \dot{\sigma}_{ij} \, \Delta \varepsilon_{ij} - (\sigma_{ij} + p \, \delta_{ij}) \\ \times \left\{ \frac{\partial}{\partial x_k} (\Delta v_i) \frac{\partial}{\partial x_j} (\Delta v_k) + \frac{\partial}{\partial x_k} (\Delta v_k) \frac{\partial}{\partial x_i} (\Delta v_j) \right\} \Big] dV = 0 \, . \end{aligned} \right\} \tag{17}$$

Bifurcation can occur only if this equation is satisfied for some non-zero continuous differentiable field representing Δv_j and hence vanishing on S_v.

For the special case of rigid/plastic solids, an analogous equation was obtained by Hill [4].

In actual applications, recourse must be made to the constitutive law connecting the stress-rate[2]) and the strain-rate. However, the above stress-rate $\dot{\sigma}'_{ij}$ (referred to fixed axes) is not suitable for the purpose, because it does not vanish in the event of a rigid body rotation. The most suitable definition of stress-rate satisfying this requirement is the one originally given by Jaumann [5]. It is written as

$$\dot{\sigma}_{ij} = \dot{\sigma}'_{ij} - \sigma_{ik} \, \omega_{jk} - \sigma_{jk} \, \omega_{ik} \tag{18}$$

where ω_{ij} is the antisymmetric part of the velocity gradient tensor, namely

$$\omega_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right).$$

From symmetry of σ_{ij} and ε_{ij} and antisymmetry of ω_{ij} it is easy to show that

$$\Delta \dot{\sigma}'_{ij} \, \Delta \varepsilon_{ij} = \Delta \dot{\sigma}_{ij} \, \Delta \varepsilon_{ij} + 2 \, \sigma_{ij} \, \Delta \varepsilon_{jk} \, \Delta \omega_{ki} \, , \tag{19}$$

which gives the leading terms in equations (16) and (17).

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[2]) The stress-rate may depend also on higher rates of deformation.

It is to be noted that Jaumann's definition corresponds to the material rate of change of the true stress with respect to axes which take part in the instantaneous rotation. It is $\dot{\sigma}_{ij}$ (not $\dot{\sigma}'_{ij}$) that enters into the constitutive law. The so-called convected derivative of the stress used by Hill [2] is unsatisfactory and is likely to introduce errors of unknown magnitude in real solids, particularly plastic solids [6].

It is easy to see that an addition of the quantity $p\,\delta_{ij}$ to the current stress tensor σ_{ij} leaves equation (19) unchanged. Equation (17) then furnishes the following theorem:

Theorem: If a body is partly constrained and the remaining surface is submitted to a uniform fluid pressure p, the condition for the occurrence of bifurcation, irrespective of the constitutive law of the material, is the same as that without the pressure, provided the stress state is assumed as that in which the actual normal stresses in the body are increased by the amount p.

If in the conventional tensile test of a certain solid, bifurcation occurs when the tensile stress reaches the value σ, the above theorem indicates that an application of a uniform fluid pressure p on the lateral surface of the specimen reduces the tensile stress to $\sigma - p$ at bifurcation. If the deformation of the solid is substantially unaffected by a uniform hydrostatic pressure, the amount of uniform strain before bifurcation therefore remains unchanged.

4. Uniqueness in Conventional Plastic Solids

Consider a conventional elastic/plastic solid for which the plastic part of the strain rate is given by the plastic potential rule, while the elastic part is given by the generalized Hooke's law. Assuming an elastic isotropy, the constitutive law for an element currently at the yield point may be written as

$$\varepsilon_{ij} = \begin{cases} \dfrac{1}{2G}\left(\dot{\sigma}_{ij} - \dfrac{\nu}{1+\nu}\,\dot{\sigma}_{kk}\,\delta_{ij}\right) + \dfrac{3}{2H}\,\dot{\sigma}_{kl}\,n_{kl}\,n_{ij}\,, & \text{when } \dot{\sigma}_{kl}\,n_{kl} \geqslant 0\,, \\ \dfrac{1}{2G}\left(\dot{\sigma}_{ij} - \dfrac{\nu}{1+\nu}\,\dot{\sigma}_{kk}\,\delta_{ij}\right), & \text{when } \dot{\sigma}_{kl}\,n_{kl} \leqslant 0 \end{cases} \tag{20}$$

where H is a positive scalar denoting the rate of hardening. It coincides with the current slope of the true stress-strain curve (without elastic strain) in uniaxial tension when the material is also plastically isotropic. n_{ij} is the outward drawn unit normal to the regular yield surface (with $n_{ii} = 0$) in a nine-dimensional stress-space. G is the shear modulus and ν Poisson's ratio.

It follows from (20) that

$$\varepsilon_{ij}\,n_{ij} = \begin{cases} \dfrac{3G+H}{2GH}\,\dot{\sigma}_{ij}\,n_{ij}\,, & \text{when } \dot{\sigma}_{ij}\,n_{ij} \geqslant 0\,, \\ \dfrac{1}{2G}\,\dot{\sigma}_{ij}\,n_{ij}\,, & \text{when } \dot{\sigma}_{ij}\,n_{ij} \leqslant 0 \end{cases}$$

so that $\varepsilon_{ij}\,n_{ij} \gtreqless 0$ for $\dot{\sigma}_{ij}\,n_{ij} \gtreqless 0$. When $\varepsilon_{ij}\,n_{ij} \geqslant 0$, the scalar product of (20) with ε_{ij} gives

$$\dot{\sigma}_{ij}\,\varepsilon_{ij} = 2G\left[\varepsilon_{ij}\,\varepsilon_{ij} - \frac{3G}{3G+H}\,(\varepsilon_{ij}\,n_{ij})^2 + \frac{\nu}{1-2\nu}\,\varepsilon_{kk}^2\right] \tag{21}$$

where the last term is identically zero for an incompressible solid. When $\varepsilon_{ij} n_{ij} \leqslant 0$, the second equation of (20) gives

$$\dot{\sigma}_{ij} \varepsilon_{ij} = 2 G \left[\varepsilon_{ij} \varepsilon_{ij} + \frac{\nu}{1 - 2\nu} \varepsilon_{kk}^2\right]. \tag{22}$$

Consider now a fictitious solid whose constitutive law is given by the first equation of (20), regardless of the sign of $\varepsilon_{ij} n_{ij}$, whenever an element is currently plastic. Since the strain-rate is then a unique linear function of the stress-rate, the new solid may be regarded as a linearized elastic/plastic solid.

If the true stress-rate corresponding to the linearized solid is denoted by $\dot{\tau}_{ij}$, the rate-equation of this solid is

$$\varepsilon_{ij} = \frac{1}{2G} \left(\dot{\tau}_{ij} - \frac{\nu}{1+\nu} \dot{\tau}_{kk} \delta_{ij}\right) + \frac{3}{2H} \dot{\tau}_{kl} n_{kl} n_{ij} \tag{23}$$

for an element currently in the plastic region and it follows from (21) and (22) that

$$\left.\begin{aligned} &\dot{\sigma}_{ij} \varepsilon_{ij} \geqslant \dot{\tau}_{ij} \varepsilon_{ij} \\ &\quad = \frac{2G}{3G+H} [H \varepsilon_{ij} \varepsilon_{ij} + 3 G \{\varepsilon_{ij} \varepsilon_{ij} - (\varepsilon_{ij} n_{ij})^2\}] + \frac{2 G \nu}{1 - 2\nu} \varepsilon_{kk}^2 \end{aligned}\right\} \tag{24}$$

where the equality holds only in the loading part of the plastic region.

In any elastic region, $\dot{\tau}_{ij}$ completely coincides with $\dot{\sigma}_{ij}$ and hence

$$\dot{\sigma}_{ij} \varepsilon_{ij} = \dot{\tau}_{ij} \varepsilon_{ij} = 2 G \left(\varepsilon_{ij} \varepsilon_{ij} + \frac{\nu}{1 - 2\nu} \varepsilon_{kk}^2\right).$$

It is to be noted that the normal component of $\dot{\tau}_{ij}$ tangential to the elastic/plastic interface must be discontinuous.

From (16) and (19) a sufficient condition for uniqueness of the linearized solid may be written as

$$\int \left[\Delta\dot{\tau}_{ij} \Delta\varepsilon_{ij} + \sigma_{ij} (\Delta\varepsilon_{kk} \Delta\varepsilon_{ij} + 2 \Delta\varepsilon_{jk} \Delta\omega_{ki}) - \sigma_{ij} \frac{\partial}{\partial x_k} (\Delta v_i) \frac{\partial}{\partial x_j} (\Delta v_k)\right] dV$$
$$- p \int \left[l_k \frac{\partial}{\partial x_j} (\Delta v_k) - l_j \frac{\partial}{\partial x_k} (\Delta v_k)\right] v_j \, dS_f > 0$$

for all continuous differentiable fields Δv_j vanishing at the constraints.

If the constraints are rigid so that $v_j = 0$ there, every difference field is a member of the admissible field v_j. The above criterion then becomes

$$\left.\begin{aligned} &\int \left[\dot{\tau}_{ij} \varepsilon_{ij} + \sigma_{ij} \left(\varepsilon_{kk} \varepsilon_{ij} + 2 \varepsilon_{jk} \omega_{ki} - \frac{\partial v_i}{\partial x_k} \frac{\partial v_k}{\partial x_j}\right)\right] dV \\ &\qquad - p \int \left(l_k \frac{\partial v_k}{\partial x_j} - l_j \frac{\partial v_k}{\partial x_k}\right) v_j \, dS_f > 0 \end{aligned}\right\} \tag{25}$$

for all continuous differentiable fields vanishing on S_v.

In view of the inequality in (24), uniqueness of the linearized solid also ensures uniqueness of the non-linear elastic/plastic solid. If some non-zero field v_j makes the functional in (25) vanish, bifurcation in the linearized solid may occur for any value of the traction-rate on S_F and $\dot{p}$ on S_f, since this field need not represent an actual mode. In the actual elastic/plastic solid, however, bifurcation occurs only for those values of the above quantities which correspond to no unloading of the current plastic region.

Splitting the tensor $\partial v_i/\partial x_k$ into the symmetric part ε_{ik} and the anti-symmetric part ω_{ik} and noting that

$$\sigma_{ij} \frac{\partial v_i}{\partial x_k} \frac{\partial v_k}{\partial x_j} = \sigma_{ij} (\varepsilon_{jk} \varepsilon_{ki} + \omega_{jk} \omega_{ki})$$

(the other triple products cancelling one-another), the uniqueness criterion may be written in the alternative form

$$\left.\begin{aligned} &\int [\dot{\tau}_{ij} \varepsilon_{ij} + \sigma_{ij} (\varepsilon_{kk} \varepsilon_{ij} + 2 \varepsilon_{jk} \omega_{ki} - \varepsilon_{jk} \varepsilon_{ki} - \omega_{jk} \omega_{ki})] \, dV \\ &\qquad - p \int [l_k (\varepsilon_{kj} + \omega_{kj}) - l_j \varepsilon_{kk}] \, v_j \, dS_f > 0 \,, \end{aligned}\right\} \tag{26}$$

which is most convenient for practical applications in any curvilinear coordinate system[3]). ω_{ij} is related to the spin-vector $\boldsymbol{\omega} = (1/2) \operatorname{curl} \boldsymbol{v}$ by the equation

$$\omega_{ij} = - e_{ijk} \omega_k$$

where e_{ijk} is the permutation symbol. More explicitly,

$$\omega_{ij} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} . \tag{27}$$

Two important special cases will be considered below.

(i) The plastic modulus H is large in comparison with the components of σ_{ij}. In this case, the terms $\sigma_{ij} \varepsilon_{ij} \varepsilon_{kk}$ and $\sigma_{ij} \varepsilon_{jk} \varepsilon_{ki}$ in (26) are negligible in comparison with similar terms in the strain-rate components occuring in (24). The uniqueness criterion (25) then reduces to

$$\int [\dot{\tau}_{ij} \varepsilon_{ij} + \sigma_{ij} (2 \varepsilon_{jk} \omega_{ki} - \omega_{jk} \omega_{ki})] \, dV - p \int [l_k (\varepsilon_{kj} + \omega_{kj}) - l_j \varepsilon_{kk}] \, v_j \, dS_f > 0 \tag{28}$$

which is essentially similar to that given by Pearson [7] for classical elastic solids. Indeed, by letting H tend to infinity in (24) the elastic constitutive equation is recovered.

It may therefore be concluded that bifurcation of the linearized elastic/plastic solid in the present case corresponds to an eigenstate and the eigenfield makes the functional in (28) an absolute minimum.

(ii) It is sometimes useful to regard the material as rigid/plastic in which the shear modulus G is infinitely large. In this case, the strain-rate vector is entirely normal (though not necessarily outward) to the yield surface and also $\varepsilon_{kk} = 0$; hence $\dot{\tau}_{ij} \varepsilon_{ij} = (2/3) H \varepsilon_{ij} \varepsilon_{ij}$.

When the principal axes of the strain-rate coincide with those of the stress, the scalar product $\sigma_{ij} \varepsilon_{jk} \omega_{ki}$ identically vanishes and the uniqueness criterion (25)

[3]) Provided ε_{ij} etc. are interpreted as the curvilinear components.

reduces to

$$\int \left[\frac{2}{3} H \varepsilon_{ij} \varepsilon_{ij} - \sigma_{ij} (\varepsilon_{jk} \varepsilon_{ki} + \omega_{jk} \omega_{ki}) \right] dV - p \int l_k (\varepsilon_{kj} + \omega_{kj}) v_j \, dS_f > 0 \tag{29}$$

with the understanding that either the stress or the velocity is known in any rigid region. The above criterion is essentially the same as that obtained elsewhere [8] for the rigid/plastic body treated on its own merit.

5. Illustrative Examples

a) *Cylindrical Shell Under External Pressure*

As a first example for the application of the preceding theory, consider a thin-walled circular cylindrical shell subjected to a uniform external pressure p. If the cylinder is sufficiently long, it is safe to neglect axial strain-rate in the choice of admissible field for the investigation of bifurcation [7]. Using cylindrical co-ordinates (r, θ, x) in the radial, circumferential and axial directions respectively, the velocity field may therefore be taken in the form

$$v_r = w \,, \qquad v_\theta = v + \frac{z}{R} (v - w') \,, \qquad v_x = 0 \tag{30}$$

where z is the radially outward distance from the middle surface of radius R. The quantities v and w are periodic functions of θ and represent the velocity of the middle surface.

The above velocity field corresponds to the usual thinshell approximation and is adequate for calculating all strain-rate except the through-thickness one. The shear strain-rates $\varepsilon_{r\theta}$ and ε_{rx}, which are conventionally negligible, actually vanish in this case. A straightforward calculation furnishes

$$\left.\begin{aligned} \varepsilon_{\theta\theta} &= \frac{1}{R} \left[(v' + w) - \frac{z}{R} (w + w'') \right], \\ \varepsilon_{xx} &= \varepsilon_{x\theta} = \varepsilon_{rx} = \varepsilon_{r\theta} = 0 \,, \\ \omega_x &= \frac{1}{R} (v - w') \,, \quad \omega_r = \omega_\theta = 0 \end{aligned}\right\} \tag{31}$$

with a minor approximation.

When the pressure acts on the lateral surface, the current state of stress is a uniaxial compression in the circumferential direction and the unit normal to the yield surface has the non-zero components (assuming isotropy)

$$n_{rr} = n_{xx} = \frac{1}{\sqrt{6}} \,, \qquad n_{\theta\theta} = -\sqrt{\frac{2}{3}}$$

and the constitutive equation (23) gives

$$\begin{aligned} \varepsilon_{rr} &= -\left(\frac{\nu}{E} + \frac{1}{2H} \right) \dot{\tau}_{\theta\theta} - \left(\frac{\nu}{E} - \frac{1}{4H} \right) \dot{\tau}_{xx} \,, \\ \varepsilon_{\theta\theta} &= \left(\frac{1}{E} + \frac{1}{H} \right) \dot{\tau}_{\theta\theta} - \left(\frac{\nu}{E} + \frac{1}{2H} \right) \dot{\tau}_{xx} \,, \\ 0 &= -\left(\frac{\nu}{E} + \frac{1}{2H} \right) \dot{\tau}_{\theta\theta} + \left(\frac{1}{E} + \frac{1}{4H} \right) \dot{\tau}_{xx} \end{aligned}$$

where E is Young's modulus. Introducing the tangent modulus

$$T = \frac{E\,H}{E + H},$$

a short calculation yields

$$\dot{\tau}_{ij}\,\varepsilon_{ij} = \dot{\tau}_{\theta\theta}\,\varepsilon_{\theta\theta} = \lambda\,E\,\varepsilon_{\theta\theta}^2$$

where

$$\lambda = \frac{1 + 3\,T/E}{(5 - 4\,\nu) - (1 - 2\,\nu)^2\,T/E}. \tag{32a}$$

If the cylinder has closed ends and the pressure is all-round, the value of λ is modified. The state of stress is then a pure shear (with superimposed hydrostatic tension) and the non-zero components of n_{ij} are

$$n_{rr} = \frac{1}{\sqrt{2}}, \quad n_{\theta\theta} = -\frac{1}{\sqrt{2}}, \quad n_{xx} = 0.$$

Using the constitutive equation, a calculation similar to above furnishes

$$\lambda = \frac{4\,T/E}{3 + (1 - 4\,\nu^2)\,T/E}. \tag{32b}$$

Since p/E is only of order $(t/R)^3$ at the bifurcation (where t is the current uniform thickness), it is evident that the criterion (28) is appropriate in the present context. Considering a unit length of the cylinder, the criterion becomes

$$\iint (\lambda\,E\,\varepsilon_{\theta\theta}^2 + \sigma_{\theta\theta}\,\omega_x^2)\,dz\,d\theta + p\int (\varepsilon_{\theta\theta}\,v_r + \omega_x\,v_\theta)_{z=0}\,d\theta > 0$$

to a sufficient accuracy, for both the loading conditions.

Substituting for $\varepsilon_{\theta\theta}$ and ω_x from (31) and observing that $\sigma_{\theta\theta} = -p\,R/t$, the critical pressure is obtained as

$$\frac{p_c}{E} = \frac{\lambda\,t}{R}\,\min\,\frac{\int_0^{2\pi}(v' + w)^2\,d\theta + (t^2/12\,R^2)\int_0^{2\pi}(w + w'')^2\,d\theta}{\int_0^{2\pi}[(w' - v)\,w' - (v' + w)\,w]\,d\theta}.$$

The minimum value is very closely obtained by setting $w = -v'$ and $v = \operatorname{Sin} 2\theta$. Hence the critical pressure is

$$\frac{p_c}{E} = \frac{\lambda}{4}\left(\frac{t}{R}\right)^3 \tag{33}$$

where λ is given by (32).

It follows that in the elastic range ($T = E$), λ has the well-known value $1/(1 - \nu^2)$ for both loading conditions. In the plastic range, however, the critical pressure for the second case is somewhat lower than that for the first case[4]). The tangent modulus formula [9], which uses the elastic value of λ but replaces E by T in (33), underestimates the critical pressure for all values of $\nu < 0.5$. The tangent modulus value coincides with the closed-end value for $\nu = 0.5$.

[4]) Except for very small values of Poisson's ratio.

In view of the remarks in Section 4, the critical pressure of the elastic/plastic cylinder (with no imperfection) cannot be lower than that given by (33). However, the pressure must increase in such a way that there is no incipient unloading of the cylinder at the bifurcation. The velocity field at the bifurcation is any linear combination of the eigenfield (given by $v = \mathrm{Sin}\,2\theta$, $w = -2\,\mathrm{Cos}\,2\theta$) and that corresponding to a uniform contraction, such that $\varepsilon_{\theta\theta} < 0$.

b) *Cylindrical Bar Under Lateral Pressure*

As a second example, consider a circular cylindrical bar of radius a subjected to a uniform fluid pressure p on the lateral surface. The ends of the bar are supported in such a way that they are free to move axially during the deformation.

Since the critical rate of hardening would be of the same order as the yield stress, the compressibility may be neglected in the present investigation. Using cylindrical co-ordinates (r, θ, x), the virtual velocity field may be taken in the form

$$v_r = -\frac{r}{2\,l}\,f'(z)\,, \quad v_\theta = 0\,, \quad v_x = f(z) \tag{34}$$

where l is the length of the bar and $z = x/l$. The velocity Field furnishes

$$\left.\begin{aligned} &\varepsilon_{rr} = \varepsilon_{\theta\theta} = -\frac{1}{2\,l}\,f'(z)\,, \quad \varepsilon_{xx} = \frac{1}{l}\,f'(z)\,, \quad \varepsilon_{rx} = -\frac{r}{4\,l^2}\,f''(z)\,, \\ &\varepsilon_{r\theta} = \varepsilon_{\theta x} = 0\,, \quad \omega_\theta = -\frac{r}{4\,l^2}\,f''(z)\,, \quad \omega_r = \omega_x = 0\,. \end{aligned}\right\} \tag{35}$$

The current state of stress is given by

$$\sigma_{rr} = \sigma_{\theta\theta} = -p\,, \quad \sigma_{xx} = \sigma_{rx} = \sigma_{r\theta} = \sigma_{\theta x} = 0$$

and the non-zero components of the unit normal n_{ij} are

$$n_{rr} = n_{\theta\theta} = -\frac{1}{\sqrt{6}}\,, \quad n_{xx} = \sqrt{\frac{2}{3}}$$

so that $\varepsilon_{ij}\,n_{ij} = \sqrt{(3/2)}\;\varepsilon_{xx}$ in view of the assumed incompressibility.

Since H is small in comparison with G, equation (24) yields

$$\dot{\tau}_{ij}\,\varepsilon_{ij} = H\,\varepsilon_{xx}^2 + 4\,G\,\varepsilon_{rx}^2$$

and the uniqueness criterion (26) becomes

$$\iint [H\,\varepsilon_{xx}^2 + 4\,G\,\varepsilon_{rx}^2 + p\,(\varepsilon_{rr}^2 + \varepsilon_{\theta\theta}^2)]\,r\,dr\,dx - p\,a \int (\varepsilon_{rr}\,v_r + 2\,\varepsilon_{rx}\,v_x)_{r=a}\,dx > 0$$

where use has been made of the fact that $\omega_\theta = \varepsilon_{rx}$.

Substituting from (34) and (35) and integrating through the cross section, we have

$$\int_0^1 \left[H(f')^2 + p\,f\,f'' + \frac{1}{8}\,G\left(\frac{a}{l}\right)^2 (f'')^2\right] dz > 0\,. \tag{36}$$

Since the shear strain-rate is purely elastic, it is evident that the last term in the above functional is small in comparison with the first term. The minimum value of the functional is then obtained (very closely) by setting $H = p$, whatever the function f. The critical rate of hardening is therefore the same as that in uniaxial tension and this result is in complete agreement with experiment [10]. A local neck is formed at the bifurcation and the deformation continues temporarily under constant pressure.

6. Concluding Remarks

A sufficient condition for uniqueness of the bilinear elastic/plastic solid is obtained by using a linearized version of the solid. The discussion by Hill was based on the assumption that the constitutive equation has a unique inverse; the present treatment is free from this restriction. The condition for uniqueness discussed here is more general than that given by Hill and brings a wide range of practical problems within the theoretical framework.

The problem of uniqueness of solids is closely related to that of stability. The difference between the criteria governing the two problems appears only through the constitutive law. If the solid responds identically to loading and unloading, the criterion for uniqueness is the same as that for stability. For solids having separate loading and unloading responses, the two criteria however differ. As a result, a point of bifurcation may be reached before an actual loss of stability.

A general derivation of the stability condition, with special reference to rigid/plastic solids, has been given by the writer [8].

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Zusammenfassung

Die Bedingung für das Auftreten einer Verzweigung bei einem beliebigen festen Körper wird allgemein für den Fall diskutiert, dass Teile der Oberfläche unter einer druckartigen Belastung stehen. Besondere Beachtung wird der Frage der Eindeutigkeit bei konventionellen plastischen Körpern geschenkt und mit Beispielen illustriert. Die Formel mit dem Tangentenmodul für das Beulen einer langen dünnwandigen Zylinderschale unter sehr flüssigem Druck wird verbessert.

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