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Second Form of the Generalized Hamilton–Jacobi Method for Nonholonomic Dynamical Systems

By René Van Dooren, Dept. of Analytical Mechanics,
Vrije Universiteit Brussel, Brussels, Belgium.

1. Introduction

In [8] the author introduced a new generalized Hamilton–Jacobi method for dynamical systems subject to nonholonomic constraints of Chetaev's type. As opposed to the potential methods established previously by Neimark [5] and Arzanyh [2] among others, this method was governed by a variational principle applied to a certain function S. The resulting variational relation was then treated by introducing some unknown multipliers in connection with the Chetaev constraint relations. After the elimination of these multipliers the generalized momenta were found to be certain functions of the partial derivatives of S with respect to the generalized coordinates, the generalized coordinates and the time. Then the partial differential equation of the classical Hamilton–Jacobi method was modified by inserting these functions for the generalized momenta in the Hamiltonian of the system.

The basic relation for the variation of the function S is derived in Section 3 of this paper. As it is well-known the method can now proceed in two directions. One possibility is the introduction of unknown multipliers in conjunction with the constraint relations which has been completely explained in [8] and [10]. But it is also possible to proceed by the use of independent variations derived from the constraint equations.

This will be the purpose of the present paper. Of course, the results obtained in this way are equivalent with the previous results as it will be clearly illustrated by Appell's example.

2. Generalized Lagrange and Hamilton Equations for Nonholonomic Systems of Chetaev's type

Let the n generalized coordinates q_i, $(i = 1, 2, \ldots, n)$ describe the position of a dynamical system whose motion is subject to s nonholonomic constraints of the form

$$f_l(q_i, \dot{q}_i, t) = 0 \quad (i = 1, 2, \ldots, n; l = 1, 2, \ldots, s; s < n), \tag{2.1}$$

in which f_l is a homogeneous function in the generalized velocities $\dot{q}_i$. Throughout this paper a dot superscript indicates differentiation with respect to time and summation over a repeated suffix is to be understood.

The virtual displacements δq_i at time t of the system are assumed to satisfy the relations

$$\frac{\partial f_l}{\partial \dot{q}_i} \delta q_i = 0, \tag{2.2}$$

the matrix $\partial f_l / \partial \dot{q}_i$ having the rank s. In addition it will be assumed that the reaction forces of the nonholonomic constraints perform no work in any virtual displacement of the system, i.e., the constraints are ideal. The constraints defined in this way will be called constraints of

Chetaev's type. However, it is emphasized that the constraints considered originally by Chetaev in [3] are of a more general nature for they are not restricted to the requirement that f_l is a homogeneous function with respect to the generalized velocities.

The generalized momenta and the Hamiltonian of the system are defined by the relations

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \tag{2.3}$$

$$H(q_i, p_i, t) = p_i\dot{q}_i - L, \tag{2.4}$$

where $L(q_i, \dot{q}_i, t)$ is the Lagrangian of the system.

When the system is nonholonomic Hamilton's variational principle has to be expressed in the form [4] [6]

$$\int_{t_0}^{t_1} \delta L \, dt = 0, \tag{2.5}$$

where the symbol δ denotes the synchronous variation referring to a displacement from the point $q_1, \ldots, q_n$ representing the real motion in the q-space to a contemporaneous point $q_1 + \delta q_1, \ldots, q_n + \delta q_n$ on the varied path. According to the usual approach of the calculus of variations, it is found that the generalized Lagrange equations for such nonholonomic systems of Chetaev's type are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \mu_l \frac{\partial f_l}{\partial \dot{q}_i}, \tag{2.6}$$

in which μ_l are s unknown multipliers introduced in conjunction with the constraint relations (2.2).

By an approach similar to the one already used by Neimark and Fufaev [5], the generalized Hamilton equations are readily found to be

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} + \mu_l \frac{\partial f_l}{\partial \dot{q}_i}. \tag{2.7}$$

Furthermore, we have

$$\dot{H} = \frac{\partial H}{\partial t} + \mu_l \frac{\partial f_l}{\partial \dot{q}_i}\dot{q}_i = \frac{\partial H}{\partial t}. \tag{2.8}$$

3. Generalized Hamilton–Jacobi Method Using Independent Variations

Consider the function S defined by the integral

$$S = \int_{P^*}^{P} (p_i \, dq_i - H \, dt), \tag{3.1}$$

taken along the real motion in the q-space. The asterisk notation will denote the initial event and its associated values.

Let us now calculate the asynchronous variation ΔS of the function S in which the point $q_1, \ldots, q_n$ occupied at time t on the real path is correlated with the point $q_1 + \Delta q_1, \ldots, q_n + \Delta q_n$, occupied at time $t + \Delta t$ on the varied path. Hereby we shall assume that $\Delta q_1, \ldots, \Delta q_n$, Δt are functions of t which are continuous and continuously differentiable. In the sequel we shall apply the following well-known formulae [4] [6] using the concept of these variations

$$\Delta F = \delta F + \dot{F}\Delta t, \; (^{\cdot} = d/dt), \tag{3.2}$$

$$\Delta \int_{t_0}^{t_1} F\,dt = \int_{t_0}^{t_1} [\Delta F + F(\Delta t)^{\cdot}]\,dt, \tag{3.3}$$

$$(\Delta q_i)^{\cdot} = \Delta\dot{q}_i + \dot{q}_i(\Delta t)^{\cdot}, \tag{3.4}$$

where F is an arbitrary function of time t. Applying Eqn. (3.3) with $F = p_i\dot{q}_i - H$ the asynchronous variation of the function S is given by

$$\Delta S = \int_{P^*}^{P} [\Delta(p_i\dot{q}_i - H)\,dt + (p_i\dot{q}_i - H)\,d\Delta t]. \tag{3.5}$$

According to Eqn. (3.4) this can be expressed as

$$\Delta S = \int_{P^*}^{P} (\Delta p_i\,dq_i + p_i\,d\Delta q_i - \Delta H\,dt - H\,d\Delta t). \tag{3.6}$$

The 2nd and the 4th term of the integrand in the right-hand-side of this equation will be integrated by parts where, of course, the integration has to be carried out with respect to the operation d. This yields

$$\Delta S = [p_i\,\Delta q_i - H\,\Delta t]_{P^*}^{P} + \int_{P^*}^{P} (\Delta p_i\,dq_i - \Delta q_i\,dp_i - \Delta H\,dt + \Delta t\,dH). \tag{3.7}$$

It is to be emphasized that a similar relation has been established previously by Synge and Griffith [7] provided the system is free of any constraints. By inserting

$$\Delta H = \frac{\partial H}{\partial q_i}\Delta q_i + \frac{\partial H}{\partial p_i}\Delta p_i + \frac{\partial H}{\partial t}\Delta t, \tag{3.8}$$

into Eqn. (3.7), we arrive at the variational equation

$$\Delta S = [p_i\,\Delta q_i - H\,\Delta t]_{P^*}^{P} + \int_{P^*}^{P} \left[\Delta p_i\left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right) - \Delta q_i\left(\dot{p}_i + \frac{\partial H}{\partial q_i}\right) + \Delta t\left(\dot{H} - \frac{\partial H}{\partial t}\right)\right] dt. \tag{3.9}$$

Taking into account the generalized Hamilton equations (2.7) and Eqn. (2.8) which describe the real motion of the system, Eqn. (3.9) reduces to

$$\Delta S = [p_i\,\Delta q_i - H\,\Delta t]_{P^*}^{P} - \int_{P^*}^{P} \mu_l \frac{\partial f_l}{\partial \dot{q}_i}\,\delta q_i\,dt, \tag{3.10}$$

according to Eqn. (3.2). If we assume that the synchronous variations δq_i constitute a virtual displacement at time t satisfying the Chetaev relations (2.2), Eqn. (3.10) can be finally written as follows

$$\Delta S = p_i\,\Delta q_i - H\,dt - p_i^*\,\Delta q_i^* + H^*\,\Delta t^*. \tag{3.11}$$

In this important basic relation for the variation of S, the asynchronous variations Δq_i are not independent. Indeed, according to the Chetaev relations (2.2) in addition to the requirement that f_l is a homogeneous function in the generalized velocities, i.e., $(\partial f_l/\partial \dot{q}_i)\dot{q}_i = 0$ and the relation (3.2), the asynchronous variations Δq_i have to satisfy the relations

$$\frac{\partial f_l}{\partial \dot{q}_i}\Delta q_i = 0. \tag{3.12}$$

The generalized Hamilton–Jacobi method can now be derived from the basic relation (3.11) in two different ways. One possibility is the introduction of unknown multipliers in conjunction with the constraint relations (3.12). This has been completely described in [8] [10]. Another method consists in the use of independent asynchronous variations. Indeed, the relations (3.12) imply the independence of n-s asynchronous variations denoted by Δq_k, $(k = s + 1, \ldots, n)$. Let us assume that from these relations the dependent asynchronous variations Δq_l, $(l = 1, \ldots, s)$ can be expressed as follows

$$\Delta q_l = F_{lk}(q_i, \dot{q}_i, t)\,\Delta q_k, \tag{3.13}$$

in which the asynchronous variations Δq_i are, of course, functions of time t. In that case the relation (3.11) is written as

$$\Delta S = [p_l F_{lk}(q_i, \dot{q}_i, t) + p_k]\,\Delta q_k - H\,\Delta t - [p_l^* F_{lk}(q_i^*, \dot{q}_i^*, t^*) + p_k^*]\,\Delta q_k^* + H^*\,\Delta t^*. \tag{3.14}$$

Since the function S is determined by the two events q_i, t and q_i^*, t^*, we have

$$\Delta S = \frac{\partial S}{\partial q_i}\Delta q_i + \frac{\partial S}{\partial t}\Delta t + \frac{\partial S}{\partial q_i^*}\Delta q_i^* + \frac{\partial S}{\partial t^*}\Delta t^*. \tag{3.15}$$

According to Eqn. (3.13), this takes the form

$$\Delta S = \left[\frac{\partial S}{\partial q_l} F_{lk}(q_i, \dot{q}_i, t) + \frac{\partial S}{\partial q_k}\right]\Delta q_k + \frac{\partial S}{\partial t}\Delta t + \left[\frac{\partial S}{\partial q_l^*} F_{lk}(q_i^*, \dot{q}_i^*, t^*) + \frac{\partial S}{\partial q_k^*}\right]\Delta q_k^* + \frac{\partial S}{\partial t^*}\Delta t^*. \tag{3.16}$$

The variations $\Delta q_k(\Delta q_k^*$ resp.) being independent, it is found from Eqns. (3.14) and (3.16) that

$$\frac{\partial S}{\partial q_l} F_{lk}(q_i, \dot{q}_i, t) + \frac{\partial S}{\partial q_k} = p_l F_{lk}(q_i, \dot{q}_i, t) + p_k, \tag{3.17}$$

$$\frac{\partial S}{\partial t} = -H, \tag{3.18}$$

$$\frac{\partial S}{\partial q_l^*} F_{lk}(q_i^*, \dot{q}_i^*, t^*) + \frac{\partial S}{\partial q_k^*} = -[p_l^* F_{lk}(q_i^*, \dot{q}_i^*, t^*) + p_k^*], \tag{3.19}$$

$$\frac{\partial S}{\partial t^*} = H^*. \tag{3.20}$$

Using the relations (2.3) between the generalized velocities and momenta, the n-s equations (3.17) and the s constraint Eqns (2.1) can be written as

$$\frac{\partial S}{\partial q_l}\bar{F}_{lk}(q_i, p_i, t) + \frac{\partial S}{\partial q_k} = p_l\bar{F}_{lk}(q_i, p_i, t) + p_k, \tag{3.21}$$

$$\bar{f}_l(q_i, p_i, t) = 0, \tag{3.22}$$

where the symbol $\bar{}$ indicates the replacement of the generalized velocities by the generalized momenta in the corresponding expressions. Now the Eqns (3.21) and (3.22) will be considered as constituting a system of n in general nonlinear equations in the n unknown functions p_i which will be solved in function of $\partial S/\partial q_j$, q_j, t, $(j = 1, \ldots, n)$ yielding

$$p_i = p_i\left(\frac{\partial S}{\partial q_j}, q_j, t\right). \tag{3.23}$$

By inserting these expressions for the generalized momenta in the Hamiltonian of the system, Eqn. (3.18) becomes

$$\frac{\partial S}{\partial t} + \tilde{H}\left(q_i, \frac{\partial S}{\partial q_i}, t\right) = 0, \tag{3.24}$$

the symbol tilda denoting the suppression of the momenta in favour of the partial derivatives of S with respect to q_i. Eqn. (3.24) will be called the modified partial differential equation of the generalized Hamilton–Jacobi method for nonholonomic systems of Chetaev's type. The partial differential equation of the classical Hamilton–Jacobi method is modified by replacing the generalized momenta in the Hamiltonian of the system by certain functions of the partial derivatives of S with respect to the generalized coordinates, the generalized coordinates and the time. Of course, this yields the same generalized Hamilton–Jacobi equation as in the method using unknown multipliers.

4. Example

The method characterized by the use of independent variations will be illustrated by Appell's well-known example [1] studying the motion of a particle of mass m in a gravitational field subject to a nonlinear nonholonomic constraint represented by the equation

$$f(\dot{x}, \dot{y}, \dot{z}) \equiv a^2(\dot{x}^2 + \dot{y}^2) - \dot{z}^2 = 0, \tag{4.1}$$

where a is a constant. The solution to this problem by the method using unknown multipliers has been presented in [8].

The generalized momenta and the Hamiltonian of the particle are given by

$$p_x = m\dot{x}, \qquad p_y = m\dot{y}, \qquad p_z = m\dot{z}, \qquad H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + mgz, \tag{4.2}$$

where g is the gravitational acceleration.

According to Eqn. (4.1) and the Chetaev constraint relations (2.2), we get for the asynchronous variation of $q_1 = z$

$$\Delta z = \frac{a^2}{\dot{z}}(\dot{x}\,\Delta x + \dot{y}\,\Delta y), \tag{4.3}$$

taking $\Delta q_2 = \Delta x$ and $\Delta q_3 = \Delta y$ as the independent variations. Hence

$$\bar{F}_{12} = a^2 p_x/p_z, \qquad \bar{F}_{13} = a^2 p_y/p_z. \tag{4.4}$$

The nonlinear system in p_z, p_x, p_y derived from Eqns. (3.21) and (3.22), is now written as

$$\begin{aligned} &p_x(1 + a^2 - a^2 S_z/p_z) = S_x, \qquad p_y(1 + a^2 - a^2 S_z/p_z) = S_y, \\ &a^2(p_x^2 + p_y^2) - p_z^2 = 0, \end{aligned} \tag{4.5}$$

where S_z, S_x, S_y denote the partial derivatives of S with respect to z, x, y.

It is readily seen that p_z, p_x, p_y are given by

$$p_z = \pm a[(S_x^2 + S_y^2)^{1/2} \pm aS_z]/(1 + a^2),$$
$$p_x = S_x[(S_x^2 + S_y^2)^{1/2} \pm aS_z]/[(1 + a^2)(S_x^2 + S_y^2)^{1/2}], \tag{4.6}$$
$$p_y = S_y[(S_x^2 + S_y^2)^{1/2} \pm aS_z]/[(1 + a^2)(S_x^2 + S_y^2)^{1/2}].$$

Now the modified Hamilton–Jacobi Equation (3.24) is found to be

$$S_t + [(S_x^2 + S_y^2)^{1/2} \pm aS_z]^2/[2m(1 + a^2)] + mgz = 0. \tag{4.7}$$

Since the Hamiltonian in Eqn. (4.7) does not involve x, y and t, the method of separation of variables yields as a complete integral of this generalized Hamilton–Jacobi equation [8]

$$S = c - \alpha_1 t + \alpha_2 x + \alpha_3 y \pm a^{-1} \int \{-(\alpha_2^2 + \alpha_3^2)^{1/2} \pm [2m(1 + a^2)(\alpha_1 - mgz)]^{1/2}\}\, dz, \tag{4.8}$$

where $\alpha_i (i = 1, 2, 3)$ and c are arbitrary constants.

The motion of the particle will be obtained from Eqns. (3.19) with $k = 2, 3$ in addition to the constraint Eqn. (4.1). The former can be written as

$$\frac{\partial S}{\partial q_1^*} \bar{F}_{12}(q_i^*, p_i^*, t^*) + \frac{\partial S}{\partial q_2^*} = \beta_2, \qquad \frac{\partial S}{\partial q_1^*} \bar{F}_{13}(q_i^*, p_i^*, t^*) + \frac{\partial S}{\partial q_3^*} = \beta_3, \tag{4.9}$$

in which β_2 and β_3 represent two arbitrary constants. By inserting the expression for S given by Eqn. (4.8), it is readily seen that these equations and the constraint Eqn. (4.1) admit the well-known solution of Appell's problem, i.e.,

$$z = \frac{\alpha_1}{mg} - \frac{ga^2}{2(1 + a^2)}(t + \gamma_1)^2, \qquad x = \gamma_2 \pm \frac{ga\alpha_2}{2(1 + a^2)(\alpha_2^2 + \alpha_3^2)^{1/2}}(t + \gamma_1)^2,$$
$$y = \gamma_3 \pm \frac{ga\alpha_3}{2(1 + a^2)(\alpha_2^2 + \alpha_3^2)^{1/2}}(t + \gamma_1)^2, \tag{4.10}$$

where $\gamma_1, \gamma_2, \gamma_3$ are some constants. The generalized momenta are then derived from (4.6).

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Summary

New light is cast on the generalized Hamilton–Jacobi method for nonholonomic dynamical systems of Chetaev's type. As opposed to the method characterized by the introduction of unknown multipliers, the method presented here proceeds using independent variations and is illustrated by Appell's example.

Résumé

Des nouveaux résultats sont obtenus sur la méthode généralisée de Hamilton–Jacobi pour des systèmes dynamiques non-holonomes du type de Chetaev. Contrairement à la méthode caracterisée par l'introduction de multiplicateurs inconnus, la méthode présentée utilise des variations indépendantes et est illustrée par l'exemple d'Appell.

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