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Beam propagation in a linear or nonlinear lens-like medium using *ABCD* ray matrices: the method of moments

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AN INVITED PAPER

We derive exact expressions for the evolution of the second order moment of the intensity distribution of an arbitrary beam propagating in a nonlinear Kerr medium with a quadratic index profile. The results can be recast in terms of the *ABCD* matrix formalism after introducing a generalized complex radius of curvature, $Q(z)$. Various definitions of the beam quality factor are introduced. Numerical simulations reveal the interest of this approach.

1. Introduction

In the past few years the so-called 'beam quality factor' has been introduced in several different ways as a standard measure of the propagation and focusing properties of an arbitrary beam. We refer to other papers in this special issue for further discussion and applications of this parameter. This beam quality factor compares the divergence of a real beam with the divergence of an ideal beam, i.e. a Gaussian beam.

For a general comparison, applicable to an arbitrary beam, the divergence is defined here in terms of the second order moment of the intensity distribution in the far field. The law of propagation of the second order moment of an optical beam (and of an optical pulse) has recently been demonstrated by several authors using different approaches [1–5]. To our knowledge, however, Vlasov *et al.* [6] were the first to show that the second order moment obeys a parabolic law of propagation in an homogeneous medium, in a two-transverse-dimensions geometry. Their proof was based on various methods, later used by other authors: integration by parts of the differential equation, the Huygens–Fresnel integral and the theory of partial coherence (Wigner transformation). Their result also covers the case of a Kerr nonlinearity, showing that, in such a medium, the law is still parabolic [6, 7]. For a linear homogeneous medium they also demonstrated that the moments of order n evolve according to a polynomial of order z^n, where z is the propagation distance.

In this paper we extend the result of Vlasov *et al.* [6] by deriving the law of propagation of the second order moment of an arbitrary beam propagating in a nonlinear medium with a quadratic index profile. In Section 2 we first consider an homogeneous medium. We emphasize an important difference between a one- and a two-transverse-dimensions geometry, showing that only in the latter case can the parabolic law of propagation be extended to the nonlinear problem. The derivation leads us to propose a first definition of the beam quality factor. In particular we stress the influence of the initial wavefront on this factor. In Section 3 we show that the results of Vlasov *et al.* can be recast in an elegant way by introducing the complex radius of curvature, $Q(z)$. This complex radius of curvature represents a further generalization of the complex radius introduced in [8] for a general beam, and it involves a more general beam quality factor.

In Section 4 we discuss briefly the question of self-trapping and the advantage of using the theory of moments instead of the approximate model of 'aberrationless approximation' to describe the nonlinear focusing of an arbitrary beam. In Section 5, in order to cover any paraxial system (i.e. those described by an *ABCD* matrix), we consider a lens-like medium with a Kerr nonlinearity. In the linear case the beamwaist of a Gaussian beam is known to evolve in a sinusoidal way as it propagates in such a medium. Here we show that the second order moment of an arbitrary beam behaves in the same way, even in the presence of a nonlinearity. Furthermore, the period of oscillation is found to be independent of the power of the beam. To our knowledge this has not been shown elsewhere.

This is followed by a presentation of numerical simulations showing the interest of defining an effective radius of curvature and an effective beamsize. This is particularly appropriate for a Gaussian beam where the wavefront remains quasiparabolic below the critical power for self-focusing. Super-Gaussian beams are also considered. Finally, we conclude by mentioning possible extensions and applications of this work.

2. Propagation of the second order moment in a nonlinear Kerr medium

In a lossless and lens-like medium $[n^2 = n_0^2(1 - \alpha^2\rho^2)]$ with a Kerr nonlinearity, the propagation of a c.w. monochromatic beam is described by the paraxial wave equation

$$\frac{1}{\rho^{d-1}}\frac{\partial}{\partial\rho}\left(\rho^{d-1}\frac{\partial u}{\partial\rho}\right) - 2ik\frac{\partial u}{\partial z} - k^2\alpha^2\rho^2 u + \gamma|u|^2 u = 0 \tag{1}$$

where $u(\rho, z)$ represents the slowly varying complex amplitude of the electric field. $k = 2\pi n_0/\lambda$, where λ is the wavelength in a vacuum. The dimension parameter d is equal to 1 or 2 for a one- or a two-transverse-dimensions geometry, respectively. ρ is the transverse coordinate: x for $d = 1$, or the radial variable r for a rotationally symmetric beam in the circular geometry $d = 2$ (the extension for an arbitrary two-dimensional transverse geometry is discussed in Section 7). γ is real and proportional to the nonlinear refractive index n_2 ($\gamma = k^2 n_2/n_0$ if one defines n_2 as $n = n_0 + \frac{1}{2}n_2|u|^2$). It will be shown that a beam does not propagate in the same way for $d = 1$ and $d = 2$.

Equation 1 has the first invariant

$$I_0 = \int_S |u|^2\rho^{d-1}\,\mathrm{d}\rho = \text{Const.} \tag{2}$$

which simply states that the power of the beam $[P = \varepsilon_0 c m_0(2\pi I_0)]$ is conserved during the propagation, as would be expected. The domain of integration, S, extends from $-\infty$ to $+\infty$ for $d = 1$ and from 0 to $+\infty$ for $d = 2$. As shown in [8], the complex radius of curvature of a Gaussian beam, $q(z)$, can be generalized to deal with an arbitrary beam. The beam size then refers to an average value corresponding to the second order moment of the intensity distribution, namely

$$W^2 = (4/dI_0)\int_S |u|^2\rho^{d+1}\,\mathrm{d}\rho \tag{3}$$

The factor $4/d$ appears in order to have $W^2 = w^2$ for a Gaussian beam $u = \exp(-\rho^2/w^2)$.

In a first step we consider an homogeneous medium. Then, using the paraxial wave equation (Equation 1) with $\alpha = 0$ and integrating by parts (assuming a well behaved function at the limit $\rho \to \infty$), the derivatives of W^2 can be evaluated versus z. For the first derivative

$$\frac{\mathrm{d}W^2}{\mathrm{d}z} = -\frac{8}{dkI_0}\,\mathrm{Im}\left[\int_S u^*\left(\frac{\partial u}{\partial \rho}\right)\rho^d\,\mathrm{d}\rho\right] \tag{4a}$$

(u^* represents for the complex conjugate of u) or, in terms of the amplitude and phase profiles A and ϕ {i.e. $u(\rho, z) = A(\rho, z)\exp[\mathrm{i}\phi(\rho, z)]$}

$$\frac{\mathrm{d}W^2}{\mathrm{d}z} = -\frac{8}{dkI_0}\int_S \rho^d A^2\left(\frac{\partial \phi}{\partial \rho}\right)\mathrm{d}\rho \tag{4b}$$

The next two deriatives are given by

$$\frac{\mathrm{d}^2W^2}{\mathrm{d}z^2} = \frac{8}{dk^2I_0}\int_S \rho^{d-1}\left(\left|\frac{\partial u}{\partial \rho}\right|^2 - \frac{d\gamma}{4}|u|^4\right)\mathrm{d}\rho \tag{5a}$$

$$= \frac{8}{dk^2I_0}\int_S \left\{\left[\left(\frac{\partial A}{\partial \rho}\right)^2 - \frac{d\gamma}{4}A^4\right] + A^2\left(\frac{\partial \phi}{\partial \rho}\right)^2\right\}\rho^{d-1}\,\mathrm{d}\rho \tag{5b}$$

and

$$\frac{\mathrm{d}^3W^2}{\mathrm{d}z^3} = \frac{2(2-d)\gamma}{dk^2I_0}\frac{\mathrm{d}}{\mathrm{d}z}\left(\int_S \rho^{d-1}|u|^4\,\mathrm{d}\rho\right) \tag{6}$$

This is a generalization of the results of Vlasov *et al.* [6] written in a compact form for both the one- and two-transverse-dimensions geometries. For a linear medium ($\gamma = 0$) the third order derivative is identically zero and, consequently, all higher order derivatives are also zero. We then obtain the known result that, in a linear homogeneous medium, the second order moment obeys a parabolic law of propagation, viz.

$$W^2(z) = W_0^2 + c_1z + c_2z^2 \tag{7}$$

where the coefficients $c_j = (1/j)(\mathrm{d}^jW^2/\mathrm{d}z^j)_{z=0}$ ($j = 1, 2$) are given by Equations 4 and 5 and $W_0 = W(z = 0)$.

For a nonlinear medium ($\gamma \neq 0$) the behaviour depends on the geometry. First, in the case of only one transverse dimension ($d = 1$) the law of propagation is no longer parabolic because the third derivative in Equation 6 is not zero in general. Equation 1 with $d = 1$ then corresponds to the nonlinear Schrödinger equation, which describes, for example, the propagation of optical pulses in a nonlinear medium such as an optical fibre [9] (as mentioned in [9], p. 87, an approximate parabolic law can also be derived for the case $d = 1$ for a Gaussian beam of limited intensity). It is well known that this equation has periodic solutions, the so-called higher order solitons. Their existence implies the absence of a parabolic law of propagation, simply because the second order moment of these solutions must be a periodic function of z as well. The third derivative is zero only for the fundamental soliton, but this represents a trivial case because the shape of this soliton is preserved during the propagation. So, for $d = 1$ the law of propagation is not parabolic in a nonlinear medium.

Hereafter this paper concentrates on the two-dimensional transverse geometry. The range of integration in the following integrals is from 0 to $+\infty$. In such a case, in contrast with the previous situation, it turns out that the third derivative is identically zero and we obtain the striking result, first pointed out by Vlasov *et al.* [6], that the second order moment evolves according to the parabolic law (Equation 7), even in a nonlinear medium. From Equation 5 this amounts to stating that the integral I_2, defined as

$$I_2 = \int \left[\left(\frac{\partial A}{\partial r} \right)^2 - \frac{\gamma}{2} A^4 + A^2 \left(\frac{\partial \phi}{\partial r} \right)^2 \right] r \, dr \tag{8}$$

represents a second invariant of the paraxial wave equation (Equation 1): $I_2 = Const.$ [7, 10].

We now introduce the effective radius of curvature, R, as a generalization of the familiar radius of curvature of a Gaussian beam. We then define [11]

$$\frac{1}{R(z)} \equiv \frac{1}{2W(z)^2} \frac{dW(z)^2}{dz} \tag{9a}$$

$$= -\frac{2}{kW^2 I_0} \int r^2 A^2 \frac{\partial \phi}{\partial r} \, dr \tag{9b}$$

The physical meaning of this parameter is the following (see also [11]): at an arbitrary plane z, consider the fields $u_1(r) = A(r) \exp[i\phi(r)]$ and $u_2(r) = A(r) \exp[-i(kr^2/2R_2)]$. Both fields have the same amplitude profile, but the phase of u_2 is parabolic whereas u_1 has an arbitrary wavefront. Using Equation 9 it is easy to show that the growth rate of the second order moment, i.e. $(1/W)(dW/dz)$, at this plane is the same for both fields if the radius of curvature of the parabolic phase profile is equal to the effective radius $R(z)$ (Equation 9b) of u_1, i.e. if $R_2 = R(z)$. We refer below to this parabolic wavefront as the 'effective wavefront'.

Combining Equations 5, 8 and 9, the law of propagation can now be written as

$$W(z)^2 = W_0^2 + (2W_0^2/R_0)z + (2I_2/k^2 I_0)z^2 \tag{10}$$

In the far field the spot size increases linearly with a divergence $\theta = W/z$ given by

$$\theta^2 = 2I_2/k^2 I_0 \tag{11}$$

In the case of a Gaussian beam with a beamwaist W_0, the divergence is $\theta_{GB} = \lambda/\pi W_0$. In free space, as is well known, the Gaussian beam represents the optimum profile as regards the far field divergence (see, for example [12], and references therein). As suggested by various authors (see, for example [13]), it could serve as a standard reference for evaluating the 'quality' of other beam profiles. A first definition of the beam quality factor of a beam profile would then correspond to the ratio of its far field divergence to the divergence of a Gaussian beam of same effective beam size W_0, i.e. $(M_Q^2)_I = \theta/\theta_{GB}$. From Equations 10 and 11 this yields

$$(M_Q^2)_I^2 = \frac{1}{I_0^2} \int A_0^2 r^3 \, dr \int \left[\left(\frac{dA_0}{dr} \right)^2 - \frac{\gamma}{2} A_0^4 + A_0^2 \left(\frac{d\phi_0}{dr} \right)^2 \right] r \, dr \tag{12}$$

where $A_0 = A(z = 0)$ and $\phi_0 = \phi(z = 0)$. For a given amplitude profile $A(r)$ it is clear, from Equation 12, that $(M_Q^2)_I$ is optimum (minimum) for a uniform phase profile $[\phi_0(r) = Const.]$. In the linear regime $(M_Q^2)_I \geqslant 1$. We also note that a focusing nonlinearity ($\gamma > 0$) reduces the divergence (inside the nonlinear medium) and then improves the beam quality factor as defined here. The dependence on the nonlinearity is better expressed if the 'critical power for self-focusing', P_{cr}, is introduced [10]. (For a detailed review of theoretical and experimental work on self-focussing and self-trapping, see [14, 15].) At this power the nonlinear focusing can balance the natural diffraction of the beam, and the second order moment remains unchanged during the propagation. P_{cr} is defined for a field with a uniform wavefront and, according to Equation 12, is given by [6]

$$P_{cr} = \frac{4\pi\varepsilon_0 c n_0}{|\gamma|} \int \left(\frac{dA_0}{dr}\right)^2 r\,dr \int A_0^2 r\,dr \Big/ \int A_0^4 r\,dr \tag{13}$$

This ratio of integrals is independent of the absolute amplitude and of the transverse scaling; only the beam profile matters. Although this phenomenon is not possible for a defocusing medium ($\gamma < 0$), P_{cr} can still be used as a normalization if $|\gamma|$ is used instead of γ. We return below to the notion of critical power when we discuss self-trapping (Section 4). The beam quality factor can now be written as

$$(M_Q^2)_{\rm I}^2 = \frac{1}{I_0^2}\left(\int A_0^2 r^3\,dr\right)\left[\left(1 - \frac{\eta P}{P_{cr}}\right)\int\left(\frac{dA_0}{dr}\right)^2 r\,dr + \int A_0^2\left(\frac{d\phi_0}{dr}\right)^2 r\,dr\right] \tag{14}$$

where $\eta = \operatorname{sgn}\gamma = +1$ for $\gamma > 0$ and -1 for $\gamma < 0$.

The Rayleigh distance of a Gaussian beam, $Z_R = kW_0^2/2$, can also be generalized. Let $z = 0$ correspond to the waist location, i.e. where W is minimum and $R_0 = \infty$. Z_R is then defined as the distance at which $W^2(z = Z_R) = 2W^2(z = 0) = 2W_0^2$. From Equations 10 and 12 it can then be concluded that

$$Z_R = \frac{kW_0^2}{2(M_Q^2)_{\rm I}} \tag{15}$$

For $\gamma > 0$ ($\gamma < 0$) the nonlinearity reduces (increases) the divergence and then increases (reduces) the Rayleigh range. It is important to realize that at the waist location the wavefront is not necessarily uniform; the condition is that the effective radius be infinite.

As defined above, the beam quality factor depends on the reference plane through the beamsize W_0 used for the divergence θ_{GB} of the Gaussian beam. This would not be the case if one used instead the *minimum* beamsize W_{min} (calculated from W_0, R_0 and l_2 using Equation 10) i.e. $\theta_{GB} = \lambda/\pi W_{min}$. Moreover, as discussed in the next section, this choice presents other advantages.

3. General complex radius of curvature, $Q(z)$

Having defined the effective radius of curvature and beamsize, R and W, we can now generalize the familiar definition of complex radius of curvature, $q(z)$, usually limited to Hermite(Laguerre)–Gauss beams. Recently [8] the more general complex radius $Q(z)$ has been introduced as a way of dealing with arbitrary beams. The above results allow the definition of Q to be generalized further in order to include the effect of a Kerr nonlinearity. The reader can easily verify that if the general complex radius of curvature $Q(z)$ is defined as

$$\frac{1}{Q(z)} \equiv \frac{1}{R(z)} - \mathrm{i}\,\frac{\lambda(M_Q^2)_{\rm II}}{\pi W(z)^2} \tag{16}$$

then the law of propagation of $R(z)$ and $W^2(z)$ can be summarized as

$$Q(z) = Q(z = 0) + z \tag{17}$$

i.e. $Q(z)$ follows the same law as the conventional $q(z)$ (see, for example [16]). In Equation

16 we have introduced a second definition of the beam quality factor, namely

$$(M_Q^2)_{\text{II}}^2 = \frac{k^2}{8}\left[W(z)^2\left(\frac{\mathrm{d}^2W(z)^2}{\mathrm{d}z^2}\right) - \frac{1}{2}\left(\frac{\mathrm{d}W(z)^2}{\mathrm{d}z}\right)^2\right] \tag{18a}$$

$$= \frac{W(z)^2}{2I_0}\left[\int\left(\left|\frac{\partial u}{\partial r}\right|^2 - \frac{\gamma}{2}|u|^4\right) r\,\mathrm{d}r\right] - \frac{1}{I_0^2}\left(\mathrm{Im}\int ru^*\frac{\partial u}{\partial r}\,r\,\mathrm{d}r\right)^2 \tag{18b}$$

$$= \frac{W_0^2}{2I_0}\left[\left(1 - \frac{\eta P}{P_{\text{cr}}}\right)\int\left(\frac{\mathrm{d}A_0}{\mathrm{d}r}\right)^2 r\,\mathrm{d}r + \int A_0^2\left(\frac{\mathrm{d}\phi_0}{\mathrm{d}r}\right)^2 r\,\mathrm{d}r\right] - \frac{1}{I_0^2}\left[\int rA_0^2\left(\frac{\mathrm{d}\phi_0}{\mathrm{d}r}\right) r\,\mathrm{d}r\right]^2 \tag{18c}$$

$$= (M_Q^2)_{\text{I}}^2 - k^2W^4/4R^2 \tag{18d}$$

We believe that this definition of the beam quality factor is more general than the first one (Equation 12). First, it can be shown that $(M_Q^2)_{\text{II}}$ is invariant, i.e. its value is independent of the reference plane z; this follows directly from the parabolic law (Equation 10) and Equation 18a. Furthermore, it is not modified if the beam goes through a lens or is reflected by a mirror. Indeed, it can be verified (using Equation 18c) that if the phase ϕ is changed to $\phi + (Const.)r^2$, then $(M_Q^2)_{\text{II}}$ remains unchanged. These properties are essential, otherwise the general complex radius of curvature $Q(z)$, as defined by Equation 16, would not be very useful. $(M_Q^2)_{\text{II}}$ is minimum for a uniform or a parabolic wavefront, as can be shown using the Schwarz inequality. Finally, we stress again the influence of the phase on the beam quality factor. People working in the field of partial coherence would not be surprised by this conclusion.

Equation 18d shows how the two definitions of the beam quality factor are related. It turns out that the two factors would be identical if the *minimum* beamsize W_{min} were used for the divergence of the reference Gaussian beam. This observation also indicates how to evaluate the beam quality factor in the laboratory; through measurement of the effective beamsize $W(z)$ at various planes, one can deduce the far-field divergence θ and the beamwaist W_{min}. Then the fundamental beam quality factor is simply given by

$$(M_Q^2)_{\text{II}} = \frac{\theta}{(\lambda/\pi W_{\text{min}})^{1/2}} \tag{18e}$$

Some years ago [17] (see also [18]) we also extended the definition of the complex radius of curvature to describe the propagation of a Gaussian beam in a Kerr medium using the 'aberrationless approximation' (see Section 4). Here the result is exact and applies to any beam profile.

Both definitions of the beam quality factor take the same minimum value for a uniform wavefront. This leads to introduce a third definition of beam quality factor, $M_{Q_0}^2$, corresponding to this minimum value

$$(M_{Q_0}^2)^2 = \left(1 - \frac{P}{P_{\text{cr}}}\right)\int A_0^2r^3\,\mathrm{d}r\int\left(\frac{\mathrm{d}A_0}{\mathrm{d}r}\right)^2 r\,\mathrm{d}r\bigg/\left(\int A_0^2r\,\mathrm{d}r\right)^2 \tag{19}$$

where the subscript Q_0 refers to $\mathrm{d}\phi_0/\mathrm{d}r = 0$, i.e. a uniform wavefront. Equation 19 shows clearly the influence of the nonlinearity on the beam quality factor

$$M_{Q_0}^2 = (1 - \eta P/P_{\text{cr}})^{1/2}(M_{Q_0}^2)_{\text{L}} \tag{20}$$

where $(M_{Q_0}^2)_{\text{L}}$ refers to the linear case.

The beam quality factor $M_{Q_0}^2$ should be used to compare ideal beams, i.e. beams with a uniform phase at the minimum beamwaist location. More fundamentally, the general beam quality factor $(M_Q^2)_{II}$ (Equation 18) serves to compare real beams and it is the one that must be introduced in order to generalize the *ABCD* matrix formalism. In the linear case, as mentioned above, the beam quality factor (for the three definitions) is minimum for a Gaussian beam with a uniform wavefront. In the nonlinear case the situation is different, and this factor can even be zero for a self-trapped beam.

4. Self-trapping

As a result of the focusing effect of the nonlinearity (if $\gamma > 0$), a particular beam profile $u_{st}(r)$ exists which preserves its shape in propagating inside the nonlinear medium [10, 14, 15, 19]. This phenomenon, known as 'self-trapping', represents a spatial solitary wave. True self-trapping requires the exact beam shape u_{st} (or the associated higher order solutions [20]); for an arbitrary beam one might simply request that the second order moment be preserved during the propagation [6]. From Equation 10 this is possible for any beam if the following conditions are satisfied:

$$R_0 = \infty \qquad I_2 = 0$$

The second condition gives the required power

$$\frac{P_{st}}{P_{cr}} = 1 + \int A_0^2 \left(\frac{d\phi_0}{dr}\right)^2 r\,dr \Big/ \int \left(\frac{dA_0}{dr}\right)^2 r\,dr \tag{21}$$

and it is minimum $(= P_{cr})$ for a uniform wavefront. It it interesting to realize that self-trapping, according to this loose definition, is also possible for a nonuniform wavefront as long as $R_0 = \infty$. Vlasov *et al.* [6] also showed that P_{cr} (Equation 13) is minimum for the spatial solitary wave $A_0(r) = u_{st}(r)$, and we refer to this minimum value as P_{min}. The exact profile $u_{st}(r)$ must be determined numerically [19] and the ratio of integrals in Equation 13 is then found to be 0.917 $(5.7637/2\pi)$ [6]. However, as shown elsewhere (see [21, 22] and references in [21]), u_{st} can be approximated well by a Gaussian or even better by a hyperbolic secant. Indeed, using Equation 13, the critical power for these beam profiles can be evaluated. The result is

$$P_{cr}^{GB}/P_{min} = 1.090 \qquad P_{cr}^{sech}/P_{min} = 1.017$$

The self-focusing of a Gaussian beam is frequently considered within the framework of the so-called 'aberrationless approximation' [10]. In this approach, in order to obtain analytical results, the beam profile is expanded around the axis and an ordinary differential equation is obtained which describes the evolution of the beamwaist. The whole beam is assumed to remain Gaussian during the propagation. In limiting the analysis too close to the axis, this approach overestimates the importance of the nonlinear refraction and yields a poor estimate P_{cr}^{aberr} of the critical power for a Gaussian beam: $P_{cr}^{aberr} = 0.25P_{cr}$ [6, 14, 15]. The constant-shape approximation is also exploited in another approach based on a variational principle [21–23]. This other method is much more accurate than the paraxial aberrationless approximation, as it predicts the same value for P_{cr} as the theory of moments [21, 24]. The advantage of the latter, however, is its generality; it gives exact results and it can be applied to arbitrary beams. In contrast, the variational method is fruitful as long as the constant-shape approximation is valid, i.e. for smooth beams such as a Gaussian or a hyperbolic secant.

If $P > P_{cr}$ a collimated beam (i.e. $\phi_0 = Const.$) will 'collapse', globally, as its second order moment goes to zero. Another definition of the critical power corresponds to the threshold above which local collapse occurs, i.e. when a focal point is formed so that the on-axis intensity increases to infinity [6, 14, 15]. This threshold P_{cr}^{local} must be determined numerically. For a Gaussian beam its value is close to what is expected using the theory of moments, i.e. P_{cr} obtained from Equation 13 ($P_{cr}^{local} \approx P_{cr}$ [6, 14, 15]). In other words, global and local collapse have nearly the same threshold for a Gaussian beam. In Section 6 it will be found that this is not general, as we simulate the propagation of super-Gaussian beams.

It must be mentioned that the self-trapping solution, u_{st}, is actually unstable (see [14, 25] and references therein). For example, assume $A_0(r) = (1 + \delta)u_{st}(r)$; then, if $\delta > 0$ the input power exceeds P_{cr} and global collapse occurs; on the other hand, if $\delta < 0$ then $P < P_{cr}$, the diffraction will dominate and the beam will spread out. This is in contrast with what prevails in one-transverse dimension ($d = 1$) where the fundamental spatial soliton of the nonlinear Schrödinger equation is known to be robust.

We also point out that near the on-axis singularity the paraxial wave equation is no longer valid and either the Helmholtz equation [26] must be used or other physical phenomena such as a saturation of the nonlinearity [14, 25] must be introduced.

5. Propagation in a lens-like medium with a Kerr nonlinearity

We now consider the propagation of an arbitrary beam in a nonlinear medium with a quadratic index profile, as described by Equation 1 with $\alpha \neq 0$. In the linear problem ($\gamma = 0$) with a Gaussian beam, the propagation gives rise to a periodic focusing and defocusing. Here we take into account the influence of the nonlinearity. Equation 1 has been treated recently in an approximate way using the aberrationless approximation [27] or the more reliable variational approach [22, 28]. It was then found that the periodicity is preserved, with the surprising property that the period is independent of the power of the beam; this was also observed numerically [22]. In the following, because we limit the analysis to the second order moment, exact analytical results can be obtained, and these confirm the approximate results just mentioned.

Repeating the analysis of Section 2 with $\alpha \neq 0$

$$\frac{\mathrm{d}W^2}{\mathrm{d}z} = -\frac{4}{kI_0}\,\mathrm{Im}\left(\int u^* \frac{\partial u}{\partial r}\, r^2\, \mathrm{d}r\right) \tag{22}$$

$$\frac{\mathrm{d}^2 W^2}{\mathrm{d}z^2} = -2\alpha^2 W^2 + \frac{4}{k^2 I_0}\left[\int\left(\left|\frac{\partial u}{\partial r}\right|^2 - \frac{\gamma}{2}|u|^4\right) r\, \mathrm{d}r\right] \tag{23}$$

$$\frac{\mathrm{d}^3 W^2}{\mathrm{d}z^3} = -4\alpha^2 \frac{\mathrm{d}W^2}{\mathrm{d}z} \tag{24}$$

Equation 24 implies a periodic behaviour of the form

$$W(z)^2 = c_1 \sin 2\alpha z + c_2 \cos 2\alpha z + c_3 \tag{25}$$

This periodic behaviour is even more surprising if one realizes that the period π/α is actually independent of the power. The nonlinearity affects only the amplitude of the oscillations. The evolution of $W(z)$ and $R(z)$ can again be reformulated in terms of the general complex radius of curvature, $Q(z)$, and with the same beam quality factor $(M_Q^2)_{\mathrm{II}}$ as before (Equation 18b; which is still an invariant of the equation, as can be checked using Equations 22 to 24 and 18b).

The independence of $(M_Q^2)_{\mathrm{II}}$ from α means that the passage of an arbitrary beam in a nonlinear medium with a quadratic index profile can be expressed in the usual way, viz.

$$Q(z) = [AQ(z = 0) + B]/[CQ(z = 0) + D] \tag{26}$$

with the conventional *ABCD* matrix

$$\begin{pmatrix} \cos \alpha z & (1/\alpha) \sin \alpha z \\ -\alpha \sin \alpha z & \cos \alpha z \end{pmatrix}$$

To our knowledge this has not been published previously. On the basis of this result and those of the previous sections [namely, the invariance of $(M_Q^2)_{\mathrm{II}}$ upon translation, lens transformation and quadratic duct propagation], it can now be concluded that the passage of an arbitrary beam through a nonlinear Kerr medium with *ABCD* paraxial elements can be described with the usual *ABCD* matrix formalism if the general complex radius of curvature $Q(z)$ (Equation 16) is used.

As an example, consider the case of a collimated Gaussian beam at the input, i.e. $u(z = 0) = \exp(-r^2/W_0^2)$. In propagating, the beam will not remain exactly Gaussian, but its effective waist will evolve according to the simple law

$$W(z)^2 = W_0^2 + (W_1^2 - W_0^2) \sin^2 \alpha z \tag{27}$$

where

$$W_1^2 = \frac{(1 - \eta P/P_{\mathrm{cr}})}{\alpha^2 (k W_0^2/2)^2} W_0^2 \tag{28}$$

In the limit $\alpha \to 0$ the parabolic law of propagation is recovered, as is easily verified. With a quadratic index profile, a distinction must be made between the power necessary for global collapse and the power required to preserve the initial effective beamwaist. From Equation 27 it can be found that collapse occurs at the same critical power, P_{cr}, as in an homogeneous medium ($\alpha = 0$). In contrast, $W^2(z)$ can remain constant at a lower power

$$\frac{\eta P}{P_{\mathrm{cr}}} = 1 - \alpha^2 (k W_0^2/2)^2 \tag{29}$$

Actually, as is well known [16], and as indicated by Equation 29, $W(z)$ can also remain constant in the linear problem ($P = 0$, in Equation 29) for a particular value of the input beamwaist. In the nonlinear case, in contrast with the self-trapped solution u_{st} discussed in Section 4, here the equilibrium is now stable (marginally) due to the converging index profile. If the input power is not the ideal one, the beamsize simply oscillates around the equilibrium position, as it does in the linear case when the input beamsize is not the correct one. This was also pointed out elsewhere [22, 28].

Equation 27 remains valid if $\alpha^2 < 0$, i.e. for a diverging index profile. The width of the beam then increases as sinh $(|\alpha| z)$ except at a particular power (Equation 29) greater than P_{cr} in order to overcome the divergence imposed by the index profile. This situation is, however, unstable.

6. Numerical simulations

To demonstrate the interest of defining an effective radius of curvature, with the corresponding effective parabolic wavefront, we solved numerically the nonlinear paraxial wave equation for the homogeneous case (Equation 1, with $\alpha = 0$). Gaussian and super-Gaussian beams with a uniform wavefront at the input were considered, i.e. $u(r, 0) = \exp[-(r/a)^m]$, where m is the order of the super-Gaussian (we use the symbol a instead of W_0 to avoid any possible confusion caused by the fact that $W \neq W_0$ for $u = \exp[-(r/W_0)^m]$ with $m > 2$). The effective radius and beamsize, $R(z)$ and $W(z)$, then evolve according to

$$R(z) = z(1 + Z_{\mathrm{R}}^2/z^2) \tag{30}$$

and

$$W(z)^2 = W_0^2(1 + z^2/Z_R^2) \tag{31}$$

where Z_R (Equation 15) can be written explicitly in terms of the gamma function, $\Gamma(z)$

$$Z_R = \frac{\{4^{(1-1/m)}[\Gamma(4/m)]^{1/2}/m\}}{(1 - \eta P/P_{cr})^{1/2}} \left(\frac{ka^2}{2}\right) \tag{32}$$

The critical power (Equation 13) can also be evaluated analytically:

$$P_{cr}/P_{min} = 4^{(1/m-1)}m/0.917 \tag{33}$$

The propagation of a Gaussian beam in a nonlinear medium has been extensively explored numerically in the past [14]. However, the emphasis was mostly on the determination of the critical power for local collapse and on the intensity profile. As far as we know, little has been done on the evolution of the wavefront, except a mention of its quasiparabolic profile below P_{cr} [14]. Here, in contrast, we are interested in precisely this aspect.

In order to compare the propagation of Gaussian and super-Gaussian beams, we find it preferable to normalize the axial distance z to Z_0, the Rayleigh range of a Gaussian beam in the linear case ($Z_0 = ka^2/2$). Similarly, the power is expressed in terms of the minimum power for self-trapping, P_{min}, instead of P_{cr} which depends on the beam profile. Also, since the theory of moments does not provide information on the on-axis phase shift, the latter is arbitrarily fixed at zero. For the present analysis only the curvature of the wavefront is relevant (see [8]). Finally, except when otherwise stated, we assume a focusing nonlinearity ($\gamma > 0$).

First we concentrate on the case of a Gaussian beam (for which $P_{cr} = 1.09P_{min}$). Figure 1 illustrates a typical example. At $P = 0.15P_{cr}$ ($=0.164P_{min}$), except for a reduced spreading, the beam behaves qualitatively as it does in the linear case (Fig. 1a). In such circumstances, as clearly evidenced by Figs 1b to e, the effective wavefront [i.e. a parabolic phase profile with a curvature given by $R(z)$, see Section 2] represents an excellent approximation to the exact phase profile and for any distance of propagation. Indeed, from the intensity profiles (Fig. 1a) it can be seen that the small discrepancy between the two wavefronts appears mainly in the wings, where the intensity is negligible.

In Fig. 2 the power is substantially increased to $P = 0.5P_{cr}$ ($=0.545P_{min}$). Figure 2a shows that the beam initially focuses on the axis but soon spreads out since the nonlinearity is insufficient to collapse the beam. The initial focusing can be explained if it is realized that $P = 0.5P_{cr}$ corresponds to $P = 2P_{cr}^{aberr}$ (Section 4); so, according to the 'aberrationless approximation', we are above the critical power and, hence, a focusing is to be expected. However, as clearly demonstrated by this example, the aberrationless approximation is valid only close to the axis and for short distances [14].

The initial on-axis focusing of the beam is best illustrated by its phase profile (Fig. 2b). Near the axis the wavefront is locally converging although globally, the beam diverges. At $z \gtrsim 0.5Z_0$ the on-axis intensity begins to decrease and, as the propagation progresses, the phase profile gradually becomes nearly parabolic and is better approximated by the effective wavefront (Figs 2c to e). Considering that this example corresponds to a significant power, it can be concluded that the effective wavefront can prove very useful when developing approximate models of the nonlinear propagation of a Gaussian beam. In particular, if $P \lesssim P_{cr}^{aberr}$ ($=0.25P_{cr}$), no initial on-axis focusing occurs and the exact phase profile is always nearly parabolic, so the effective wavefront represents a good approximation at any distance z.

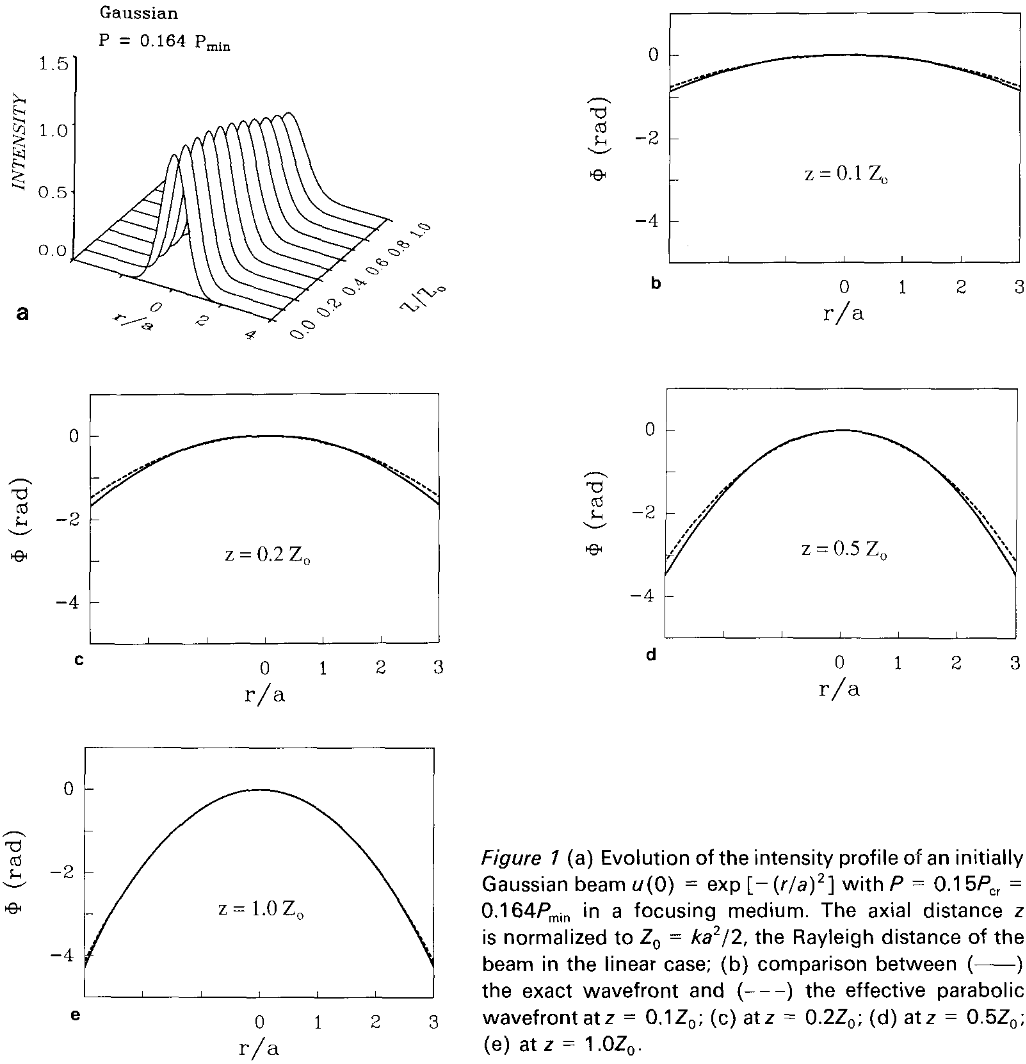
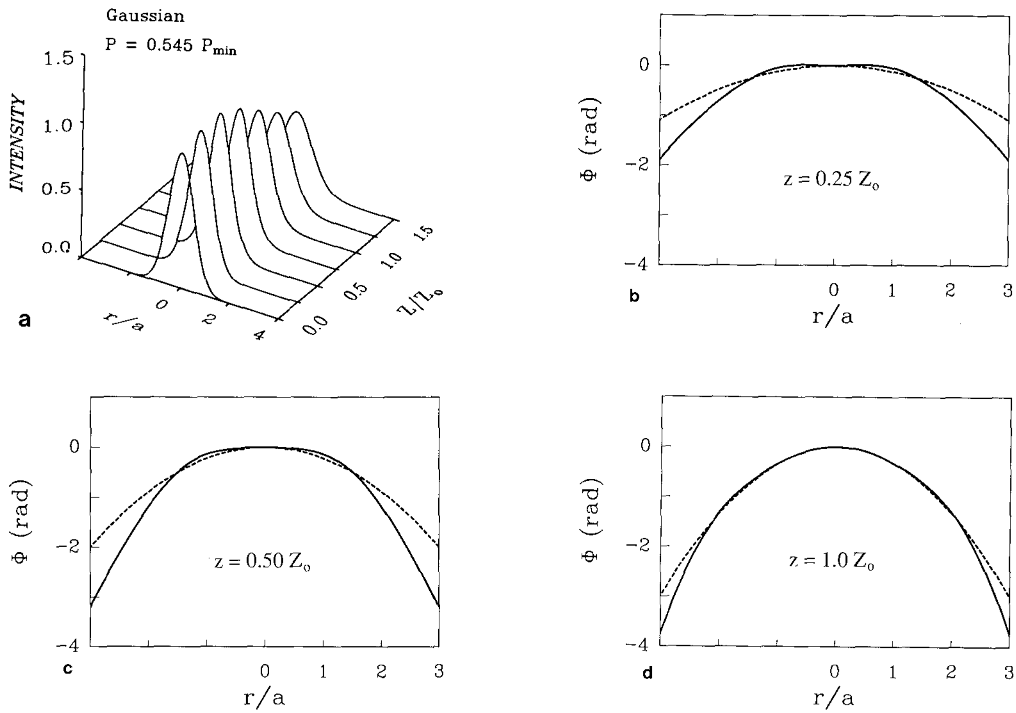


Figure 1 (a) Evolution of the intensity profile of an initially Gaussian beam $u(0) = \exp[-(r/a)^2]$ with $P = 0.15P_{cr} = 0.164P_{min}$ in a focusing medium. The axial distance z is normalized to $Z_0 = ka^2/2$, the Rayleigh distance of the beam in the linear case; (b) comparison between (——) the exact wavefront and (– – –) the effective parabolic wavefront at $z = 0.1Z_0$; (c) at $z = 0.2Z_0$; (d) at $z = 0.5Z_0$; (e) at $z = 1.0Z_0$.

If we increase the power further (Fig. 3, $P = 0.87P_{min}$), the conclusions are similar but, because the initial on-axis convergence is more important, it is necessary to go to larger distances before the wavefront becomes nearly parabolic (Fig. 3d).

For curiosity we illustrate in Fig. 4 what happens at the critical power for a Gaussian beam ($P = P_{cr} = 1.09P_{min}$). Because P_{cr} is slightly above P_{cr}^{local}, local collapse occurs, i.e. the on-axis intensity keeps increasing. Notice, however, that despite the local collapse the effective beamsize, W, remains constant during the propagation. Figures 4b and c are enlightening; near the axis the wavefront is evidently converging, but in the wings the curvature of the wavefront is changed and a fraction of the power disperses away. In the average, there is no global focusing and W does not change. Again for curiosity, we show in Fig. 5 that above P_{cr} ($P = 2P_{cr} = 2.18P_{min}$) the whole beam collapses and catastrophic focusing occurs at a shorter distance. Notice that, in practice, in order to avoid optical damage, one operates below the critical power so that the last two examples should not be



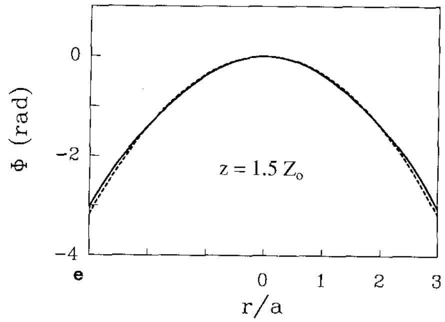


Figure 2 (a) Same as Fig. 1a, but now the power is increased to $P = 0.5P_{cr} = 0.545P_{min}$; (b) (——) exact and (- - -) effective wavefronts of $z = 0.25Z_0$; (c) at $z = 0.50Z_0$; (d) at $z = 1.0Z_0$; (e) at $z = 1.5Z_0$.

viewed as counterexamples to the practical usefulness of the averaged description of beam propagation.

As a final example with a Gaussian beam, Figs 6 illustrates the propagation in a defocusing nonlinear medium ($\gamma < 0$). In such a case the nonlinearity and the diffraction act together to widen the beam more rapidly. In the absence of initial on-axis focusing a good correspondence can then be expected between the exact and the effective wavefronts even at short distances. This is confirmed in Fig. 6b and c and for a power as high as $0.8P_{cr}$ ($0.87P_{min}$). The comparison between Figs 6c and 3b, which correspond to the same power and same distance but with opposite nonlinearities, is particularly convincing. In the defocusing case the effective wavefront becomes a very good approximation at shorter distances in comparison with the focusing case at high power.

We also simulated the nonlinear propagation of super-Gaussian beams of order $m = 3$ ($P_{cr} = 1.30P_{min}$) and $m = 6$ ($P_{cr} = 2.06P_{min}$). The case $m = 3$ is considered in Figs 7 to 9.

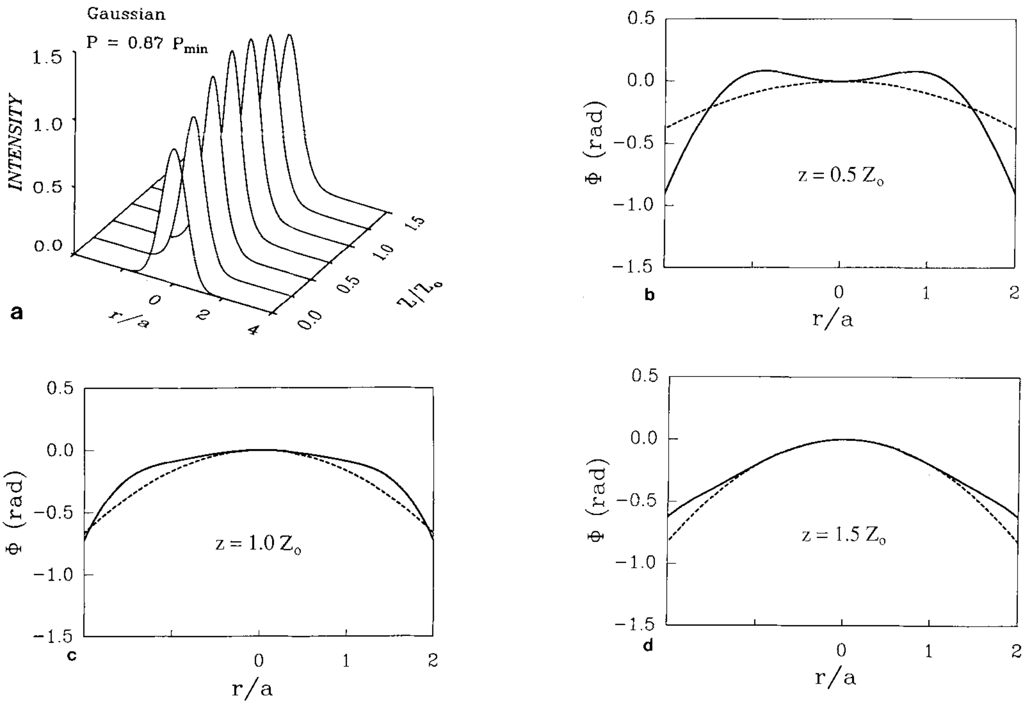


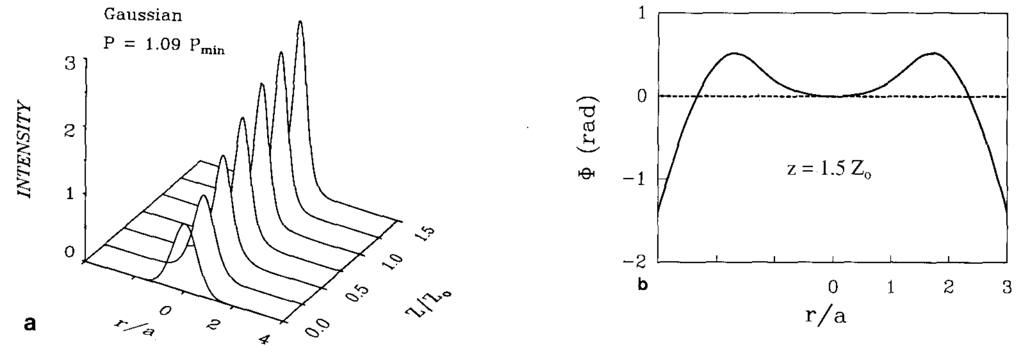
Figure 3 Same as Fig. 1a, but now the power is increased to $P = 0.8P_{cr} = 0.87P_{min}$; (b) (——) exact and (– – –) effective wavefronts at $z = 0.5Z_0$; (c) at $z = 1.0Z_0$; (d) $z = 1.5Z_0$.

First, Fig. 7a depicts the behaviour in free space; in contrast with a Gaussian beam there is an initial increase in the on-axis intensity even in the absence of nonlinearity. This intrinsic initial focusing enhances the nonlinear lensing (Fig. 7b) and larger distances (in comparison with a Gaussian beam of same power, see Fig. 2) must be considered before the wavefront looks parabolic, as a comparison between Figs 7c and 2c indicates. At $z = Z_0$ the wavefront is parabolic near the axis and is approximated well by the effective wavefront (Fig. 7d). The agreement extends further off-axis at longer propagation lengths (Fig. 7e).

At $P = P_{cr}$ $(=1.30P_{min})$ the beam collapses locally (Fig. 8) although its second order moment remains constant. In this case local and global collapses have very different thresholds. As a crude approximation, it can be estimated that $P_{cr}^{local} \approx P_{min}$. The exact value depends on the details of the evolution of the beam and can only be determined numerically. We defer this investigation to future work.

The propagation of the super-Gaussian in a defocusing medium ($\gamma < 0$) is illustrated in Fig. 9. At $P = 0.42P_{cr}$ $(=0.545P_{min})$ and for short distances the defocusing nearly balances the intrinsic focusing mentioned above and the on-axis intensity remains quasiconstant. As for a Gaussian beam, Fig. 9c reveals a good agreement between the exact and the effective wavefronts at shorter distances (compare Fig. 7c).

The propagation of a super-Gaussian of order $m = 6$ $(P_{cr} = 2.06P_{min})$ is also simulated, in Fig. 10, with the same power $(0.545P_{min})$ as for the Gaussian beam of Fig. 2 and the super-Gaussian of order 3 (Fig. 7). The behaviour of both super-Gaussian beams is found to be similar, as regards the correspondence between the exact and the effective wavefronts. After a transient reshaping the beam becomes smoother, the on-axis intensity gradually



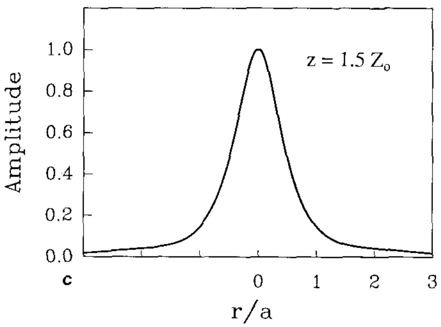


Figure 4 (a) Evolution of a Gaussian beam at the critical power P_{cr} ($=1.09P_{min}$). The effective beamwaist remains constant but local collapse occurs. (b) (——) exact phase profile and (– – –) effective wavefront at $z = 1.5Z_0$; (c) normalized amplitude profile at $z = 1.5Z_0$.

decreases and the phase profile is approximately parabolic near the axis. Then, where the intensity is significant, the effective wavefront represents a good approximation of the exact phase profile. The exact shape of the wavefront is also similar to that observed in the linear regime [29].

With these numerical simulations, the interest of defining an effective wavefront that approximates the exact phase profile of an arbitrary beam propagating in a linear or nonlinear medium can be realized. More precisely, the following conclusions can be drawn. First, in a defocusing medium the wavefront of Gaussian and super-Gaussian beams rapidly becomes nearly parabolic and is approximated well by the effective wavefront, particularly when considering long propagation lengths or reasonable powers. For a focusing nonlinearity the validity of the approximation depends on the power, on the distance of propagation and on the beam profile. In the case of a Gaussian beam the

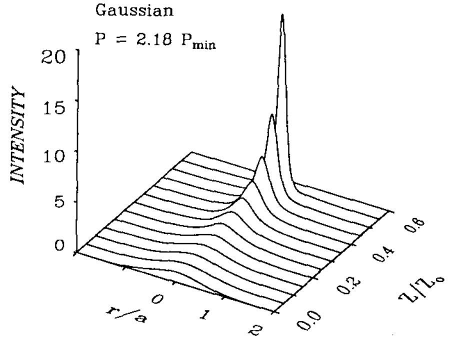
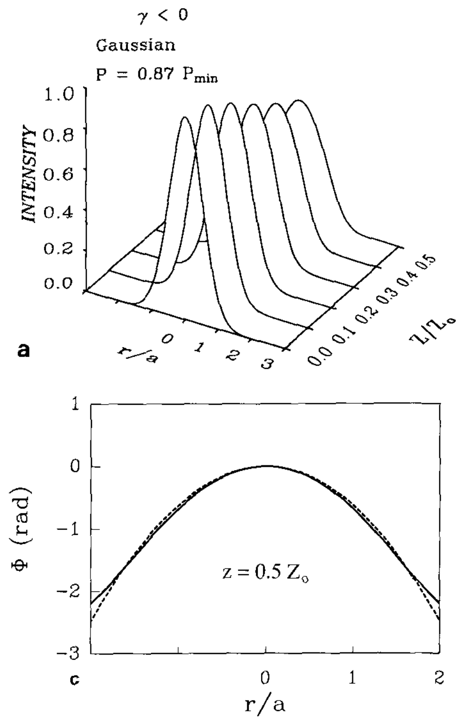


Figure 5 Global collapse of an initially Gaussian beam at twice the critical power: $P = 2P_{cr} = 2.18P_{min}$.



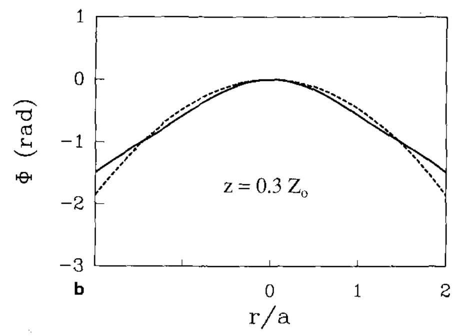


Figure 6 (a) Evolution of a Gaussian beam in a defocusing medium ($\gamma < 0$) at $P = 0.8P_{cr} = 0.87P_{min}$; (b) (——) exact and (– – –) effective phase profiles at $z = 0.3Z_0$; (c) at $z = 0.5Z_0$.

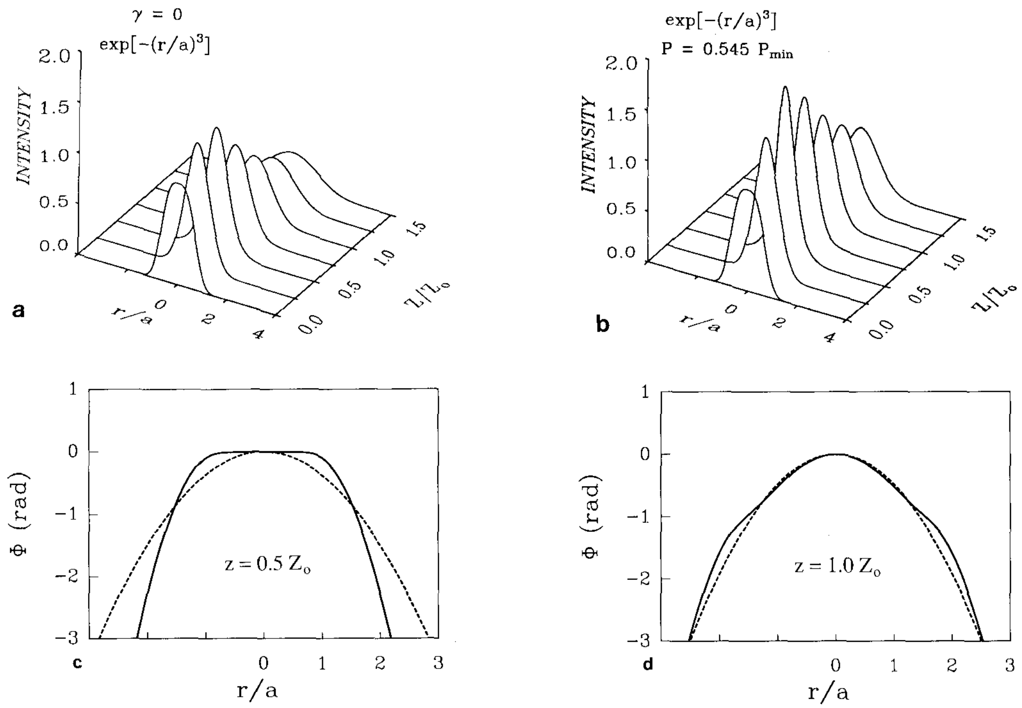
approximation is still very good at a power as high as $0.6P_{min}$ for $z \gtrsim 0.5Z_0$. In general the agreement begins to be good at shorter distances as we decrease the power. Also, if $P \lesssim P_{cr}^{aberr}$ $(=0.25P_{cr})$ the effective wavefront matches closely the exact wavefront at any distance. Similar conclusions should also apply to other smooth beams such as a hyperbolic secant, for example.

The effective beamwaist, $W(z)$, is also a relevant parameter. This can be observed in Fig. 11, which shows the normalized intensity profiles depicted in Fig. 10a at various positions. No quantitative assessment, such as the encircled energy within $r < W(z)$, has been done. Our purpose here is simply to show that in the absence of local collapse this averaged description of the beam size gives a reliable idea of the evolution of the beam spreading.

7. Discussion and conclusion

In dealing with nonlinear beam propagation, it is usually necessary to resort to either numerical tools or approximate analytical methods. Indeed, except for a few particular cases (see [21, 30–32] and references therein) where, for example, symmetry considerations can prove fruitful [22, 30, 31], exact analytical solutions of the nonlinear paraxial wave equation (Equation 1) are not available. Approximate methods, such as the variational approach [21–23], are accurate as long as the constant-shape approximation is justified, which is certainly not the case for a super-Gaussian, for example. This is why the exact results offered by the averaged description of the theory of moments can be of practical importance.

In including a quadratic index profile, one of the main results of this paper was to show that, thanks to a generalization of the complex radius of curvature, the propagation of an



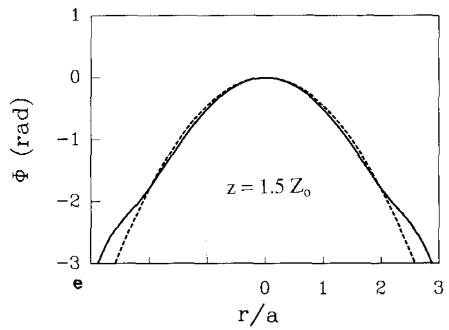


Figure 7 (a) Evolution of the intensity profile of an initially super-Gaussian beam of order 3, i.e. $u(0) = \exp[-(r/a)^3]$, in a linear medium. The axial distance z is still normalized to $Z_0 = ka^2/2$, the Rayleigh distance of a Gaussian beam (see text); (b) same as (a) but in a focusing medium with the same power as the Gaussian in Fig. 2, i.e. $P = 0.545P_{min}$ ($=0.42P_{cr}$); (c) comparison between (——) the exact wavefront and (- - -) the effective parabolic wavefront at $z = 0.5Z_0$; (d) at $z = 1.0Z_0$; (e) at $z = 1.5Z_0$.

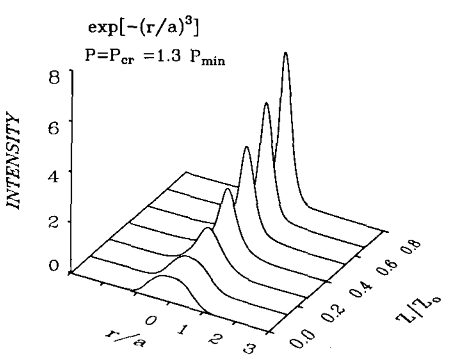
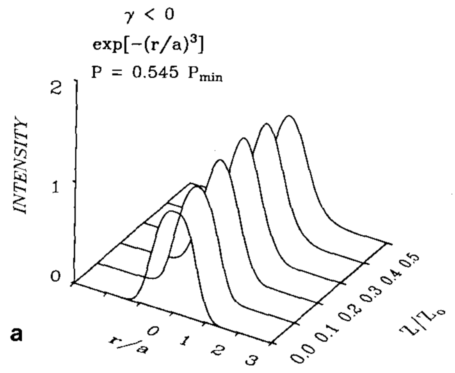
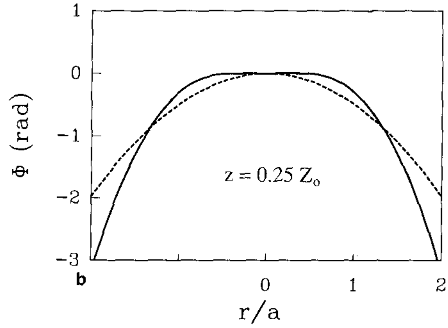


Figure 8 Same as Fig. 7a, but now the power is increased to the critical power P_{cr} of the super-Gaussian: $P = 1.0P_{cr} = 1.30P_{min}$; $W(z)$ is constant but local collapse occurs.





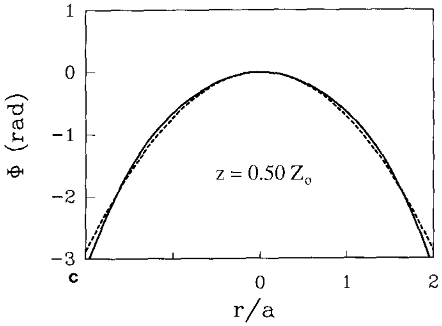
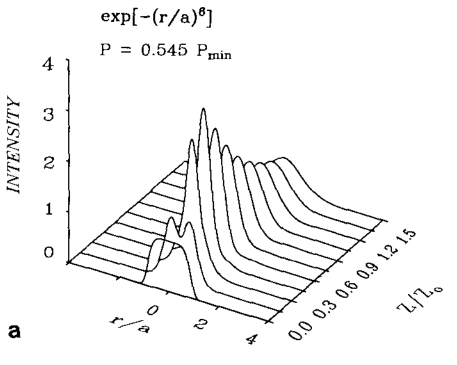
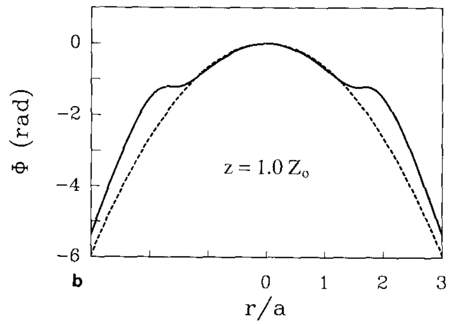


Figure 9 (a) Evolution of the same super-Gaussian as in Fig. 8 but in a defocusing medium at $P = 0.545P_{\mathrm{min}}$; (b) (——) exact phase profile and (- - -) effective wavefront at $z = 0.25Z_0$; (c) at $z = 0.50Z_0$.





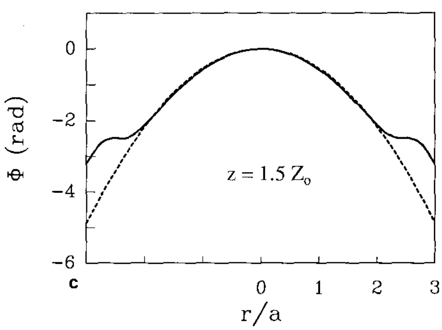
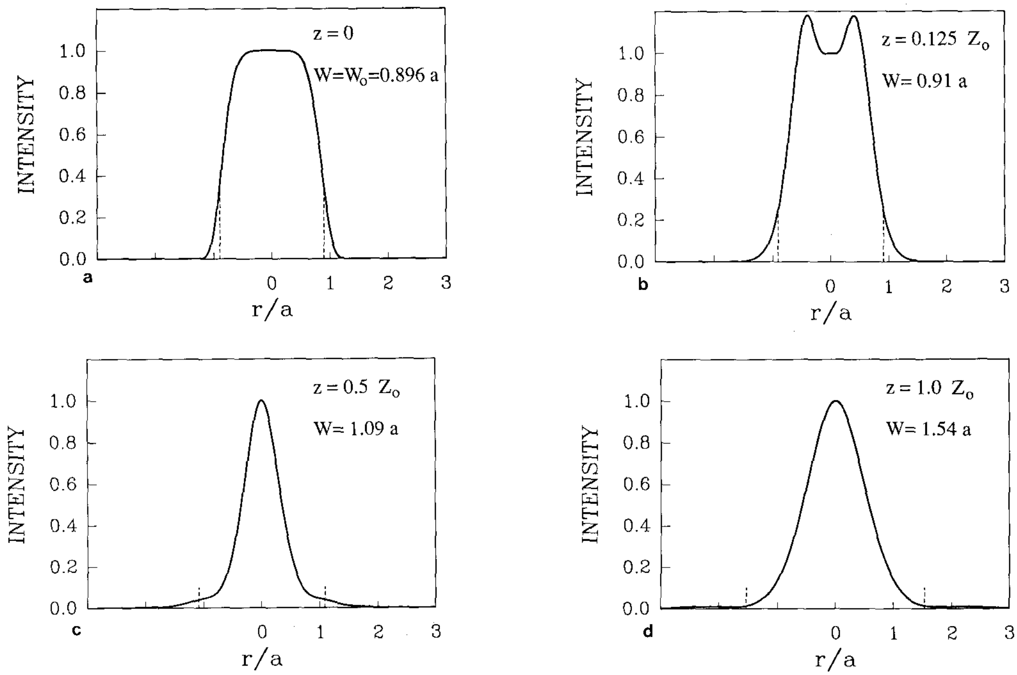


Figure 10 (a) Evolution of the intensity profile of an initially super-Gaussian beam of order 6, i.e. $u(0) = \exp[-(r/a)^6]$, in a focusing medium with the same power as the Gaussian in Fig. 2, i.e. $P = 0.545P_{\mathrm{min}}$ ($=0.264P_{\mathrm{cr}}$); (b) comparison between (——) the exact wavefront and (- - -) the effective parabolic wavefront at $z = 1.0Z_0$; (c) at $z = 1.5Z_0$.



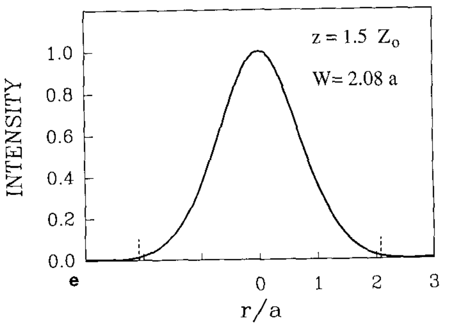


Figure 11 Normalized intensity profiles for the propagation shown in Fig. 10a at various planes: (a) $z = 0$; (b) $z = 0.125Z_0$; (c) $z = 0.50Z_0$; (d) $z = 1.0Z_0$; (e) $z = 1.5Z_0$. Vertical broken lines are drawn at $r = W(z)$, showing that this parameter gives a reliable account of the average spreading of the beam: (a) $W = W_0 = 0.896a$; (b) $W = 0.91a$; (c) $W = 1.09a$; (d) $W = 1.54a$; (e) $W = 2.08a$.

arbitrary beam in a linear or nonlinear medium can be treated in terms of the familiar *ABCD* ray matrices. Considering the widespread utilization of the latter, this certainly adds a pedagogical value to the present results. However, care should be exercised in the application of the matrix formalism for the treatment of cascaded optical systems when one of the elements is nonlinear. It must be borne in mind that the beam quality factor depends on both the power and the beam profile. It is z-invariant in a linear or a nonlinear medium, but does not take the same value in the two media. The variation in the beam quality factor at the entrance and the exit of the nonlinear medium adds another step to the conventional matrix treatment. This represents little work for a Gaussian beam of moderate power. Indeed, as noted in Section 6, the beam remains nearly Gaussian during the propagation, so the beam quality factor is then closely approximated by $(M_Q^2)_{\mathrm{II}} \approx (1 - \eta P/P_{\mathrm{cr}})^{1/2}$ (from Equation 18c) in the nonlinear medium and remains close to unity in the linear medium. The extended matrix analysis is still advantageous and fruitful.

Concerning the beam quality factor, the reader will notice how it naturally appears within the present description of beam propagation. It has been shown that the wavefront should be considered when comparing various beam profiles. The different definitions introduced here all have their own justification, and the use of one or another depends on what is really meant by 'beam quality'. For instance, the fact that the generalized beam quality factor can be smaller than unity (for $\eta > 0$; see Equation 20, for example) does not imply that the nonlinearity 'improves' the 'quality' of a beam beyond the ideal Gaussian beam limit. This comes simply from the generalization of the definition of the beam quality factor. When the beam leaves the nonlinear medium this factor is necessarily increased and greater than or equal to unity. The nonlinearity reduces the rate at which the effective beam width increases in the nonlinear medium, but does not necessarily improve the coherence of the field. The advantage of the generalization is the extension of the *ABCD* matrix approach (Equation 16). Clearly, the latter is invalidated for $P > P_{cr}$, as it implies an imaginary value for the beam quality factor and a purely real Q parameter. The evolution (Equation 10), however, is still valid and it correctly predicts the focusing eventually leading to an on-axis singularity. Near that singularity, however, the paraxial equation is incomplete as mentioned in Section 6. This difficulty is absent below the critical power.

In this paper, in order to simplify the presentation, we have assumed a cylindrical symmetry. However, the generalization is straightforward and terms such as $r\partial u/\partial r$ must then be interpreted as $\boldsymbol{r} \cdot \nabla \boldsymbol{u}$, etc. [7]. It was also implicitly assumed in our analysis that the beam was aligned. Otherwise, the first order moment represents another interesting and physically meaningful parameter to consider, as it describes the trajectory of the 'centre of mass' of the beam [6]. In a lens-like medium, for example, it would predict a periodic crossing of the z-axis.

The parabolic law of propagation, as derived in Section 2 for the cylindrical geometry, is directly related to the second invariant, I_2, of the paraxial wave equation (Equation 1). Of course, the absence of a similar law for the one-dimensional case does not contradict the existence of other invariants for the nonlinear Schrödinger equation [33]. None of these invariants, however, corresponds to the second derivative of the second order moment.

The numerical simulations presented here confirm the interest in defining an effective radius of curvature and an effective beamwaist. In the absence of local collapse, which is the case in practice as one wants to avoid optical damage, these parameters offer a fair description of the gross characteristics of a beam. A detailed comparison between the propagation of Gaussian and super-Gaussian beams in a linear or nonlinear medium is now in progress and will be reported elsewhere. Application of the extended *ABCD* formalism to the problem of a nonlinear optical resonator is also presently being investigated. Finally, we mention that approximate laws of propagation in an active medium have recently been derived [34].

Acknowledgements

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