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SURVEY OF SOLVED AND OPEN PROBLEMS IN THE DEGENERACY PHENOMENON

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1. Introduction

Degeneracy is a phenomenon that may arise, e.g., in linear programming (LP for short), bottleneck LP, multiparametric LP, linear vectormaximization, etc. If it does arise then it certainly influences any vertex-searching method for mathematical models based on a system of linear inequalities and in some cases it leads to misinterpretation of optimal solutions.

Consider a vertex $x^0 \in X$, $X := \{x \in \mathbb{R}^n : Ax \leqslant b, x \geqslant 0\}$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Transforming X by means of slack variables $u_i = x_{n+i} \geqslant 0$, $i = 1, \ldots, m$, into

$$\bar{X} := \{y \in \mathbb{R}^{m+n} \mid \bar{A}y = b, y \geqslant 0\}, \quad y = \begin{pmatrix} x \\ u \end{pmatrix}, \bar{A} = (A \mid I_m),$$

a basis B (i.e. an $m \times m$ submatrix) of $\bar{A}$ is one-to-one assigned to x^0, provided that x^0 is nondegenerate.

If $x^0 \in X$ is a degenerate vertex, i.e., it is overdetermined (more than n hyperplanes pass through x^0), a set of bases $B^0 := \{B_u^0 | u = 1, \ldots, U\}$, $U > 1$, is associated with x^0. Denote by $y^{(0)} \in \mathbb{R}^{m+n}$ the basic feasible solution of $\bar{A}y = b$, $y \geqslant 0$ that corresponds to x^0. Further, denote by $\sigma \in \{1, \ldots, m\}$ the *degeneracy degree* of x^0, i.e., σ is the number of zero-elements of $y^{(0)}$ that occur among the basic variables of $y^{(0)}$.

From the historical point of view, in 1976-77 the problem arose to find all or a part of neighbouring (adjacent) vertices of a given degenerate vertex $x^0 \in X$ with minimal effort [10]. We tried then to approach this problem from a graph-theoretical point of view.

Assuming $X \neq \emptyset$, and, for simplicity, bounded, the set X defines obviously a convex polytope. Suppose there is no degenerate vertex $x \in X$. Then it is possible to represent any convex polytope by the so called *graph of the polytope* $G'(X)$ [20].

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The nodes of $G'(X)$ correspond to vertices of X (or: to bases of $\bar{X}$) and every edge of $G'(X)$ is one-to-one assigned to an edge (1-face) of X. Any basis B can be also represented by a simplex-type tableau (without the reduced costs row). "Travelling" from a vertex $x^i \in X$ (associated with the basis B_i) to another vertex $x^{i'} \in X$, $i \neq i'$ (associated with the basis $B_{i'}$) along an edge $(x^i, x^{i'}) \subset X$ corresponds to a unique basis-exchange B_i to $B_{i'}$ (or vice versa). In tableau representation this corresponds to a Gauss–Jordan elimination step with a positive pivot (because of $y \geqslant 0$). Denote this exchange by $B_i \overset{+}{\leftrightarrow} B_{i'}$.

Suppose now that x^0 is degenerate. Then the graph of the polytope, $G'(X)$, misses essential information about the degeneracy structure of x^0. We therefore introduced a so called (positive or proper) *degeneracy graph* (DG for short), $G_+^0 := G_+^0(x^0)$, the nodes of which correspond to the bases B_u^0 associated with x^0 and the edges of which are defined as $\{B_u^0, B_{u'}^0\}$, $u, u' = 1, \ldots, U$, $u \neq u'$, iff $B_u^0 \overset{+}{\leftrightarrow} B_{u'}^0$. Embedding G_+^0 into $G'(X)$ we obtain the so called *representation graph*, $G(X)$, of the polytope X. If $U = 1$ then $G'(X) = G(X)$.

Consider as an example a cube $X := \{x \in \mathbb{R} \mid x_1 \leqslant 1,\ x_2 \leqslant 1,\ x_3 \leqslant 1,\ x_{1,2,3} \geqslant 0\}$ (Fig. 1.1a) and its graph representation (Fig. 1.1b).

We now add the constraint $x_1 + x_2 + x_3 \leqslant 3$ which preserves X, but now the vertex x^0 is overdetermined (Fig. 1.2a). Symbolizing by a frame the DG G_+^0 of x^0 we obtain the representation graph $G(X')$ of the "new" cube

$$X' := \{x \in \mathbb{R}^3 \mid x_1 \leqslant 1, x_2 \leqslant 1, x_3 \leqslant 1, x_1 + x_2 + x_3 \leqslant 3, x_{1,2,3} \geqslant 0\}$$

(Fig. 1.2b).

Let us have a closer look at a hypothetical graph G_+^0 (Fig. 1.3) of a degenerate vertex that has 3 neighbours corresponding to B^1, B^2 and B^3. Hypothetical because G_+^0 of x^0 in the above cube X' would be a too simple graph with only 4 nodes. The nodes in frames are the so called *transition nodes* with the property that they connect G_+^0 with an "outer" node, i.e., with a node belonging to $G(X)$ but not to G_+^0. The remaining nodes of G_+^0 are called *internal nodes.*

In this paper we shall give a concise survey of results concerning the theory and some applications of the DG's.

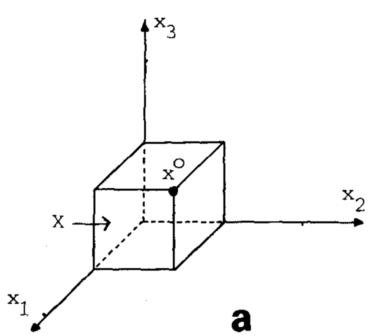


Fig. 1.1.a.

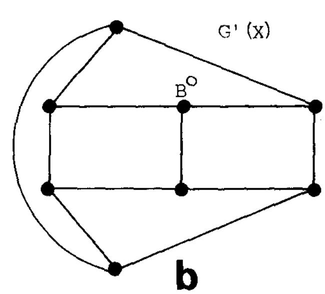


Fig. 1.1.b.

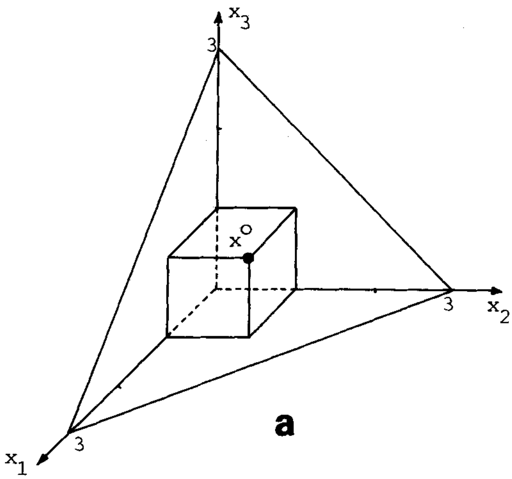
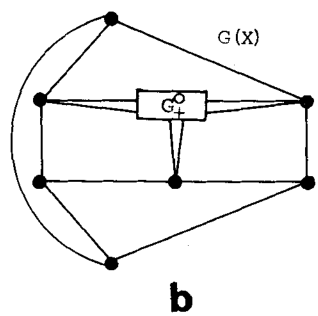


Fig. 1.2.a.



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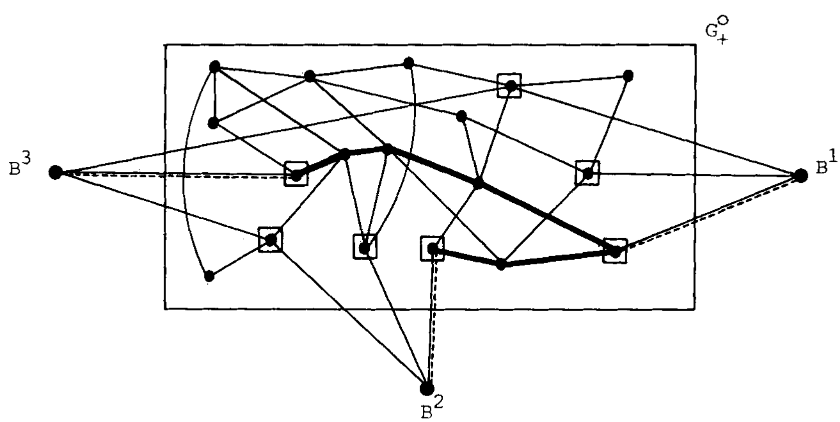

Fig. 1.3

2. Some theoretical results

We proved [10, 12] that in G_+^0 there exists a tree $\bar{G}^0 \subset G_+^0$ that connects x^0 with all its neighbours. In Fig. 1.3 such a tree is illustrated by bold lines. Thus having found such a tree the initial problem to find all neighbours of a degenerate x^0 is solved.

In order to determine efficiently a tree $\bar{G}^0$ we developed a method called the N-tree-algorithm (N for neighbour) [15, 17]. It is is a special tree-algorithm which is combined with a lexicographic selection rule. We found that the degree of the transition nodes might play a role in selecting the "best" starting node for the N-tree-algorithm. This might lead to the determination of a so called minimal tree $\bar{G}^0_{\min}$ (a tree with minimal number of nodes). We conjectured that the greatest integer smaller or equal to $U/(\sigma+1)$ is an upper bound on the number of nodes needed to form a minimal tree. A series of tests is running with a corresponding subroutine that finds a transition node with the maximal degree as starting node. Recent computing results confirm the above conjecture and they show a possible direction for further improvement of the procedure [16]. We hope for new clues to determine $\bar{G}^0_{\min}$ from further research into the theory of DG's. We enlarged our research to DG's in which the edges can be defined via basis-exchange using negative pivots. Allowing negative pivots only we called the corresponding DG the *negative DG* G^0_-. If any nonzero pivot is admissible we call the corresponding DG the *general DG* G^0.

We found [27] that the general DG is always connected, while the proper or negative DG's may be unconnected.

Investigating the bounds of the number U of nodes B^0_u of B^0 the upper bound is obviously

$$U_{\max}=\binom{n+\sigma}{\sigma}$$

and we found [27, p. 50] that the lower bound is

$$U_{\min}=2^{\sigma-1}(n-\sigma+2).$$

Currently a series of empirical tests is running in order to find $U\in[U_{\min}, U_{\max}]$ in real cases. The preliminary results show that U is not much larger than $U_{\min}$ even though $U_{\min}$ is itself a substantial number. For example, when $n=50$ and $\sigma=5$ then $U_{\min}$ and $U_{\max}$ have the values 752 and 3479761 respectively.

We studied the general structure of DG's and we started with the construction principle for the case $\sigma\leqslant 2$ [17, 34]. The results are summarized in Theorem 2.1.

Given the main characteristics σ and n ($x\in\mathbb{R}^n$ with the degeneracy degree σ) of a DG we concisely write $\sigma\times n$-DG. In Theorem 2.1 the notation of a line graph $L(G)$ appears. $L(G)$ is a graph the nodes of which correspond to the edges of G, and two nodes of $L(G)$ are adjacent nodes iff the corresponding edges of G are adjacent edges. $K(p_1,\ldots,p_r)$ is the complete r-partite graph such that the cardinalities of the node sets in the partition are $p_1,\ldots,p_r$ (for details see [17] or [34]).

Theorem 2.1. *A graph G is a* $2\times n$*-DG iff G is isomorphic to a line graph*

$$L(K(p_1,\ldots,p_r))$$

where $r, p_1,\ldots,p_r$ *are positive integers with* $r\geqslant 2, \sum_{i=1}^r p_i=n+2$ ($p_1,p_2>1$ for $r=2$)

The properties of a $2\times n$-DG are summarized in:

Theorem 2.2. *The* $2\times n$*-DG of Theorem* 2.1. *has the following properties*:

(i) *the diameter is* $\leqslant 2$.

(ii) *the number of nodes is* $\frac{1}{2}((n+2)^2-\sum_{i=1}^r p_i^2)$.

(iii) *the number of edges is* $\sum_{i=1}^{r} p_i\binom{n+2-p_i}{2}$.
(iv) *the connectivity is* $2n+2-p_{r-1}-p_r$.

For $\sigma=1$ the DG's are complete graphs with $n+1$ nodes [27, p. 116]. Further investigations into the structure of DG's with $\sigma>2$ are currently being performed.

3. Degeneracy graphs and linear programming

3.1. Linear programming and simplex-cycling

Consider the linear program

$$\text{(LP)} \qquad \max_{x\in X} c^{\mathrm{T}}x, \quad c\in\mathbb{R}^n.$$

Speaking about the connection between (LP) and degeneracy the first idea coming to one's mind is simplex-cycling [4, 21]. Several anticycling methods have been developed during the past 35 years (see, e.g., [3, 5, 7, 33]) since Charnes [6] published the first known perturbation scheme in 1952. Anticycling devices are embedded into professional LP-software despite a discussion in the literature [18, 25, 26, 29, 32] on whether cycling appears in real-world applications or not. In our opinion more attention should be paid to the connection between the structure of the matrix $(A\,|\,b)$ enlarged by the row $(c^{\mathrm{T}},0)$ and cycling than to attempt to improve or invent new anticycling methods. In other words, $(A\,|\,b)$ implies B^0 and hence induces G_+^0, and in G_+^0 there exist in general closed lines (circuits for short — see also Fig. 1.3). The question is then to identify the properties of G_+^0 that cause simplex cycling. We approached this problem from two points of view: firstly using known properties of the DG's and secondly introducing new concepts. For the first approach it was necessary to enlarge the notion of a DG by the so called LP-DG [28]. We then found that a circuit of G_+^0 is a simplex cycle iff the circuit can be enlarged to a so

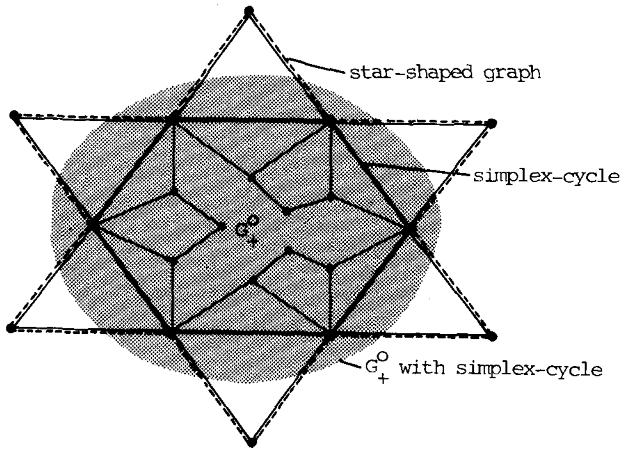

Fig. 3.1

called star-shaped graph embedded into the LP-DG (Fig. 3.1); the depicted graph is based on the Beale cycling example [4].

The second approach is not exclusively based on the theory of the DG's. Given (A, b, c) and a circuit C of G_+^0; then it is possible to induce in a certain way a point-set $M(C)$ such that C is a simplex-cycle iff the gradient of the objective function satisfies $c \in M(C)$ [35]. We are currently trying to develop a method which solves the problem: given (A, b, c) and a degenerate vertex x^0, determine whether $c \in M(C)$ for at least one C of G_+^0. If so, simplex-cycling occurs. We are at the very beginning of finding an efficient method to solve this problem.

3.2. Degeneracy in an optimal solution

Suppose that (LP) has an optimal solution with optimal degenerate vertex x^0. Let us select those nodes $\tilde{B}_k^0$ of B^0 which are associated with optimal bases. In general only a part $\tilde{B}^0 \subseteq B^0$ forms the set of optimal bases associated with x^0. The subgraph $\tilde{G}^0 \subset G^0$ induced by $\tilde{B}^0$ is then called an o-DG (o for optimal). We studied various properties of $\tilde{G}^0$.

We first investigated the number of nodes in $\tilde{B}^0$. We proved [28] some results which are summarized in:

Lemma 3.1

(i) *The case that* $\tilde{B}^0 = B^0$ *exists.*

(ii) *There is a triple* (A, b, x^0) *to which no objective function* $z = c^{\mathrm{T}}x$ *can be assigned such that* $\tilde{B}^0 = B^0$.

(iii) *The case exists that one and only one basis of* x^0 *is an optimal basis.*

(iv) *For the case that one and only one basis of* x^0 *is an optimal basis the necessary condition is that* $\tilde{G}_-^0$ *has at least one isolated node.*

Denote by $\tilde{G}_+^0$ and $\tilde{G}_-^0$ the positive and negative o-DG's, respectively.

We studied the connectivity properties of the o-DG's [28] and found — among other things — that a simultaneous dual degeneracy plays an important role in this connection which corresponds to very recent results by Greenberg [19]. Our results [28] are summarized in:

Lemma 3.2.

(i) *Two optimal bases* $\tilde{B}_k^0, \tilde{B}_{k'}^0 \in \tilde{B}^0$, $k \neq k'$, *for which* $\tilde{B}_k^0 \overset{+}{\leftrightarrow} \tilde{B}_{k'}^0$, *are always dual degenerate.*

(ii) $\tilde{G}_+^0$ *is trivial (i.e., any component of* $\tilde{G}_+^0$ *consists of exactly one node) iff there is no dual degeneracy.*

(iii) $\tilde{G}_-^0$ *can be unconnected.*

3.3. Sensitivity analysis under degeneracy

Sensitivity analysis with respect to the right hand side ("RHS-ranging") or with respect to the objective function coefficients ("Cost-ranging") has now become a constituent part of commercial LP-software. However, in case of degeneracy the corresponding subprograms yield erroneous results [9, 13, 24].

Performing sensitivity analysis in a nondegenerate case means [11] considering a scalar parameter λ, e.g., in the RHS, to determine the parameter interval Λ such that for all $\lambda \in \Lambda$ the optimal basis associated with the optimal vertex $x \in X$ does not change (remains optimal). However, if $x^0 \in X$ is a degenerate optimal vertex, the set $\tilde{B}^0$ is assigned to x^0. What would then be invariant? In [9, 13, 24] it is suggested to determine Λ_k^0 for each $\tilde{B}_k^0$ associated with x^0. Then the parameter interval is the union $\Lambda = \bigcup_{k=1}^{K} \Lambda_k^0$, K being the number of all bases in $\tilde{B}^0$. Changing the RHS implies, however, that $\tilde{B}^0$ and B^0 change.

We propose, therefore, the following formulation for sensitivity analysis with respect to the RHS b under degeneracy:

Determine Λ such that for all $\lambda \in \Lambda$ at least one basis $\tilde{B}_u^0 \in \tilde{B}^0$ remains optimal. With respect to the cost ranging the corresponding formulation reads:

Determine T (the overall parameter interval) such that for all $t \in T$ at least one $\tilde{B}_u^0 \in \tilde{B}^0$ remains optimal, or equivalently, x^0 remains the optimal vertex.

We are studying the economic impact of these formulations. An interesting contribution to these questions is found in [19]. The above "definitions" can easily be enlarged to the case with parameter-vectors.

3.4. Shadow prices under degeneracy

The determination of shadow prices under (primal) degeneracy is closely related to sensitivity analysis. It is known (see, e.g., [1, 2, 8, 9, 19, 23, 31]) that under primal degeneracy there does not exist *the* shadow price of a resource b_i. It has been proved that in such a case there exist two shadow prices, one for "buying one unit" and one for "selling one unit" of b_i. For details see [13]. Hence, the problem is not any more the existence of shadow prices under (primal) degeneracy but how to determine them. In [24] methods have been proposed, unfortunately without complete proofs.

Therefore, we are currently investigating the questions [30]: If for instance the o-DG $\tilde{G}_-^0$ is not connected (see Lemma 3.2(iii)) is it sufficient to consider only one component (maximal connected subgraph) of $\tilde{G}_-^0$? If yes, which of the components should be chosen? Or does it not matter which one?

3.5. Degeneracy and redundant constraints

Degeneracy is closely related to weakly redundant constraints. A weakly (w. for short) redundant constraint passing through $x^0 \in X$ causes its overdetermination and thus degeneracy. Hence w. redundancy is a sufficient condition for degeneracy because the latter is not necessarily caused by w. redundancy only. Concerning the sources of degeneracy, compare also [19].

From a formal point of view it is known [22] that a w. redundant constraint can be omitted without influencing the set X. If degeneracy is caused by w. redundancy would then this circumstance not simplify sensitivity analysis and the determination of shadow prices?

We found that in the case of sensitivity analysis with respect to b and shadow prices it is not possible to omit the w. redundant constraints and *then* determine *the* shadow price or *the* parameter interval in the usual sense. We proved [14] that omitting the w. redundant constraints leads to false results in general. This is not true for sensitivity analysis with respect to c.

This is, incidentally, in contrast to formal methods for determining redundant constraints [22]. These methods aim to determine the redundancies in order to omit them. Hence, the number of constraints is reduced which reduces the CPU time for computing an optimal solution of (LP). However, omitting the w. redundant constraints may heavily influence the economic interpretation of the optimal solution, the determination of shadow prices and sensitivity analysis with respect to b.

Note added in proof

After the manuscript of this paper had been submitted the authors found a "proof" that degeneracy problems are not by a long way out of consideration [36, 37, 38] and wish to hint to a very recent state-of-the-art of degeneracy [39].

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