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Propagation of Mechanical and Temperature Acceleration Waves in Thermoelastic Materials

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1. Introduction

Recently, there has been considerable interest in acceleration waves in thermoelastic materials, see e.g. [2–10] and the references therein. In this paper we study the behaviour of acceleration waves as predicted using the nonlinear thermoelasticity theory of Green and Lindsay [1]. This theory, which was developed with the aid of an entropy inequality due to Green and Laws [11], allows heat to travel with a finite wavespeed: the present study, therefore, may be regarded as an extension of work on the conventional theory of thermoelasticity which, from the nonlinear acceleration wave viewpoint taken here, was revived in 1961 by Truesdell [9] and continued by Chen [5, 6] and Chadwick and Currie [2, 3, 4].

Green [8] has investigated acceleration waves in the linear isotropic theory of thermoelasticity of Green and Lindsay [1]. He demonstrated that the theory allowed for two coupled waves, which may be thought of as arising due to discontinuities in the acceleration of the body and an 'acceleration', $\ddot{\theta}$, of the temperature θ. The present writers [12] continued this work by examining the behaviour of a thermal wave in a rigid conductor of Green and Laws [11] type. Since no linearity was assumed the amplitude of the thermal wave could become infinite in a finite time; an effect which may be associated with shock wave formation. We here take the work a stage further and investigate the behaviour of an 'acceleration' discontinuity in the coupled nonlinear theory of thermoelasticity presented in [1]. As with [12], the possibility of an infinite amplitude is encountered, but we here find there are two coupled waves, both of which exhibit nonlinear behaviour.

Several other models allowing for a thermal wave of finite speed (usually known as a second sound effect) appropriate to thermoelastic media have been based on the Maxwell–Cattaneo relation, the more general temperature-gradient history approach of Gurtin and Pipkin or an idea of phonon diffusion; see [7, 13–19]. It is highly likely, as the work of [7] would indicate, that the Chen–Gurtin–Pipkin theory will predict similar results to those obtained here. However, we believe that the theory of Green–Laws–Lindsay is more tractable in that less cumbersome algebraic manipulations are involved since the constitutive variables are quantities defined at the present time only and do not employ histories as in [7, 20]. It might be mentioned, at this point, that it would be interesting to investigate nonlinear wave propagation according to each of the aforementioned theories and compare the various results. However, this broad objective is beyond the scope of the present paper.

2. A Brief Review of the Generalized Temperature Theory of Thermoelasticity

For completeness, points of the theory of thermoelasticity presented by Green and Lindsay [1] relevant to the present study, are collected here.

Thermodynamic arguments are based on the inequality

$$\frac{d}{dt}\int_V \rho_0 \eta \, dV - \int_V \frac{\rho_0 r}{\phi}\, dV + \int_{\partial V} \frac{Q_A N_A}{\phi}\, dA \geq 0, \tag{2.1}$$

which holds for all sub-volumes, V, of some reference configuration, and where η, ρ_0, r, $\phi(>0)$ and $\mathbf{Q}$ are, respectively, the specific entropy, reference density, externally supplied heat, temperature function and the heat flux vector acting over the surface at time t but measured per unit area of ∂V. ϕ is a function of the Kelvin temperature θ and all other independent variables in the constitutive theory. It transpires that ϕ reduces to a function of θ and $\dot{\theta}$, the functional form of ϕ to be determined by interpretation of analysis in accordance with physics (cf. the problem of determining the functional form of the strain energy in classical nonlinear elasticity). However, the canonical form for ϕ may be motivated from a statistical mechanical viewpoint and may be thought of as a continuum attempt to explain the idea of heat being conveyed by molecular collisions.

Standard indicial notation (see e.g. Truesdell and Toupin [21]) is employed throughout with x_i denoting spatial coordinates and X_A reference coordinates. The momentum, energy and mass conservation equations are then

$$p_{Ai,A} + \rho_0 \mathscr{F}_i = \rho_0 \ddot{x}_i, \tag{2.2}$$

$$\rho_0 \dot{\varepsilon} - \rho_0 r - \dot{x}_{i,A} p_{Ai} + Q_{K,K} = 0, \tag{2.3}$$

$$j\rho = \rho_0, \tag{2.4}$$

where $j = \det(x_{i,A})$ and where ε, $\mathbf{p}$ and $\mathscr{F}$ are the specific internal energy, Piola–Kirchoff stress tensor and body force, respectively. (The Cauchy stress tensor, $\mathbf{t}$, and the usual heat flux vector, $\mathbf{q}$, are given by $jt_{ik} = x_{i,A} p_{Ak}$ and $jq_i = x_{i,A} Q_A$.)

A free energy function $\psi\ (= \varepsilon - \eta\phi)$ is introduced and constitutive equations are assumed such that

$$\psi,\ \phi,\ \eta,\ \mathbf{Q} \text{ and } \mathbf{p} \tag{2.5}$$

are functions of

$$\theta,\ \theta_{,A},\ \dot{\theta} \text{ and } F_{iA},$$

where $F_{iA} = x_{i,A}$. (It is assumed throughout this work that the reference body is homogeneous and so $\mathbf{X}$ is omitted from the above list. This is not essential to the development of the theory, see [1].) Then, with the aid of (2.1), the following relations are deduced:

$$\phi = \phi(\theta, \dot{\theta}), \tag{2.6}$$

$$\eta = -\frac{\partial\psi}{\partial\theta}\Big/\frac{\partial\phi}{\partial\dot{\theta}}, \tag{2.7}$$

$$Q_A = -\rho_0\phi\frac{\partial\psi}{\partial\theta_{,A}}\Big/\frac{\partial\phi}{\partial\dot{\theta}}, \tag{2.8}$$

$$p_{Ai} = \rho_0\frac{\partial\psi}{\partial x_{i,A}}. \tag{2.9}$$

The function ϕ is restricted in that $\phi|_E = \theta$, where a line followed by a subscript E indicates evaluation in a state of constant temperature ($\dot{\theta} = \theta_{,A} = 0$). Moreover, we note for later use that if the material is in a static configuration at uniform deformation and temperature and if the initial body has a centre of symmetry, then

$$\frac{\partial p_{Ai}}{\partial\theta_{,K}}, \quad \frac{\partial Q_K}{\partial F_{iA}}, \quad \frac{\partial\varepsilon}{\partial\theta_{,A}}, \quad \frac{\partial Q_K}{\partial\dot{\theta}}, \tag{2.10}$$

are identically zero.

3. Acceleration Waves in Anisotropic Materials

For the notation employed here the reader is referred to Chen [5], Sections 4 and 5.

The waves considered in this paper are propagating singular surfaces across which the discontinuities of lowest order are the second derivatives of displacement and temperature. The wave normal $\mathbf{n}$ and the local speed of propagation $U = u_n - \dot{\mathbf{x}}\cdot\mathbf{n}$ of an acceleration wave $\sigma(t)$ are respectively the unit vector normal to $\sigma(t)$ in the direction of travel and the component in the direction of $\mathbf{n}$ of the velocity with which the wave moves relative to the material. $\sigma(t)$ is the spatial representation of the surface and a material representation $\Sigma(t)$ is also useful, together with its associated normal $\mathbf{N}$ and the speed of propagation U_N. The wave amplitudes $\mathbf{a}(t)$ and $\alpha(t)$ are defined by

$$\mathbf{a} = [\ddot{\mathbf{x}}] = \ddot{\mathbf{x}}^- - \ddot{\mathbf{x}}^+ \quad \text{and} \quad \alpha = [\ddot{\theta}], \tag{3.1}$$

and we suppose throughout that $\mathbf{a} \neq \mathbf{0}$, $\alpha \neq 0$.

It should be observed that thermoelastic waves defined as above are different from the analogous ones in the classical theory of thermoelasticity, see Chadwick and Currie [3], and as may be expected they lead to different results to those predicted by the classical theory.

It is supposed the body force and heat supply are zero. The calculation of the equation governing wavespeeds is now a routine application of compatibility relations given in e.g. Chen [6]. (The analysis leading to the propagation and growth of amplitude equations is now well known and so we merely present a statement of the final results together with the relations needed to calculate the coefficients appearing in them. However, complete details are given in [22], available from the writers.) In fact, the propagation conditions are

$$\begin{aligned} \chi_{ij}^{(1)} a_j - \chi_i^{(3)} \alpha &= 0, \\ \chi^{(2)} \alpha - \chi_i^{(3)} a_i &= 0, \end{aligned} \tag{3.2}$$

where $\boldsymbol{\chi}^{(1)}$, $\chi^{(2)}$ and $\boldsymbol{\chi}^{(3)}$ are defined by

$$\begin{aligned} \chi_{ij}^{(1)} &= \rho_0 U_N^2 \delta_{ij} - \hat{Q}_{ij}, \\ \chi^{(2)} &= \left(\frac{\partial \phi}{\partial \dot{\theta}}\Big/ \phi\right) \left(\rho_0 \frac{\partial \varepsilon}{\partial \dot{\theta}} U_N^2 - A - U_N \left\{\rho_0 \frac{\partial \varepsilon}{\partial \theta_{,A}} N_A + \frac{\partial Q_K}{\partial \dot{\theta}} N_K\right\}\right), \\ \chi_i^{(3)} &= N_A N_K \frac{\partial p_{Ai}}{\partial \theta_{,K}} - U_N N_A \frac{\partial p_{Ai}}{\partial \dot{\theta}}, \end{aligned} \tag{3.3}$$

and where A and $\hat{Q}_{ij}(\mathbf{N})$ are given by

$$A = -N_A N_K \frac{\partial Q_K}{\partial \theta_{,A}} \quad \text{and} \quad \hat{Q}_{ij} = N_A N_B \frac{\partial p_{Ai}}{\partial F_{jB}}. \tag{3.4}$$

From (3.2) it follows that

$$(\chi_{ij}^{(1)} \chi^{(2)} - \chi_i^{(3)} \chi_j^{(3)}) a_j = 0. \tag{3.5}$$

We suppose henceforth that the material ahead of the wave is an isothermal region at rest in a fixed homogeneous configuration, and that the reference body has a centre of symmetry.

By Chen [6], (4.10), we have

$$N_A = F_{iA} \frac{|\nabla \mathbf{x}^\sigma|}{|\nabla \mathbf{X} \Sigma|} n_i, \tag{3.6}$$

and so defining β_{ij} by

$$\beta_{ij} = \frac{|\nabla \mathbf{x}^\sigma|}{|\nabla \mathbf{X} \Sigma|} \left(\frac{\partial p_{Ai}}{\partial \dot{\theta}} F_{jA}\right)\bigg|_E, \tag{3.7}$$

we may, noting the symmetry properties of the body, write $(3.2)_1$ as

$$(Q_{ij}|_E(n) - \rho_0 U_N^2 \delta_{ij}) a_j - U_N \alpha \beta_{ij} n_j = 0, \tag{3.8}$$

where $Q_{ij}(n)$ is the tensor $\hat{Q}_{ij}$ represented now as a function of $\mathbf{n}$ rather than $\mathbf{N}$.

Equation (3.8) is similar to equation (2.1) of Chadwick and Currie [3] and the term $\boldsymbol{\beta}$ plays an analogous, important rôle to the corresponding term in [3]. As with [3] we shall require $\boldsymbol{\beta}$ to be nonsingular, and since $\det \mathbf{F} \neq 0$ is a basic assumption of the theory this reduces to requiring $\det(\partial p_{Ai}/\partial \dot{\theta}) \neq 0$. A physical motivation for this requirement may be obtained as in [4], p. 305: the contact force on an arbitrary surface element is always changed when the temperature velocity, $\dot{\theta}$, is changed at constant deformation.

Theorem 2 of [3] may now be used to show there is at least one direction $\mathbf{n}^*$ such that $\boldsymbol{\beta}\mathbf{n}^*$ is an eigenvector of $\mathbf{Q}$. Since the wave is propagating into an isothermal, homogeneous region at rest, $\mathbf{Q}$ is a constant matrix and so $\boldsymbol{\beta}\mathbf{n}^*$ is fixed. Furthermore, since $\boldsymbol{\beta}$ is a constant matrix it follows $\mathbf{n}^*$ must be a fixed direction. We may, therefore, consider the propagation of a plane acceleration wave in the direction $\mathbf{n}^*$, with amplitude in the direction $\boldsymbol{\beta}\mathbf{n}^*$. Following Chadwick and Currie [3], these waves may be termed generalized longitudinal waves.

Let the unit vector in the direction of $\boldsymbol{\beta}\mathbf{n}^*$ be $\boldsymbol{\nu}$, and so $\mathbf{a} = a\boldsymbol{\nu}$. Since $a \neq 0$ (3.5) leads to the wavespeed equation

$$(U_N^2 - U_M^2)(U_N^2 - U_T^2) + KU_N^2 = 0, \tag{3.9}$$

where

$$U_T^2 = \left(A/\rho_0 \frac{\partial \varepsilon}{\partial \dot{\theta}}\right)\bigg|_E, \qquad U_M^2 = (Q_{ij}\nu_i\nu_j/\rho_0)|_E, \tag{3.10}$$

and

$$K = -\left\{\left(\rho_0^2 \frac{\partial \eta}{\partial \dot{\theta}} \frac{\partial \phi}{\partial \dot{\theta}}\right)^{-1} N_A^* \frac{\partial p_{Ai}}{\partial \dot{\theta}} N_B^* \frac{\partial p_{Bi}}{\partial \dot{\theta}}\right\}\bigg|_E. \tag{3.11}$$

Equation (3.9) clearly admits two solutions for U_N^2 under suitable conditions on the coefficients and, indeed, appears superficially to be the same as the equation for the wavespeeds of harmonic waves in a classical linear elastic material, see e.g. Sneddon [23], p. 43. However, unlike the classical case, (3.9) admits two real solutions for U_N^2. (Other areas in which two distinct wavespeeds exist are in granular media, Nunziato and Walsh [24]; mixtures, Bowen and Chen [25] and Bowen and Wright [26]; and the present writers have observed a similar phenomenon with temperature-twist waves in the Ericksen–Leslie theory of Nematic liquid crystals based on the thermodynamics of Green and Laws [11], these waves being an extension of the isothermal twist waves studied by Shahinpoor [27].)

Equation (3.9) has real solutions if either

$$\text{(i) } K \geq (U_M + U_T)^2 \quad \text{or} \quad \text{(ii) } K \leq (U_M - U_T)^2.$$

However, (i) is inconsistent with the fact that $U_N^2 > 0$, and so we find it necessary that (ii) holds. We have already assumed det $(\partial p_{Ai}/\partial \dot{\theta}) \neq 0$, and we suppose also that the derivatives of ϕ and η with respect to $\dot{\theta}$ do not vanish. Thus, $K \neq 0$. If $\eta = \eta(\phi, F_{iA})$, then sgn $K = -$sgn $(\partial\eta/\partial\phi)$, and we may expect from this that $K < 0$, although we still examine the possibility of $K > 0$.

$\underline{K < 0.}$

For this case.

$$U_N^{(2)2} < (U_M^2, U_T^2) < U_N^{(1)2}. \tag{3.12}$$

U_M is the speed of an acceleration wave in an elastic material neglecting thermal effects, see Chen [6], whereas U_T is the speed of a temperature wave in a rigid heat conductor, i.e. neglecting elastic effects, see Lindsay and Straughan [12]. Therefore, (3.9) gives two solutions, one which represents a wave travelling with speed greater than either an isothermal acceleration wave or a temperature wave in a rigid heat conductor, and the other slower than either of these. (Of course, this occurs only at the instant of generation of the wave, for, after that the slower wave advances into a nonequilibrium region.) The speed of the second wave has then to be calculated from Eqns. (3.2). A similar situation was encountered by Nunziato and Walsh [24] for one-dimensional waves in granular materials. We can employ their ideas to show that the second wave cannot intersect with the first. For, the coefficients in (3.5) in the region between the waves are continuous functions of **x**, t and so, therefore, will be $U_N^{(2)}$. If the second wave intersects the first, then at the point of contact the normal to the second wave is $\mathbf{n}^*$, and so by (3.12) $U_N^{(2)} < U_N^{(1)}$ which is inconsistent with the fact that $U_N^{(2)}$ is a continuous function of **x**, t and the second wave is overtaking the first. Clearly, for this approach to be applicable it is necessary that the second wave remains a smooth surface and we have, therefore, to neglect the interesting possibility of a caustic forming on the second wave with possible interaction of both waves.

$\underline{K > 0.}$

For this case either

$$\min(U_T^2, U_M^2) < U_N^{(2)2} < U_N^{(1)2} < \max(U_T^2, U_M^2), \tag{3.13}$$

or

$$K = (U_M - U_T)^2, \tag{3.14}$$

which represents a single wave propagating with constant speed U_N given by $U_N = (U_M U_T)^{1/2}$.

We shall now proceed to give details of the growth or decay of amplitude of the first wave. The question of amplitude behaviour of the second wave is beyond the scope of this paper as it requires a detailed knowledge of the deformation behind the leading wave. This point raises the question as to whether the second wave can develop into a shock wave before the first wave and whether the shock wave can then interact with the leading wave.

Since the calculations now concern the fast wave only the coefficients appearing are all for the equilibrium region ahead of the first wave and so we shall omit the $|_E$ notation.

The differential equation governing the evolutionary behaviour of the amplitude, **a**, is obtained by differentiating (2.2) and (2.3) with respect to time and evaluating the resulting expressions at the surface of discontinuity. The resulting equation is

$$\mathscr{B}_{ij}C_j U_N^2 + 2\mathscr{D}_i a_j\left(\rho_0^2 \frac{\partial \varepsilon}{\partial \theta} U_N^4 \delta_{ij} - A\hat{Q}_{ij}\right) + \alpha\left(\chi^{(2)} N_A U_N \Lambda \frac{\partial p_{Ai}}{\partial \theta} + \chi_i^{(3)} \rho_0 \frac{\partial \varepsilon}{\partial \theta} U_N^2\right)$$
$$+ \alpha^2\left(\chi^{(2)}\Lambda\left\{N_A U_N \frac{\partial^2 p_{Ai}}{\partial \theta^2} + \frac{N_A N_C N_K}{U_N} \frac{\partial^2 p_{Ai}}{\partial \theta_{,C}\, \partial \theta_{,K}}\right\}\right.$$

$$+ \chi_i^{(3)}\left\{2N_C N_K \frac{\partial^2 Q_K}{\partial\theta_{,C}\,\partial\dot{\theta}} + \rho_0 U_N^2 \frac{\partial^2\varepsilon}{\partial\dot{\theta}^2} + \rho_0 N_A N_C \frac{\partial^2\varepsilon}{\partial\theta_{,A}\,\partial\theta_{,C}}\right\}\Bigg)$$

$$- \alpha a_j\Bigg(2\Lambda\chi^{(2)} N_A N_B \frac{\partial^2 p_{Ai}}{\partial\dot{\theta}\,\partial F_{jB}}$$

$$+ \chi_i^{(3)} N_A\left\{\frac{2N_B N_K}{U_N} \frac{\partial^2 Q_K}{\partial\theta_{,B}\,\partial F_{jA}} + 2\rho_0 U_N \frac{\partial^2\varepsilon}{\partial\dot{\theta}\,\partial F_{jA}} - U_N \frac{\partial p_{Aj}}{\partial\dot{\theta}}\right\}\Bigg)$$

$$+ a_j a_m\left(\frac{\chi^{(2)}\Lambda}{U_N} R_{ijm} + \chi_i^{(3)}\left\{\rho_0 N_A N_B \frac{\partial^2\varepsilon}{\partial F_{mB}\,\partial F_{jA}} - \hat{Q}_{jm}\right\}\right) = 0, \quad (3.15)$$

where

$$\Lambda = \theta\Big/\frac{\partial\phi}{\partial\dot{\theta}}, \quad (3.16)$$

$$\mathscr{B}_{ij} = (\chi_{ij}^{(1)}\chi^{(2)} - \chi_i^{(3)}\chi_j^{(3)})\Lambda,$$

$$C_i = N_A N_B[\dot{F}_{iA,B}], \quad (3.17)$$

$$R_{ijm} = N_A N_B N_G \frac{\partial^2 p_{Ai}}{\partial F_{jB}\,\partial F_{mG}}, \quad (3.18)$$

and $\mathscr{D}_t$ is the 'displacement derivative' denoted in [6] by $\delta_D/\delta t$. It should be observed that (3.15) may be obtained for *any* plane wave with amplitude in a direction ν_i and which conforms with the condition $\mathscr{B}_{ij}\nu_j = 0$, not necessarily the one travelling in the $\mathbf{n}^*$ direction, and so we have presented the equation in its fuller generality. Since $\mathscr{B}$ is a symmetric singular matrix, forming the scalar product of (3.15) with ν_i leads to a growth equation of the form

$$\mathscr{D}_t a = -\beta a + \omega a^2, \quad (3.19)$$

where $a = |\mathbf{a}|$.

The solution of (3.19) with the initial value of wave amplitude $a(0) = a_0$ is

$$a(t) = \beta/\omega\{(\beta/\omega a_0 - 1)\,e^{\beta t} + 1\}. \quad (3.20)$$

The evolutionary behaviour of $a(t)$ may now be deduced as in e.g. Chadwick and Currie [2], p. 152. The details are similar to those of Lemma 5.1 which deals with the equivalent problem for cylindrical and spherical waves. However, it is worth observing that whenever β and ω are such that $t_{\text{crit}} = -\beta^{-1}\log(1 - (\beta/\omega a_0)) > 0$, then $|a(t)| \to \infty$, $t \to t_{\text{crit}}^-$; a phenomenon which is thought to be associated with shock wave formation.

For completeness, we give the values of ω, β for the generalized longitudinal wave propagating in the $\mathbf{n}^*$ direction:

$$\beta = \frac{(U_N^2 - U_M^2)\left\{U_N^2 \dfrac{\partial\varepsilon}{\partial\theta} - \Lambda\dfrac{\partial\eta}{\partial\dot{\theta}}(U_N^2 - U_T^2)\right\}}{2\dfrac{\partial\varepsilon}{\partial\dot{\theta}}(U_N^4 - U_M^2 U_T^2)}, \quad (3.21)$$

$$\omega = \left\{-1/2\rho_0 \frac{\partial \varepsilon}{\partial \dot{\theta}}(U_N^4 - U_M^2 U_T^2)\right\}$$

$$\times \left\{\Lambda \nu_i N_A (U_N^2 - U_M^2)\left(U_N \frac{\partial^2 p_{Ai}}{\partial \dot{\theta}^2} + \frac{N_C N_K}{U_N} \frac{\partial^2 p_{Ai}}{\partial \theta_{,C}\, \partial \theta_{,K}}\right)\right.$$

$$+ \frac{\chi_i^{(3)} \nu_i}{\chi^{(2)}}(U_N^2 - U_M^2)\left(\rho_0 U_N^2 \frac{\partial^2 \varepsilon}{\partial \dot{\theta}^2} + 2 N_C N_K \frac{\partial^2 Q_K}{\partial \dot{\theta}\, \partial \theta_{,C}} + \rho_0 N_A N_C \frac{\partial^2 \varepsilon}{\partial \theta_{,A}\, \partial \theta_{,C}}\right)$$

$$+ \chi_i^{(3)} \nu_i \left(N_A N_B \nu_j \nu_m \frac{\partial^2 \varepsilon}{\partial F_{mB}\, \partial F_{jA}} - U_M^2\right)$$

$$+ \frac{\chi^{(2)} \Lambda}{\rho_0 U_N} R_{ijm} \nu_i \nu_j \nu_m - \frac{2\Lambda}{\rho_0} \chi_m^{(3)} \nu_m N_A N_B \nu_i \nu_j \frac{\partial^2 p_{Ai}}{\partial \dot{\theta}\, \partial F_{jB}}$$

$$- (U_N^2 - U_M^2)\left(\frac{2}{U_N} \nu_j N_A N_B N_K \frac{\partial^2 Q_K}{\partial \theta_{,B}\, \partial F_{jA}} + 2 \rho_0 U_N N_A \nu_j \frac{\partial^2 \varepsilon}{\partial \dot{\theta}\, \partial F_{jA}}\right.$$

$$\left.\left. - U_N \nu_j N_A \frac{\partial p_{Aj}}{\partial \dot{\theta}}\right)\right\}. \tag{3.22}$$

With the aid of (2.7)–(2.9), ω may clearly be rewritten in terms of derivatives of the Helmholtz free energy ψ. In fact, the expression essentially consists of a combination of third derivatives of ψ with respect to the independent variables, cf. the corresponding expression in the classical theory of elasticity, e.g. [2], p. 152.

The behaviour of the thermal amplitude α may now be deduced with the aid of equation (3.2). Since $a(t)$ and $\alpha(t)$ do not change sign, and $\operatorname{sgn} \alpha(0) = -\operatorname{sgn} a(0) \times \operatorname{sgn}(-K)$, it is, therefore, likely that $\operatorname{sgn} \alpha(t) = -\operatorname{sgn} a(t)$.

In the next section we examine briefly acceleration waves in isotropic materials.

4. Principal Waves in Isotropic Materials

The theory appropriate to the present paper for an isotropic thermoelastic material was developed by Green [8] in the linear case and by Lindsay [28] for the nonlinear case. We review the presentation of [28]. The continuity, momentum and energy equations are in the current frame,

$$\dot{\rho} + \rho \dot{x}^i_{,i} = 0, \tag{4.1}$$

$$\rho \ddot{x}^p = \rho \mathscr{F}^p + t^{kp}_{,k}, \tag{4.2}$$

$$\rho \dot{\varepsilon} = t^{ik} d_{ik} - q^i_{,i} + \rho r, \tag{4.3}$$

where $d_{ik} = \frac{1}{2}(\dot{x}_{i,k} + \dot{x}_{k,i})$ are the covariant components of the rate of deformation tensor, the other quantities being defined as in Section 2.

The constitutive equations for an isotropic thermoelastic material are ([28])

$$t^i_j = h_0 \delta^i_j + h_1 b^i_j + h_2 b^i_a b^a_j + h_3 b^i_r b^k_j g^r g_k + h_4 g^i g_j, \tag{4.4}$$

$$q^i = (k_0\delta^i_j + k_1 b^i_j + k_2 b^i_a b^a_j)g^j, \tag{4.5}$$

where $g_i = \theta_{,i}$, $b^{ij} = g^{AB}x^i_{,A}x^j_{,B}$, g^{AB} denote the metric tensor with respect to the **X** coordinates and $h_0, \ldots, h_4$, k_0, k_1 and k_2 are functions of θ, $\dot{\theta}$ and the invariants I_k, where

$$\begin{aligned} I_1 &= b^r_r, & I_2 &= b^i_j b^j_i, & I_3 &= \det \mathbf{b}, \\ I_4 &= g^i g_i, & I_5 &= b^i_j g_i g^j, & I_6 &= b^i_j b^j_k g^k g_i. \end{aligned} \tag{4.6}$$

In fact, in terms of the free energy function ψ, $h_0, \ldots, k_2$ have the explicit forms:

$$\begin{aligned} h_0 &= 2\rho_0 I_3^{1/2}\frac{\partial\psi}{\partial I_3}, & h_1 &= 2\rho_0 I_3^{-1/2}\frac{\partial\psi}{\partial I_1}, \\ h_2 &= 4\rho_0 I_3^{-1/2}\frac{\partial\psi}{\partial I_2}, & h_3 &= 2\rho_0 I_3^{-1/2}\frac{\partial\psi}{\partial I_6}, \\ h_4 &= -2\rho_0 I_3^{-1/2}\frac{\partial\psi}{\partial I_4}, & k_0 &= -2\rho_0 I_3^{-1/2}\frac{\partial\psi}{\partial I_4}\left(\phi\Big/\frac{\partial\phi}{\partial\dot{\theta}}\right), \\ k_1 &= -2\rho_0 I_3^{-1/2}\frac{\partial\psi}{\partial I_5}\left(\phi\Big/\frac{\partial\phi}{\partial\dot{\theta}}\right), & k_2 &= -2\rho_0 I_3^{-1/2}\frac{\partial\psi}{\partial I_6}\left(\phi\Big/\frac{\partial\phi}{\partial\dot{\theta}}\right). \end{aligned} \tag{4.7}$$

In Section 3 we have seen that there is at least one direction in which a plane acceleration wave may propagate. We now wish to consider the case when this direction is an eigenvector of the tensor **b**. Moreover, we again suppose the region, $\mathscr{R}$, ahead of the wave is an isothermal one and is at rest in a homogeneous configuration. It then follows from (4.4) that the principal axes of stress and strain coincide in $\mathscr{R}$. The propagation conditions are again (3.2), and we shall rewrite $(3.2)_1$ as follows

$$(\rho U^2\delta^p_m - Q^p_m)a^m + \alpha U n_k\frac{\partial t^{kp}}{\partial\dot{\theta}} = 0, \tag{4.8}$$

where the acoustic tensor **Q** is given by

$$Q^p_m = 2\frac{\partial t^{kp}}{\partial b^{mn}}b^{ns}n_s n_k. \tag{4.9}$$

Suppose now $\mathbf{n}^{(1)}$ is an eigenvector of **b**, corresponding to eigenvalue (principal stretch) $\lambda^2_{(1)}$. Then, using (4.4) we see that the vector with components $n_k(\partial t^{kp}/\partial\dot{\theta})$ is in the direction of $\mathbf{n}^{(1)}$. Moreover, **Q** given by (4.9) is the mechanical acoustic tensor for an isothermal region (see Truesdell and Noll [10], Section 74) and has the same representation as (74.2) of [10], and so we may apply the argument given there to (4.8), to establish Truesdell's theorem for principal waves in our theory; namely: 'In an isotropic material, the acoustic axes for principal waves coincide with the principal axes; in particular, every principal wave is either longitudinal or transverse'.

The wavespeeds of the longitudinal waves are obtained by contracting (3.5) with $n^{(1)}_i$ and it is worth noting that the waves in this case are also longitudinal in the conventional sense. If we let U_{11} be the wavespeed and write $K_{(1)} = k_0 + \lambda_{(1)}^2 k_1 + \lambda_{(1)}^4 k_2$, then (3.5) may be written in the form

$$\frac{\partial\phi}{\partial\theta}\left(\rho U_{11}^2 - 2\lambda_{(1)}^2 \frac{\partial t_{(1)}}{\partial\lambda_{(1)}^2}\right)\left(\rho\frac{\partial\varepsilon}{\partial\theta} U_{11}^2 + K_{(1)}\right) - \theta U_{11}^2\left(\sum_{\Delta=0}^{2} \lambda_{(1)}^{2\Delta}\frac{\partial h_\Delta}{\partial\theta}\right)^2 = 0, \quad (4.10)$$

where it is to be remembered ε and $t_{(1)}$, the principal stretch in the $\mathbf{n}^{(1)}$ direction, are evaluated at constant temperature.

As in Section 3, the wave amplitudes a and α may be obtained and the coefficients given by (3.21) and (3.22) are easily calculated employing (4.4)–(4.7). Details of these routine calculations are given in [22].

Transverse waves

By Truesdell's theorem we may consider a transverse principal wave travelling say in the $\mathbf{n}^{(1)}$ direction. We consider only the case where the amplitude is in the direction of one of the other principal axes of stress, $\mathbf{n}^{(2)}$ or $\mathbf{n}^{(3)}$. For definiteness, suppose $\mathbf{a} = a\mathbf{n}^{(2)}$. For this case, the propagation conditions (3.2) reduce to

$$\begin{aligned}&(\rho U_{12}^2 \delta_m^p - Q_m^p(\mathbf{n}^{(1)}))a^m n_p^{(2)} = 0,\\ &\left(\rho\frac{\partial\varepsilon}{\partial\theta} U_{12}^2 + n_j^{(1)} n_i^{(1)} \frac{\partial q^i}{\partial g_j}\right)\alpha = 0,\end{aligned} \quad (4.11)$$

where U_{12} denotes the wavespeed. These equations may be interpreted as implying that initially the wave separates into two waves, with wavespeeds

$$\begin{aligned}U_{12}^{(1)2} &= Q_j^i n_i^{(2)} n_{(2)}^j/\rho,\\ U_{12}^{(2)2} &= -n_i^{(1)} n_j^{(1)} \frac{\partial q^i}{\partial g_j}\Big/\rho\frac{\partial\varepsilon}{\partial\theta},\end{aligned} \quad (4.12)$$

where on the (1) wave $a \neq 0$, $\alpha = 0$ and on the (2) wave $\alpha \neq 0$, $a = 0$. We shall suppose $U_{12}^{(1)} > U_{12}^{(2)}$ at the instant the wave is created, although the modifications for the other cases are obvious. For this case the (1) wave is a mechanical wave and travels into the equilibrium region, whereas the (2) wave travels behind into a nonequilibrium region and so the coefficients in $(4.12)_2$ have to be calculated for such a region. Again, the Nunziato–Walsh proof employed in Section 3 may be used to show the second wave cannot 'catch up' with the first.

After deriving the amplitude equations it is found that the amplitude of the first wave is that obtained by Chen [6], i.e. $\mathscr{D}_t a = 0$ and so $a(t) \equiv a(0)$. However, the amplitude equation for the second wave is essentially that of a thermal wave entering an isothermal region, cf. [12], although the waveshape has first to be *calculated* before an attempt is made to find the amplitude (see Section 6).

We have seen that for transverse waves a second sound effect is again present, although the theory essentially breaks down into that for a classical thermoelastic material, studied extensively by Chadwick and Currie [4] and Chen [6], and further details may be obtained using the methods developed in these works.

5. Curved Waves

In the previous sections, only plane acceleration waves were considered, although it is clear that non-plane waves will certainly be of importance. In order to study such waves, we examine the situation in which the stress in the isothermal region ahead of the wave is a hydrostatic pressure. The waveshape is allowed to be arbitrary, but smooth, although the wave amplitude is assumed to be uniform over the surface. Considering only longitudinal waves, we find as before two waves are present and explicit expressions for the wavespeeds are given. The solution of the general amplitude equation depends on the mean curvature of the waveshape and details are given in [22]. However, we here present explicit solutions for the physically important cases of cylindrical and spherical waves (cf. Chen [5, 6]).

The stress is hydrostatic and so $\mathbf{t} = -p\mathbf{I}$ where the pressure p is a function of ρ, θ and $\dot{\theta}$. Suppose the body has undergone a deformation of the form $\mathbf{x} = \lambda\mathbf{X}$; then, since the region ahead of the wave is isothermal, the arguments of Chen ([6], Section 10) concerning the existence of longitudinal and transverse waves may be shown to continue to hold. In particular, if the region ahead of the wave is homogeneous and at rest, longitudinal waves have the same constant speed throughout the body and may propagate in every direction.

The wavespeed, U, of an arbitrary shaped acceleration wave moving into such a region satisfies the same propagation conditions as those for principal waves and in this case the equation for the wavespeeds of longitudinal waves of arbitrary shape, may be written as

$$\left\{\rho U^2 - \tfrac{4}{3}\lambda^2(h_1 + 2\lambda^2 h_2) - \rho\frac{\partial p}{\partial \rho}\right\}\left\{\rho\frac{\partial \varepsilon}{\partial \dot{\theta}}U^2 + \sum_{\Delta=0}^{2}\lambda^{2\Delta}k_\Delta\right\}\frac{\partial \phi}{\partial \dot{\theta}} - U^2\left\{\sum_{\Delta=0}^{2}\lambda^{2\Delta}\frac{\partial}{\partial \dot{\theta}}h_\Delta\right\}^2\theta = 0. \quad (5.1)$$

The same procedure as in Section 3 is again used to obtain the amplitude equation although since now the waves may be curved the complete compatibility expressions (e.g. Chen [6], (4.14), (4.16), (5.9)) are necessary.

For cylindrical and spherical waves, the mechanical amplitudes are given by, cf. Chen [5], p. 247,

$$a_{\mathrm{cyl}}(t) = t^{-1/2}\left\{\frac{e^{-\mu(t-1)}}{a_1^{-1} - \xi\mu^{-1/2}e^{\mu}(\gamma(\tfrac{1}{2},\mu t) - \gamma(\tfrac{1}{2},\mu))}\right\}, \quad (5.2)$$

$$a_{\mathrm{sph}}(t) = t^{-1}\left\{\frac{e^{-\mu(t-1)}}{a_1^{-1} - \xi\, e^{\mu}(E_1(\mu) - E_1(\mu t))}\right\}, \quad (5.3)$$

where $a_1 = a(1)$, and $\gamma(.,.)$ and E_1 are the incomplete gamma function and exponential integral defined by

$$\gamma(p, x) = \int_0^x t^{p-1} e^{-t}\, dt$$

and

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\, dt.$$

The coefficients μ and ξ are determined from (3.21) and (3.22) and are given explicitly in [22].

Concerning the behaviour of the amplitude we may prove from (5.2) and (5.3) the following lemma concerning the decay of the wave amplitude or possible formation of a shock wave ($|a| \to \infty$).

Lemma 5.1. *Suppose $\mu > 0$. The amplitude of a cylindrical wave is such that:*

I. *If* $\operatorname{sgn} a_1 = -\operatorname{sgn} \xi$, *or if*

$$|a_1^{-1}| > |\xi|\, e^{\mu} \mu^{-1/2}\{\sqrt{\pi} - \gamma(\tfrac{1}{2}, \mu)\},$$

then $|a| \to 0$, $t \to \infty$. *Otherwise,*

II. $|a| \to \infty$, $t \to t_\infty^-$, *where* t_∞ *satisfies*

$$\gamma(\tfrac{1}{2}, \mu t_\infty) = \gamma(\tfrac{1}{2}, \mu) + \sqrt{\mu}(a_1 \xi)^{-1} e^{-\mu}. \tag{5.4}$$

The amplitude of a spherical wave is such that:

III. *If* $a_1 = -\operatorname{sgn} \xi$, *or if*

$$|a_1^{-1}| > |\xi|\, e^{\mu} E_1(\mu),$$

then $|a| \to 0$, $t \to \infty$. *Otherwise,*

IV. $|a| \to \infty$, $t \to t_\infty^-$, *where* t_∞ *satisfies*

$$E_1(\mu t_\infty) = E_1(\mu) - (a_1 \xi)^{-1} e^{-\mu}. \tag{5.5}$$

Suppose now $\mu < 0$. The amplitude of a cylindrical wave is such that:

V. *If* $\operatorname{sgn} \xi = \operatorname{sgn} a_1$ *then* $|a| \to \infty$, $t \to t_\infty$, *where* t_∞ *satisfies* (5.4). *Otherwise,*

VI. *a remains bounded for all t and*

$$a(t) \to \frac{\mu}{\xi}, \quad t \to \infty. \tag{5.6}$$

The amplitude of a spherical wave is such that:

VII. *If* $\operatorname{sgn} \xi = \operatorname{sgn} a_1$ *then* $|a| \to \infty$, $t \to t_\infty$, *where* t_∞ *satisfies* (5.5). *Otherwise*

VIII. **VI** *applies.*

The behaviour of the thermal amplitude, α, may again be inferred from (3.2).

6. Concluding Remarks

So far we have been mainly concerned with the fast wave. However, the second wave will in general be as important as the first unless its behaviour can be shown to be negligible. If one can determine the deformation and its gradients in the region behind the first wave then one can in principle determine the shape of the second wave by using a bicharacteristic method developed for elasticity by Varley and Dunwoody [29]. In principle one can construct the solution behind the first wave by Taylor series, although the actual construction may be difficult. It appears physically obvious that the spherical wave in Section 5 will create a spherically symmetric deformation field behind it, and since $\dot{x}^{i}_{,j}{}^{-} = -a^{i}n_{j}/U$, and $n^{1} = 1$, $n^{2} = n^{3} = 0$, $b^{ij} = \lambda^{2}g^{ij}$, we deduce from the propagation condition $(3.2)_1$ that $\dot{x}^{1}_{,1}{}^{-}$ is the only nonzero component of $\dot{x}^{i}_{,j}{}^{-}$. However, to deduce the velocity behind the wave is radially symmetric one needs a knowledge of $x^{i}_{,jk}{}^{-}$ and higher gradients at the wave and the solution to this problem is not obvious to the present writers. If one can show that $\dot{x}^{i} \equiv (\dot{x}^{1}(r), 0, 0)$ behind the first wave then it is straightforward to show the second wave is spherical (cf. Section 9.3 of Lindsay and Straughan [30] where a similar problem is solved). Nevertheless, the method of bicharacteristics would appear to be a possible means of investigating the second wave.

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Abstract

The behaviour of acceleration waves in the nonlinear theory of thermoelasticity of Green and Lindsay [1] is investigated systematically. It is shown that two coupled waves may propagate with different finite wavespeeds. For waves entering isothermal homogeneously strained regions, explicit results are obtained for the wavespeeds and wave amplitudes, and the possibility of shock-wave formation is discussed.

Zusammenfassung

Das Verhalten von Beschleunigungswellen in der nichtlinearen thermoelastischen Theorie von Green und Lindsay [1] wird systematisch untersucht. Es wird gezeigt, dass zwei gekoppelte Wellen mit verschiedenen endlichen Wellengeschwindigkeiten wandern können. Für Wellen, welche in isotherme, homogen verzerrte Gebiete eintreten, werden die Wellengeschwindigkeiten und Amplituden explizit bestimmt, und es wird die Möglichkeit der Bildung von Stosswellen diskutiert.

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