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WEBER'S PROBLEM AND WEISZFELD'S ALGORITHM IN GENERAL SPACES

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For solving the Euclidean distance Weber problem Weiszfeld proposed an iterative method. This method can also be applied to generalized Weber problems in Banach spaces. Examples for generalized Weber problems are: minimal surfaces with obstacles, Fermat's principle in geometrical optics and brachistochrones with obstacles.

Key words: Weber's Problem, Weiszfeld's Method, Optimal Location.

> ···*Wenn aber schon einmal Theorie getrieben werden soll, (man möchte die Vorliebe dafür angesichts gewisser missglückter Erscheinungen ja allerdings manchmal zum Teufel wünschen) so ist als eine ihrer Formen auch diejenige nötig, die die Abstraktion auf die Spitze treibt.*
>
> Alfred Weber, 1909

1. Introduction

The classical Weber problem of mathematical economics can be formulated as follows: Given points $a_1, \dots, a_n \in \mathbf{R}^d$, find a point $x \in \mathbf{R}^d$ minimizing the functional

$$\sum_{i=1}^{n} w_i \cdot \|x - a_i\|, \tag{WP}$$

where w_i are positive weights and $\|\cdot\|$ is the Euclidean vector norm of $\mathbf{R}^d$. This problem was stated in the context of location theory in the celebrated book of Alfred Weber in 1909 [44] (see also [33]). The same problem was stated as a pure mathematical problem by Fermat (see [32]), Cavalieri (see [38, Kap. IX]), Steiner (see [10, Vol. I, Ch. IV, §1.1.c]), Fasbender [21] and Sturm [40] and many others. In [4] many references to related publications of the 19th century are given.

The problem (WP) can be extended and generalized in many respects in order to adapt it to different practical needs. We give some possibilities for generalization:

— Using any other norm rather than the Euclidean norm [23, 35, 47].
— Replacing the w_i by more general functions of the norm [46].
— Minimizing the weighted sum of square roots of more general quadratic functions [5, 16].
— Minimizing the sum of squared Euclidean distances (gravity problem) [20].
— Imposing constraints on x [12, 16].
— Optimal location of more than one point [24].
— Optimal location with randomly located destinations a_i [30].

Related problems are:

— The convex minimax problem $\min_x \max_i w_i \cdot \|x - a_i\|$ (see [14, 19]).
— The nonconvex maximin problem $\max_x \min_i w_i \cdot \|x - a_i\|$ (Maximin: bishop at Trier in 342) [13, 36].
— The nonconvex problem of geometrically nonlinear frameworks [2, 41, 18].
— Optimal location with integer conditions [6].
— Optimal location on graphs [15, 27, 39].

A detailed investigation of the problems connected with Weber's problem is given in the book of Francis and White [26]. Francis and Goldstein [25] gave an extensive bibliography. The duality theory for Weber's problem is treated in [3], [5], and [31].

2. Generalization of the Euclidean distance Weber problem

Now we are going to generalize the original Weber problem (WP) (cf. [16, 17]). It will become evident later on that there is a large class of problems which can be formulated as generalized Weber problems.

Let X be a reflexive Banach space. For an open set $\Omega \subseteq \mathbf{R}^d$ let $Y = L^1(\Omega)$ and $\mathbf{P}_y = \{y \in Y \mid y \geq 0$ almost everywhere in $\Omega\}$. Let $A : X \times X \to y$ be a symmetric bilinear continuous operator (see [7, §6.1]), $L : X \to Y$ a linear continuous operator, $q_0 \in Y$ fixed and $w : Y \to \mathbf{R}$ a continuous linear functional.

The following conditions are assumed to hold:

(Cl) $(A(x, y))^2 \leq A(x, x) \cdot A(y, y)$ almost everywhere in Ω and for all x and y in X (generalized Schwarz inequality).

(C2) The quadratic operator $q(x) := A(x, x) + L(x) + q_0$ is bounded from below $q(x) \geq \alpha^2$ almost everywhere in Ω for all $x \in X$, where $\alpha > 0$.

(C3) The continuous linear functional w is represented by a function in $L^\infty(\Omega)$ which is denoted by $w(t)$. It is assumed that there is a fixed positive constant w_0 such that $0 < w_0 \leq w(t)$ almost everywhere in Ω.

(C4) The quadratic term of q is norm-coercive, i.e. there is a positive constant

$$m \cdot \|x\|^2 \leq w(A(x, x)) \quad \text{for all } x \in X$$

($\|\cdot\|$ is the norm in X). This means that we have a classical variational problem in X which is split into two pieces, namely w and q.

Now we define another operator $\sigma : \mathbf{P}_Y \to \mathbf{P}_Y$ which is nonlinear and is placed between w and q. σ is defined by

$$\sigma(y(t)) = \sqrt{y(t)} \quad \text{for almost all } t \in \Omega.$$

Condition (C2) implies that Ω has finite measure, thus σ is well-defined.

For a fixed nonempty closed convex set $K \subseteq X$ we state the generalized Weber problem

$$\begin{array}{l}\text{Find } x \in K \text{ such that} \\ \varphi(x) := w(\sigma(q(x))) \text{ is minimal.}\end{array} \tag{GWP}$$

(C4) yields $\varphi(\mu \cdot x) \geq w_0 \cdot |\mu| \cdot \sqrt{m}$ for $\|x\| = 1$, consequently for fixed $x_0 \in K$ the set

$$\{x \in K \mid \varphi(x) \leq \varphi(x_0)\} \tag{*}$$

is bounded and closed and thus weakly compact. (C1), (C3) and the nonnegativity of $q(x)$ together imply $\varphi(x)$ being a convex functional and thus this set is also convex. We conclude that (GWP) has a solution ([11, Corollary 1.4.1]). By (C2) we can even prove strict convexity of $\varphi(x)$ so that there is only one solution of (GWP) (see [11, Theorem 1.5.2]).

3. The algorithm of Weiszfeld

In 1937 Weiszfeld (A. Vazsonyi) [45] proposed a remarkable algorithm for solving (WP). This algorithm was generalized in [16]. We now attempt to generalize it to (GWP).

For any given $y \in Y$ satisfying $y \geq \alpha$ a.e. in Ω (see (C2) for definition of α) we state the auxiliary problem

$$\begin{array}{l}\text{Find } x \in K \text{ such that} \\ w(q(x)/y) \text{ is minimal.}\end{array} \tag{$A(y)$}$$

The auxiliary problem means the minimization of a quadratic functional over K which can be regarded as a standard problem in numerical analysis.

The generalized algorithm of Weiszfeld is

Step 0: Start with $y_0 \in Y$ satisfying $y_0 \geq \alpha$ a.e. in Ω, put $r := 0$.
Step 1: Let x_{r+1} be a solution of $(A(y_r))$.
Step 2: Put $y_{r+1} = \sigma(q(x_{r+1}))$.
Step 3: $r := r + 1$, go to Step 1.

When $(A(y_r))$ has always a solution, the algorithm produces an infinite sequence of functions x_r. It is not difficult to prove the existence of a solution to $(A(y_r))$ but this will become clear anyway later on (Corollary 1). (C2) implies $y_r \geq \alpha$ a.e. in Ω for all r.

In order to simplify the argumentation we reformulate the problem as follows. Define in $Z = L^2(\Omega)$

$$K_0 = \{\eta \in Z \mid \eta \geq \sigma(q(x)) \text{ a.e. in } \Omega \text{ for some } x \in K\}.$$

Since K is a nonempty convex set and $\sigma(q(x))$ is a convex function, K_0 is a nonempty convex set. We restate the problems (GWP) and (A(y)):

$$\text{Find } \eta \in K_0 \text{ such that } w(\eta) \text{ is minimal} \tag{RGWP}$$

and

$$\text{Find } \eta \in K_0 \text{ such that } w(\eta^2/y) \text{ is minimal.} \tag{RA(y)}$$

It is easily seen that (GWP) and (RGWP) as well as $(A(y))$ and $(RA(y))$ are equivalent in the sense that for each solution x of one of the original problems $\eta = \sigma(q(x))$ solves the restated problem. To each solution η of one of the restated problems there exists an $x \in K$ such that $\eta \geq \sigma(q(x))$ which solves the corresponding original problem.

The first theorem states the monotonicity of Weiszfeld's method:

Theorem 1. $\varphi(x_{r+1}) \leq \varphi(x_r)$.

Proof (see Weiszfeld [45]).

$$y_{r+1}^2 \geq 2 \cdot y_r \cdot y_{r+1} - y_r^2 \Rightarrow w(y_{r+1}^2/y_r) \geq 2 \cdot w(y_{r+1}) - w(y_r).$$

Since $w(y_{r+1}^2/y_r) \leq w(y_r)$ is

$$\varphi(x_{r+1}) = w(y_{r+1}) \leq w(y_r) = \varphi(x_r).$$

The set $\{x \mid w(q(x)/y_r) \leq w(q(x_r)/y_r)\}$ is nonempty, bounded, closed and convex, hence by [11, Corollary 1.4.1] we have

Corollary 1. *All problems* $(A(y_r))$ *have a solution.*

The following theorem states the main convergence result:

Theorem 2. *Let x^* be a solution of* (GWP). *Then there exists a number $\gamma > 0$ such that*

$$0 \leq \varphi(x_r) - \varphi(x^*) \leq \gamma \cdot \sqrt{(\varphi(x_r))^2 - (\varphi(x_{r+1}))^2}.$$

Proof. Let $y = y_r$ and $y^* = \sigma(q(x^*))$. Define the linear transformation $T: Z \to Z$

by $T\eta = \eta \cdot \sqrt{w}/\sqrt{y}$. We put $y(\mu) = Ty + \mu \cdot (Ty - Ty^*)$ and denote by μ_0 the parameter value which minimizes $\|y(\mu)\|$ for $0 \le \mu \le 1$. By convexity is $y(\mu_0) \in TK_0$. Now we maximize $w(\eta)$ on $M_0 = \{\eta \mid \|T\eta\|^2 \le \|y(\mu_0)\|^2\}$ in order to find an upper bound for w. This maximization can be performed explicitly since M_0 is a sphere in TY. We get $w(y) \le \|\sqrt{w \cdot y}\| \cdot \|y(\mu_0)\|$ for all $\eta \in M_0$. If y_1 is a solution of (RA(y)), then $y_1 \in M_o$ since $w(\eta^2/y) = \|T\eta\|^2$. As we know an upper bound for $w(\eta)$ on M_0 we have

$$(w(y_1))^2 \le \|\sqrt{w \cdot y}\|^2 \cdot \|y(\mu_0)\|^2 = w(y) \cdot \|y(\mu_0)\|^2.$$

An elementary calculation yields ($\langle \cdot, \cdot \rangle$ scalar product in Z)

$$\|y(\mu_0)\|^2 = \|Ty\|^2 - \frac{\langle Ty, Ty - Ty^* \rangle^2}{\|Ty - Ty^*\|^2}$$

Using $\|Ty\|^2 = w(y)$ and $\langle Ty, Ty^* \rangle = w(y^*)$ we get

$$w(y) - w(y^*) \le \frac{\|Ty - Ty^*\|}{\sqrt{w(y)}} \cdot \sqrt{(w(y))^2 - (w(y_1))^2}.$$

Since $\|Ty - Ty^*\|^2 = w(y) - 2 \cdot w(y^*) + \|Ty^*\|^2$ we get the estimate of the theorem where the constant γ depends on an upper bound and a positive lower bound for $w(y)$ an K_0.

Theorems 1 and 2 together imply convergence of $\varphi(x_r)$ to $\varphi(x^*)$ which implies weak convergence of x_r to x^*.

Theorem 3. $x_r \to x^*$ *(strongly)*.

Proof. The second Fréchet derivative $\varphi_x^{\|}(h, h)$ of φ can be easily calculated. By (C1), (C2) and (C4) we get

$$\varphi_x^{\|}(h, h) = \|h\|^2 \cdot \delta(x),$$

where $\delta(x)$ is bounded from below on each bounded set. Hence, by [11, Theorem 1.6.3], lim $x_r = x^*$.

4. Discussion of the assumptions

In order to adapt the generalized Weber problem to different practical situations, the basic assumptions (C1–4) have to be modified.

In the original Weber problem (WP) the set Ω consists of isolated single points $\Omega = \{a_1, \ldots, a_n\}$. In this case is $Y = \mathbf{R}^n$ equipped with the L^1 vector norm. Similarly Y can be adapted if Ω has a more complicated structure, e.g. the union of an open set with isolated points and arcs (see Example 4 below).

In many applications is w not a continuous linear functional on the whole space Y. It is only necessary for w to be continuous on the set $\{\sigma(q(x)) \mid x \in K\}$ (see Examples 2 and 4).

The most serious assumption is (C2). If there is an $x_0 \in X$ such that $A(x_0, x_0) > 0$ almost everywhere in Ω, then $A(x, x) \geq 0$ almost everywhere in Ω for all $x \in X$ by (C1). If, in addition, L is the zero operator and $q_0 \geq \alpha^2$ almost everywhere, then (C2) is true [17] (see Examples 1 and 3).

If $\alpha = 0$, then the solution of (GWP) is not necessarily unique (see [45] for examples). This implies that Theorem 3 is no longer true, one can only guarantee that each accumulation point of $\{x_r\}$ is a solution of (GWP). Moreover, one has to find additional conditions to make sure that the problems (A(y_r)) make sense. For the original Weber problem (WP) $\alpha = 0$ (see Example 4).

5. Applications

We consider some illustrative examples.

Example 1. As a simple problem having also practical relevance we consider the classical Fermat principle of geometrical optics: Given in the atmosphere a refraction index $n(h)$ changing with height h. We want to find the path of light emitted from A and going to B (see Fig. 1). This path can be found by

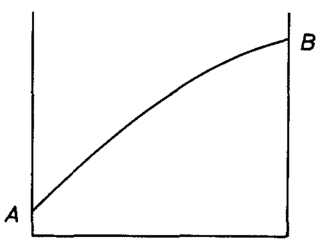

Fig. 1. Fermat's principle.

minimizing the integral

$$\int_0^1 n(h) \cdot \sqrt{1 + (\mathrm{d}h/\mathrm{d}x)^2}\, \mathrm{d}x.$$

Changing the variables yields

$$\int_A^B n(h) \cdot \sqrt{(\mathrm{d}x/\mathrm{d}h)^2 + 1}\, \mathrm{d}h \to \text{Minimum}$$

which is a generalized Weber problem. We take $\Omega = (A, B)$, X the Sobolev space $W^{1,2}(\Omega)$ of all those functions whose first generalized derivative is square integrable,

$$K = \{x \in X \mid x(A) = 0, x(B) = 1\}, \quad q(x) = (\mathrm{d}x/\mathrm{d}h)^2 + 1, \quad w(h) = n(h).$$

The problem is of interest in the study of the behaviour of radar waves in the atmosphere and in computations related to the transmission of signals through optical fibers [43].

Example 2. A quite similiar problem is given by the brachistochrone with "obstacle" as formulated in [34] (see Fig. 2). Here one wants to find a path from

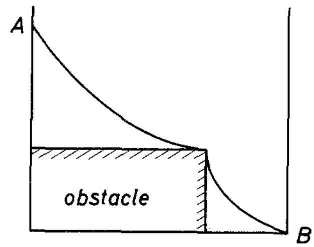

Fig. 2. Brachistochrone with obstacle.

A to B which is traversed under the influence of gravity (constant g) in minimal time with the additional restriction that an obstacle is avoided by the path. The problem is to minimize

$$\int_0^1 \left[\frac{1+(\mathrm{d}y/\mathrm{d}x)^2}{2gy}\right]^{1/2} \mathrm{d}x.$$

Change of variables again results in a generalized Weber problem which differs from the problem described above mainly in K being the set of all functions fulfilling the boundary conditions and avoiding the obstacle. (The problem might be of interest in designing body-slides.)

Example 3. Another interesting problem is the minimal surface problem. Since this problem is a very aesthetic one and since it is a good test problem for methods solving nonlinear boundary value problems, a vast amount of literature exists. For the aesthetical point of view the reader is referred to [1]. Minimal surface problems with obstacles are treated e.g. by Nitsche [37] and Titov [42]. Many articles are devoted to the numerical solution of minimal surface problems (see e.g. [8] and [28]). A closely related problem is the capillarity problem [22]. The author has performed numerical tests in applying Weiszfeld's method to discretized minimal surface problems with obstacles [16]. In Fig. 3 the solution of the following example problem is shown:

$$\text{Minimize} \int_{-1}^{+1}\int_{-1}^{+1} \sqrt{1+(\mathrm{d}u/\mathrm{d}x)^2+(\mathrm{d}u/\mathrm{d}y)^2}\,\mathrm{d}y\,\mathrm{d}x,$$

$$\text{subject to } u=0 \quad \text{for } y=\pm 1, -1\le x\le 1,$$

$$u=\sqrt{1+y^2} \qquad \text{for } x=\pm 1, -1\le y\le 1,$$

$$u\ge\sqrt{1-x^2-y^2} \qquad \text{for } x^2+y^2\le 1.$$

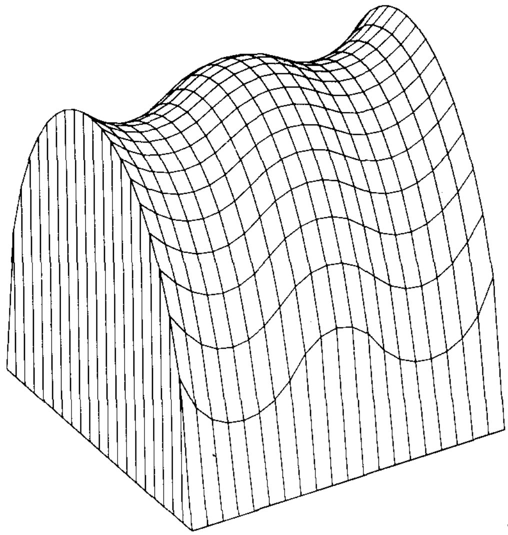

Fig. 3. Minimal surface with obstacle.

Example 4. Returning to the original Weber problem, we consider the task of locating a service center (e.g. a fire station or a heliocopter station etc.) in a contiuous environment. Let $\Omega \subseteq \mathbf{R}^d$ ($d = 2$ or 3) be a bounded open set. For each $\xi \in \Omega$ we denote by $w(\xi)$ the probability that service is needed in ξ. If the service station is located at the point x, the mean distance to the point where service is needed is given by

$$\varphi(x) = \int_\Omega w(\xi) \cdot \|x - \xi\| \, \mathrm{d}\xi.$$

Here is $X = \mathbf{R}^d$ and $q(x) = \sum_{j=1}^{d} (x_j - \xi_j)^2$. In this problem $\alpha = 0$ as in the original Weber problem. The functional to be minimized in $(A(y_r))$ is

$$\psi(x) = \int_\Omega w(\xi) \cdot \|x - \xi\|^2 / \|x_r - \xi\| \, \mathrm{d}\xi.$$

In $\mathbf{R}^2$ and $\mathbf{R}^3$ is $1/\|x_r - \xi\|$ integrable. Since all other terms in the integral are bounded in Ω, the integral exists and $\psi(x)$ is a continuous function.

6. Numerical considerations

We add some remarks on the numerical properties of Weiszfeld's algorithm. Katz [29] showed that its convergence is linear when applied to (WP) if the solution does not coincide with a destination a_j. Katz's proof cannot be extended to (GWP) and, especially in the presence of constraints, one cannot expect more than linear convergence. Application of the method is, therefore, adviceable if the following three conditions are met:

— The solution of (GWP) is only needed with moderate accuracy. Usually in practical problems of the type described above, an accuracy of some percent is sufficient.
— An efficient computer program for minimizing the quadratic functional in (A(y)) is readily available. Note that minimizing a quadratic functional is equivalent to solving a linear operator equation.
— A good starting solution is known. It should be mentioned here, that there is a close relationship between the minimal surface problem (formulated as (GWP)) and Dirichlet's problem ((A(1)) in this context) (cf. [9]). This relationship is expressed by the fact that the solution of (A(y*)) also solves (GWP). The solution of (A(1)) can therefore be used as a reasonable starting solution for Weiszfeld's method.

In [16] a more detailed discussion of the problem of minimal surfaces with obstacles is given.

A related problem is the computation of the static equilibrium of geometrically nonlinear networks in structural mechanics. An impressive example is the Olympic tent in München. Such problems can be solved numerically by a method similar to Weiszfeld's [18].

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