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ADDENDUM★

ON THE WIDTH–LENGTH INEQUALITY*†

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This paper was written at the Rensselaer Polytechnic Institute in 1963. A lecture based on it was given at the Rand Corporation in 1965 and this version is in the form in which Ray Fulkerson received it at Rand.

The paper is the underpinning for results on resistor network inequalities (Reference [4]) which has not been published. A specific example, however, appears in *Proceedings of the IEEE* 51 (1963) 1047–1048. There is also a parallel theory of the abstract assignment problem; every W–L matrix being an assignment matrix.

More is known about minimal non-W–L matrices. U. Peled has found several additional classes of matrices which are also classically known in other contexts. A consequence of a recent matrix theorem is apparently that the degenerate projective planes are the only minimal matrices requiring unequal weights.

Key words: Networks, Graphs, Resistor Networks, Max-flow Min-Cut.

1. Introduction

Consider a finite connected graph in which two of the vertices are distinguished. A *path* of the graph is a set of edges which connect the two vertices. A *cut* is a set of edges which has at least one edge in common with each path. Hence the deletion of a set of edges which constitutes a cut disconnects (cuts) the two vertices. Consider the number of edges in each path and in each cut. The smallest of the path numbers is called the *length* of the graph and the smallest of the cut numbers is called the *width*. Moore and Shannon [5] have shown that the product of the length and width cannot exceed the total number of edges of the graph. Their proof uses the cost algorithm [3, pp. 130–134] to construct a set of disjoint cuts—one for each member of the shortest path connecting the vertices. The existence of such a partition implies the inequality.

★Due to unfortunate circumstances this article was originally published (*Mathematical Programming* 16(2)(1979) 245–259) without the proof corrections. The *correct* version is printed herewith. Please use this version when referring to the paper.

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† *Editor's Note*: This paper, although written in 1963, was sought for inclusion in *Mathematical Programming Study* 8—Polyhedral Combinatorics, which was dedicated to the memory of D.R. Fulkerson. Unfortunately, an editorial mishap prevented its inclusion. Nevertheless, the historical importance of the paper, the fact that it has been widely referenced and influenced Fulkerson's and others' work in the area has convinced the Editor that it should be published and hence made readily available. Alfred Lehman has given his assent to this despite his reluctance to publish a paper which is not current.

More generally, suppose that a pair of non-negative numbers, called the edge-length and edge-width, is given for each edge of the graph. Consider the sum of edge-lengths corresponding to each path and the sum of edge-widths corresponding to each cut. The smallest of the path sums is called the *length* of the graph and the smallest of the cut sums is called the *width*. This definition includes the Moore–Shannon definition as a special case (edge length and width both equal to 1). The product of this length and width cannot exceed the product of the length and width of each edge, summed over the edges of the graph. Duffin [1, Theorem 6] has shown this result using an argument based on Kirchoff's laws. It can also be shown by a modification of the cost algorithm decomposition used by Moore and Shannon.

Now consider not a graph but a finite abstract set and a non-null collection of non-null subsets of this set. The members of the set are called *edges* and the distinguished subsets are called *paths*. A *cut* is again a collection of edges which intersects each path. Given a pair of non-negative numbers associated with each edge, the *length* and *width* are defined as before to be the smallest path and cut sums. If, for every possible assignment of edge weights, the product of the width and length does not exceed the product of the edge width and length summed over all the edges, then the system of paths is said to satisfy the *width–length inequality*. For example, suppose that there are three edges and that each pair of edges constitutes a path. Then for edge width and length equal to 1 the system has width and length equal to 2. Since there are only three edges the width–length inequality fails. The problem is to determine, without assigning various edge weights, whether a given collection of paths satisfies the width–length inequality.

The collection of paths can also be given as an incidence matrix. Assume that the edges and paths are simply ordered. The entry in row i and column j is 1 if the jth path contains the ith edge. Otherwise the entry is 0. A criterion for the width–length inequality will be given in terms of the non-singular submatrices of this matrix. For this reason the paper will employ matrix notation.

2. W–L Matrices

Let M be a given matrix whose entries are limited to the integers 0 and 1. It is also assumed that the integer 1 occurs in every column. Suppose that A is any non-singular (square) submatrix. A was obtained from M by deleting certain rows and columns. Let A^* be the submatrix of M obtained by deleting the same rows and the complementary columns. A^* (but not A) is allowed to be empty. Thus each row of M is either deleted or is split between A and A^*. By a permutation of the rows and columns of M, A can be placed in the lower left hand corner of M. Then the relation between M, A, and A^* is displayed by

$$M \text{ (permuted)} = \left(\frac{\qquad}{A \mid A^*}\right).$$

It will suffice to consider only those submatrices A which are at least 3 by 3.

e and e^* will denote row matrices whose entries are all 1's and whose length is the number of rows and columns of A^* respectively.

Definition. M is said to be a *W–L matrix* if
for each non-singular submatrix A such that

$$eA^{-1} > 0 \qquad \text{(i.e. every entry of } eA^{-1} \text{ is positive)}$$

and

$$eA^{-1}A^* \geq e^* \quad \text{(i.e. every entry of } eA^{-1}A^* \text{ is at least 1)}$$

hold,
then A is a permutation matrix.

(A permutation matrix has a single 1 in each row and column. All other entries are 0's. Equivalently, $eA = e$.)

Suppose that M contains two columns such that for every entry 1 in the first column, there is a corresponding entry 1 in the second column. Then M is a W–L matrix if and only if the submatrix obtained by deleting the second column is a W–L matrix. The proof is as follows. Consider the submatrix A. Since A is non-singular and eA^{-1} has positive entries, at most one of the two columns enters into A. If the two columns agree on the rows constituting A and A^*, the deletion of one of them from A^* will not affect the validity of $eA^{-1}A^* \geq e^*$. Otherwise it must be the second column which enters into A^* and its deletion also does not affect the validity of $eA^{-1}A^* \geq e^*$.

Thus in determining whether M is a W–L matrix it suffices to consider the reduced matrix consisting of those columns which contain a minimal set of 1's. Also any duplicate columns can be deleted. All of the examples given in this paper are already reduced. Finally it should be noted that any permutation of the rows and columns of a W–L matrix is also a W–L matrix.

If M is unimodular (that is, every square submatrix has determinant 0, +1, or −1) then it is a W–L matrix. This is the case since unimodularity implies that A^{-1} has integer entries; eA^{-1} has positive entries and hence A can have at most a single 1 in each column. Since A is non-singular it must be a permutation. (This observation for the width–length inequality was made by Duffin and Hoffman [2].)

It can be verified that the following two matrices, each of rank 4, are W–L matrices, but that their transposes are not. Consequently neither matrix is unimodular.

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \left(\begin{array}{ccccc} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ \cline{1-3} 1 & 1 & \multicolumn{1}{c|}{0} & 0 & 1 \\ 1 & 0 & \multicolumn{1}{c|}{1} & 1 & 0 \\ 0 & 1 & \multicolumn{1}{c|}{1} & 0 & 1 \end{array}\right)$$

If one or more rows are deleted from a W–L matrix and if the resultant matrix has a 1 in each column then it is also a W–L matrix. The corresponding property does not hold for columns: Consider the 3 by 3 submatrix A in the lower left-hand corner of the second matrix. $eA^{-1} > 0$ holds but $eA^{-1}A^* \geq e^*$ does not and A is not a permutation matrix. Thus by definition the submatrix consisting of the first three columns is not a W–L matrix. Furthermore this example shows that the condition $eA^{-1}A^* \geq e^*$ is a necessary part of the definition of a W–L matrix.

$e^{\#}$ and $e^{\&}$ will denote row matrices whose entries are all 1's and whose length is the number of rows and columns of M respectively.

Given M, let l and w be row and column matrices whose entries are non-negative real numbers and whose length (number of columns and rows respectively) is the number of rows of M. c and p denote row and column matrices whose entries are limited to 0's and 1's and whose length also is the number of rows of M. Furthermore c is to satisfy $cM \geq e^{\&}$ and p is to be a column of M. Since M has no zero columns, at least one c, namely $c = e^{\#}$, always exists.

Definition. M is said to satisfy the *width–length inequality* if for each pair of matrices l and w,

$$\left(\min_p lp\right)\left(\min_c cw\right) \leq lw$$

holds.

In the terminology of the introduction, p and c are paths and cuts, l and w are the edge lengths and widths, and $\min_p lp$ and $\min_c cw$ are the length and width of M. The basic result of this section is that M is a W–L matrix if and only if it satisfies the width–length inequality. This is shown, in circular fashion, by the following three lemmas.

Lemma 1. *If M satisfies the width–length inequality, then it must be a W–L matrix.*

Proof. Given A, where $eA^{-1} > 0$ and $eA^{-1}A^* \geq e^*$ hold, let l have the entries of eA^{-1} in those positions corresponding to the rows of M which are in A. The remaining entries in l are 0's. Thus $\min_p lp = 1$.

Let w have the entries of Ae^{T} (T denotes transposition) in those positions corresponding to the rows of M which are in A. The remaining entries in w can be any real numbers greater than ee^{T}. By the width–length inequality, $\min_c cw \leq lw = ee^{\mathrm{T}}$ and hence there exists c_0 such that $c_0 w \leq ee^{\mathrm{T}}$. From the construction of w it follows that c_0 can have non-zero entries only in those positions corresponding to rows of M which are in A. Furthermore, since $c_0 M \geq e^{\&}$ and

$c_0 w \leq ee^{\mathrm{T}}$ hold, $c_0 M$ must have 1's in those positions corresponding to the columns of M which are in A. Consequently eA^{-1} is a submatrix of c_0. Since eA^{-1} is positive, $eA^{-1} = e$, $eA = e$, and hence A is a permutation matrix.

Definition. M has the *cut property* if
for any matrix l such that $\min_p lp = 1$ there exists a matrix c such that

c has 0's in every position in which l has 0's and
for any p satisfying $lp = 1$, then $cp = 1$.

Clearly the same property results if $\min_p lp = 1$ and $lp = 1$ are replaced by $\min_p lp = k$ and $lp = k$ where k is any positive constant. This modified definition is used in the proof of Lemma 3.

Lemma 2. *If M is a W–L matrix, then it has the cut property.*

Proof. Suppose l is given and $\min_p lp = 1$. Consider all l_0's such that $l_0 p = 1$ for all p such that $lp = 1$ and l_0 has 0's in all positions in which l has 0's.

Among these l_0's chose one (l_{00}) for which the set of p's such that $l_0 p = 1$ and the set of 0 entries are jointly maximal (i.e., neither can be increased without decreasing the other.) Suppose l^0 also has this maximal set of 0's and (tight) p's. (It is *not* required that $\min_p l^0 p = 1$ hold.) For a sufficiently small positive number ϵ, $(1+\epsilon)l_{00} - \epsilon l^0$ also has the properties of l_{00}. If $l^0 \neq l_{00}$, then ϵ can be increased until either $((1+\epsilon)l_{00} - \epsilon l^0)p = 1$ holds for some additional p or $(1+\epsilon)l_{00} - \epsilon l^0$ has an additional 0 entry. Since neither is possible, l^0 is identical with l_{00}.

Now construct A as follows. Delete those rows of M which correspond to 0 column entries in l_{00} and delete those columns p for which $l_{00} p > 1$. By the uniqueness argument concerning l_{00}, a subset (possibly all) of the remaining (truncated) columns from a non-singular square matrix A. Since M is a W–L matrix and $eA^{-1} > 0$ and $eA^{-1}A^* \geq e^*$ hold, A is a permutation matrix and hence $eA^{-1} = e$. Equivalently the non-zero entries of l_{00} are all 1's. Thus $c = l_{00}$ is the required cut.

Lemma 3. *If M has the cut property, then it also satisfies the width–length inequality.*

Proof. Let l be given. If $\min_p lp = 0$, then the inequality is satisfied. Otherwise, there exists c_1 given by the cut property. For sufficiently small positive ϵ, $l - \epsilon c_1$ has non-negative entries and

$$\min_p lp = \min_p (l - \epsilon c_1)p + \min_p \epsilon c_1 p = \min_p (l - \epsilon c_1)p + \epsilon.$$

Consequently, for any w,

$$lw - \left(\min_p lp\right)\left(\min_c cw\right) = (l - \epsilon c_1)w - \left(\min_p (l - \epsilon c_1)p\right)\left(\min_c cw\right) + \epsilon\left(c_1 w - \min_c cw\right)$$
$$\geq (l - \epsilon c_1)w - \left(\min_p (l - \epsilon c_1)p\right)\left(\min_c cw\right).$$

Thus the validity of the width–length inequality for $l - \epsilon c_1$ implies the validity of the inequality for l. Furthermore, for any p_0 such that $lp_0 = \min_p lp$ holds,

$$(l - \epsilon c_1)p_0 = \min_p lp - \epsilon c_1 p_0 = \min_p lp - \epsilon = \min_p (l - \epsilon c_1)p.$$

ϵ can be increased until either some additional entry of $l - \epsilon c_1$ is 0 or some p_0 such that $lp_0 > \min_p lp$ satisfies $(l - \epsilon c_1)p_0 = \min_p (l - \epsilon c_1)p$. Let this maximum ϵ be denoted by ϵ_1. If $\min_p (l - \epsilon_1 c_1)p = 0$, then the inequality holds. Otherwise, application of the cut property to $l - \epsilon_1 c_1$ yields c_2, ϵ_2 and so on. Eventually, since M is finite, the process must terminate in $l - \sum_i \epsilon_i c_i$ where $\min_p (l - \sum_i \epsilon_i c_i)p = 0$ and consequently

$$lw - \left(\min_p lp\right)\left(\min_c cw\right) \geq \left(l - \sum_i \epsilon_i c_i\right) w \geq 0.$$

It should be noted that the c_i's and ϵ_i's are independent of w.

This concludes the proof of the equivalence of the width–length and W–L properties.

An examination of the proof of Lemma 3 shows that if M has the cut property, then for any l there exist c_i's and (non-negative) ϵ_i's such that $\sum_i \epsilon_i c_i \leq l$ and $\sum_i \epsilon_i \geq \min_p lp$. Moreover the existence of these c_i's and ϵ_i's imply that

$$lw - \left(\min_p lp\right)\left(\min_c cw\right) \geq \sum_i \epsilon_i c_i w - \sum_i \epsilon_i \left(\min_c cw\right)$$
$$= \sum_i \epsilon_i \left(c_i w - \min_c cw\right) \geq 0.$$

Also, since $c_i M \geq e^{\&}$, $\sum_i \epsilon_i e^{\&} \leq \sum_i \epsilon_i c_i M \leq lM$ holds and hence $\sum_i \epsilon_i \leq \min_p lp$. The above together with the equivalence of the W–L, width–length and cut properties yields the following result.

Lemma 4. *M is a W–L matrix if and only if given l, there exist $c_1, \ldots, c_n$ together with non-negative numbers $\epsilon_1, \ldots, \epsilon_n$ such that $\sum_i \epsilon_i c_i \leq l$ and $\sum_i \epsilon_i = \min_p lp$.*

This is the analogue, for path collections, of the max-potential min-work theorem given in [1, p. 207].

Given M, let $M^{\#}$ be the matrix whose columns are the matrices c^{T} satisfying $cM \geq e^{\&}$. The columns of $M^{\#}$ may be arranged in any order.

Lemma 5. *$M^{\#}$ is a W–L matrix if and only if M is a W–L matrix.*

Proof. First note that $p^{\mathrm{T}}c^{\mathrm{T}} = (cp)^{\mathrm{T}} \geq 1$ holds for all p's and c's. Thus each p^{T} is a "cut" ($p^{\mathrm{T}}M^{\#} \geq e^{\&\#}$) with respect to the edge-path matrix $M^{\#}$. Furthermore for any cut with respect to $M^{\#}$ there is a column p of M such that the cut has 1's in every position in which p^{T} has 1's. This is a consequence of the duality of Boolean functions. Given l and w, the width–length inequality for $M^{\#}$ assumes the form $(\min_c lc^{\mathrm{T}})(\min_p p^{\mathrm{T}}w) \leq lw$. This is equivalent to $(\min_p w^{\mathrm{T}}p)(\min_c cl^{\mathrm{T}}) \leq w^{\mathrm{T}}l^{\mathrm{T}}$ which is the width–length inequality for w^{T}, l^{T}, and M. Thus $M^{\#}$ satisfies the width–length inequality if and only if it is satisfied by M.

$M^{\#}$, which is the dual of M, will be used in the proof of Lemma 6. The use of $M^{\#}$ instead of M essentially interchanges the paths and cuts.

Let s be a column matrix whose entries are non-negative numbers and whose length is the number of columns of M. In addition, s is required to satisfy the inequality $Ms \leq w$.

Definition. M is said to satisfy the *max-flow min-cut equality* if for each matrix w,

$$\max_s e^{\&}s = \min_c cw$$

holds.

(This formulation of the max-flow min-cut theorem is due to Duffin.)

By definition, $cM \geq e^{\&}$ and $Ms \leq w$ hold. Hence $e^{\&}s \leq cMs \leq cw$ and consequently $\max_s e^{\&}s \leq \min_c cw$ is automatic. The max-flow min-cut property thus asserts, for each w, the existence of s such that $e^{\&}s \geq \min_c cw$. In the terminology of the introduction s ascribes a "flow" to each path such that the sum of flows corresponding to a given edge does not exceed the width (given by w) of that edge, and the sum of all flows is the width ($\min_c cw$) of the collection of paths given by M. When discussing flows, the term "capacity" is customarily used in place of "width". A detailed treatment of the max-flow min-cut property for graphs can be found in [3, Chapter I].

A subtle but important distinction should be made between the usual concept of flow in a network and the one used in this paper. In a network, flows (such as currents) admit superposition. This requires that the flows associated with each edge of a graph have both a magnitude and an orientation (direction). For the flows considered here there is a magnitude but no orientation and, in general, no orientation is possible. Superposition of flow magnitudes is subject only to a triangule inequality. Nevertheless, in the problem of maximizing the flow between two vertices of a graph, the magnitudes of the oriented flows agree with the ones given here. The reason is that, by superposition, any assignment of oriented path flows can be transformed into an assignment where all flows corresponding to any edge have the same orientation. This property is exploited in [4] in the derivation of a system of inequalities for resistor networks.

For the reasons just given, the following result generalizes only the *unoriented* version of the usual max-flow min-cut theorem [3, p. 11].

Lemma 6. *M satisfies the max-flow min-cut equality if and only if it is a W–L matrix.*

Proof. Since the c_i's in Lemma 4 need only satisfy $\sum_i \epsilon_i c_i \leq l$, they can be chosen from those c's having a maximal set of 0 entries. Hence in applying Lemma 4 to $M^{\#}$ the c_i's can be replaced by p^{T}'s. Thus $M^{\#}$ is a W–L matrix if and only if, given w^{T}, there exist p_i^{T}'s and ϵ_i's such that $\sum_i \epsilon_i p_i^{\mathrm{T}} \leq w^{\mathrm{T}}$ and $\sum_i \epsilon_i = \min_c w^{\mathrm{T}} c^{\mathrm{T}}$; and hence if and only if, given w, there exists s (whose non-zero components are the ϵ_i's) such that $Ms \leq w$ and $\max_s e^{\&} s = \min_c cw$. The result now follows from Lemma 5.

Lemmas 1, 2, 3 and 6 are summarized in the following theorem. Note that the only property which does not involve either l or w is that of being a W–L matrix.

Theorem 1. *The following four assertions are equivalent:*

(i) *M is a W–L matrix,*
(ii) *M satisfies the width–length inequality,*
(iii) *M has the cut property,*
(iv) *M satisfies the max-flow min-cut equality.*

3. Graphs

In this section it is shown that the path collection of any finite directed or undirected graph satisfies the width–length inequality. The proof given here uses a weakened form of the cut property.

Definition. M has the *weak cut property* if for any matrix l such that $\min_p lp = 1$ there exists a matrix l^* such that

the entries of l^* are 0's and 1's,
l^* has 0's in every position in which l has 0's, and
for any p satisfying $lp = 1$, then $l^*p = 1$.

(Note that l^* need not be a cut.)

Lemma 7. *M has the weak cut property if and only if it has the cut property.*

Proof. Since the cut c given by the cut property satisfies the requirements on l^*, the cut property implies the weak cut property. Now assume the validity of the weak cut property and consider A such that $eA^{-1} > 0$ and $eA^{-1}A^* \geq e^*$. Let l be defined as in the proof of Lemma 1. Then by the given properties of l^*, $eA = e$ must hold. Thus M is a W–L matrix and hence has the cut property.

As a consequence of Lemma 7, the weak cut property can be added to the list given in Theorem 1.

Let R be a relation on the set of integers $\{0, 1, \dots, n\}$ (that is, $R \subset \{0, 1, \dots, n\}^2$) such that $(0, 0)$ is not a member of R and

(∗) there exists $k_1, \dots, k_m$ satisfying $0Rk_1, \dots, k_jRk_{j+1}, \dots, k_mR0$.

A *path* is a subset P of $\{1, \dots, n\}$ such that R restricted to the set $\{0\} \cup P$ satisfies (∗). (Note that any superset of a path is also a path.)

By assumption, the set $\{1, \dots, n\}$ is a path and no path is empty. (Hence n is at least 1.) It will be shown that the collection of paths of R satisfies the width–length inequality.

Given R and an indexing of its paths, the corresponding incidence matrix M has the entry 1 in row i and column j if the jth path contains the number i. Otherwise the entry is 0.

Theorem 2. *M, as described above, is a W–L matrix.*

Proof. Given l such that $\min_p lp = 1$, let l^* be defined as follows. Consider any $k_1, \dots, k_m$ satisfying $0Rk_1, \dots, k_jRk_{j+1}, \dots, k_mR0$. The set path $P = \{k_1, \dots, k_m\}$ yields a matrix path p. If $lp = 1$ then select the first k_j in $k_1, \dots, k_m$ which corresponds to a positive entry in l. Let the corresponding entry in l^* be 1. Assign all possible 1's in this manner. The remaining entries of l^* are 0's. From the construction of M and l^* it can be verified that $l^*p > 1$ holds only if $lp > 1$. Hence M has the weak cut property and by Lemma 4 and Theorem 1 it is a W–L matrix. An alternate proof of this theorem can be derived from the cost algorithm procedure used by Moore and Shannon.

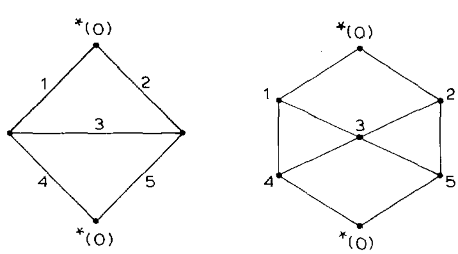
R can also be considered as a directed graph. The vertices of this graph are indexed by $0, 1, \dots, n$ and the directed edges are the pairs (i, j) (initial and terminal vertices) such that iRj holds. The paths corresponding to R are those collections of vertices (not including 0) forming a directed path which initiates and terminates at the 0 vertex. In this graph the weights l and w are assigned to the vertices instead of the edges as was the case in the introduction.

The width–length inequality still holds when weights are assigned to both edges and vertices: Consider a finite connected graph in which two vertices are distinguished. The edges can individually be either directed or undirected. Lengths and widths are assigned to each edge and to each vertex, excepting the distinguished vertices. A path is a collection of edges and vertices which connect the two distinguished vertices. For each pair of adjacent edges in the path, the common vertex is assumed to be a member of the path. In the case of directed edges the two distinguished vertices are called "initial" and "terminal" and each directed edge of the path must be oriented from the initial and towards the terminal vertex. A cut is any collection of edges and vertices which intersects each path. The length and width are defined as in the introduction except that the sums include both edge and vertex weights.

An equivalent collection of paths can also be derived from an appropriate relation R: Let both distinguished vertices be assigned the number 0 and let the edges and remaining vertices be labeled $1, \dots, n$. iRj will denote that the edge labeled i is incident on the vertex labeled j or that the vertex i is incident on the edge j. In the case of directed edges iRj denotes that vertex i is the initial vertex of the edge j or that the vertex j is the terminal vertex of the edge i. It is clear that the collection of paths of R is isomorphic with the collection of paths of the graph. Thus, by Theorems 1 and 2, both path collections satisfy the width–length inequality.

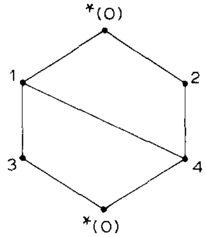
If weights are assigned only to certain edges and vertices then the width–length inequality still holds. The only modification is that the path sets are restricted to those edges and vertices carrying assigned weights. In the case of planar graphs it is possible to assign weights to edges and faces. Since this is the same as the weighting of edges and vertices in the dual graph it will not be discussed further.

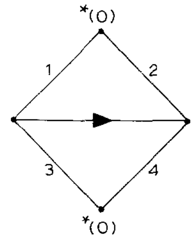
It is always possible to replace a graph with weighted edges by a graph with weighted vertices. In the first of the following examples the edges, indexed by $1, \dots, 5$, are weighted. In the second example the same weights are ascribed to the corresponding vertices. Both graphs yield the same incidence matrix.



$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

The transformation of a graph with weighted vertices into one with weighted edges may require the introduction of unweighted directed edges. For example consider the following vertex and edge-weighted graphs and their common incidence matrix. This matrix cannot be realized by weighting only the edges of an undirected graph.





$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

The W–L property is also applicable to bipartite graphs [3, pp. 49–50]. The following transport example comes from a problem posed by L.S. Joel. It requires the use of Theorem 1 instead of Theorem 2.

A continuous commodity has m sources and n destinations. Each destination is directly connected to at least one source and receives an aggregate shipment of one unit from its sources. The cost of these shipments is a charge v_{ij} per unit shipped from source i to destination j plus a charge w_i per unit for the maximum shipment from source i. It is desired to assign flows x_{ij} from source i to destination j so as to minimize the total cost. Alternatively suppose that the per unit charge w_i is replaced by a fixed charge w_i which is levied for any (non-zero) use of the source i, regardless of the amount of shipment. Does this alternative necessarily increase the total cost?

In matrix notation the problem is this. One is given an m by n matrix v where $0 \leq v_{ij} \leq \infty$ (an infinite cost v_{ij} is assigned if there is no connection between source i and destination j) and an m by 1 matrix w where $0 \leq w_i < \infty$. In addition, each column of v has a finite entry. Let x denote any m by n matrix such that $0 \leq x_{ij} < \infty$ and $\sum_i x_{ij} = 1$ holds for all j. It is desired to minimize first,

$$\sum_{ij} v_{ij}x_{ij} + \sum_i w_i \max_j x_{ij}$$

and second,

$$\sum_{ij} v_{ij}x_{ij} + \sum_i w_i\delta_i \quad \text{where } \delta_i = \begin{cases} 0 & \text{if } x_{i1}, \dots, x_{in} \text{ are all } 0, \\ 1 & \text{otherwise.} \end{cases}$$

Are these minima equal? (Clearly the second minimum is never less than the first.)

Let M be the m by n matrix obtained from v by replacing each finite v_{ij} by 1 and each infinite v_{ij} by 0. It will be shown that a necessary condition for equality over the class of all v's and w's which yield M is that M be a W–L matrix. In the case that the finite v_{ij}'s depend only on j, the W–L property is both necessary and sufficient. The proof is by the following argument.

The restriction of the v_{ij}'s means that the value of $\sum_{ij} v_{ij}x_{ij}$, being finite, must be the constant $\sum_j \min_i v_{ij}$. Hence it suffices to minimize only the summations containing the w_i's. Also, neglecting the v_{ij}'s, the cost is not changed if each x_{ij} is replaced by $\max_j x_{ij}$—and $\sum_i x_{ij} = 1$ is replaced by $\sum_i x_{ij} \geq 1$. Thus the first problem is to find l (where $l = (\max_j x_{1j}, \dots, \max_j x_{nj})$) which minimizes lw subject to the constraint $lM \geq e^{\&}$.

Suppose that M is a W–L matrix. Then by the width–length inequality $\min_c cw \leq (\min_p lp)(\min_c cw) \leq lw$ and hence there exists c_0 such that $c_0 w \leq lw$ holds for all admissible l. Consequently lw can be minimized by a 0–1 matrix and thus the minimum value of $\sum_i w_i \delta_i$ is exactly the same as the minimum of $\sum_i w_i \max_j x_{ij}$. Now conversely suppose that the minima are the same; that is, that lw can be minimized by a 0–1 matrix l_0. Since $l_0 M \geq e^{\&}$ holds, l_0 is a cut. Hence $\min_c cw \leq l_0 w \leq lw$. Thus for all l's such that $\min_p lp = 1$ and for all w's, $(\min_p lp)(\min_c cw) \leq lw$ holds. This is a scaled version of the width–length inequality and hence M is a W–L matrix.

Finally, to show that the two minima need not be equal, it suffices to exhibit a non-W–L matrix. In terms of the original problem let v be

$$\begin{pmatrix} \infty & 1 & 1 \\ 1 & \infty & 1 \\ 1 & 1 & \infty \end{pmatrix}$$

and w be

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Then the first minimum cost is $\frac{9}{2}$ while the second, which must necessarily be an integer, is 5. These costs are realized by

$$x = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

respectively.

4. Discussion

Those path collections which satisfy the width–length inequality have been characterized combinatorially by the W–L matrix property. Another approach to the problem is to characterize non-W–L matrices which in a certain sense are minimal:

If a row is deleted from a W–L matrix the resulting matrix is W–L. If a row together with all columns which have an entry 1 in that row are deleted, the resulting matrix is also W–L. In each case it is assumed that the resulting matrix has an entry 1 in each column. The validity of the first assertion is a consequence of the definition of a W–L matrix while the validity of the second assertion follows from the same argument applied to the dual $M^{\#}$. These deletions are analogous to the deletion of an edge of a graph by short-circuiting or open-circuiting respectively.

Now suppose that M is not a W–L matrix. Further suppose that the deletion of any row or a row and its associated columns (as mentioned above) yields a W–L matrix. Such matrices will be called *minimal*. Equivalently, M is minimal if there exist an l and w such that $(\min_p lp)(\min_c cw) > lw$ and any such l and w have no zero entries. Hence the dual $M^{\#}$ of a minimal matrix M is also a minimal matrix.

A matrix M is a W–L matrix if and only if it cannot be reduced, by the previous row and column deletions, to a minimal matrix. Thus the role of the minimal matrices in the width–length inequality is similar to that of the two Kuratowski (non-planar) graphs in characterizing planar graphs. Unfortunately the number of minimal matrices is infinite. Three types are displayed below.

$$\begin{pmatrix} 1&1&1&0&0&0&0\\ 1&0&0&1&1&0&0\\ 1&0&0&0&0&1&1\\ 0&1&0&0&1&0&1\\ 0&1&0&1&0&1&0\\ 0&0&1&0&1&1&0\\ 0&0&1&1&0&0&1 \end{pmatrix} \begin{pmatrix} 0&1&1&1&1&1&1\\ 1&1&0&0&0&0&0\\ 1&0&1&0&0&0&0\\ 1&0&0&1&0&0&0\\ 1&0&0&0&1&0&0\\ 1&0&0&0&0&1&0\\ 1&0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} 0&0&0&0&0&1&1\\ 0&0&0&0&1&1&0\\ 0&0&0&1&1&0&0\\ 0&0&1&1&0&0&0\\ 0&1&1&0&0&0&0\\ 1&1&0&0&0&0&0\\ 1&0&0&0&0&0&1 \end{pmatrix}$$

The first of the above examples is also the incidence matrix—points versus lines—of the seven point projective plane. The second example is one of a class of degenerate projective planes. The other members of this class are the analogous m by m matrices beginning with $m = 3$. They satisfy the two axioms: two points determine a line and two lines determine a point, but fail to satisfy the axiom: there exist four points no three of which are on a line. These projective plane examples are self-dual in the sense that $M^{\#\#}$ has the same columns as $M^{\#}$. (The only self-dual W–L matrix, in reduced form, is the matrix (1). This is shown, using minimization methods, in [4].) The third example is based on a cycle of odd length. The other members of this class are the analogous $2n+1$ by $2n+1$ matrices beginning with $n = 2$. All of these examples possess the geometric duality $M = M^{\mathrm{T}}$ (invariance under interchange of points and lines). However the members of the third class are not self-dual, that is they fail to satisfy $M^{\#\#} = M^{\#}$. In fact, the reduced form of the dual of the $n = 4$ matrix, given below, is not even square.

$$\begin{pmatrix} 1&0&1&0&1&0&1&0&1&1&0&1\\ 1&1&0&1&0&1&0&1&0&1&1&0\\ 0&1&1&0&1&0&1&0&1&0&1&1\\ 1&0&1&1&0&1&0&1&0&1&0&1\\ 0&1&0&1&1&0&1&0&1&1&1&0\\ 1&0&1&0&1&1&0&1&0&0&1&1\\ 0&1&0&1&0&1&1&0&1&1&0&1\\ 1&0&1&0&1&0&1&1&0&1&1&0\\ 0&1&0&1&0&1&0&1&1&0&1&1 \end{pmatrix}$$

Other than these three types and their permutations and duals, no additional minimal matrices have been found. Whether or not this list is complete, the multiplicity of minimal matrices seems to preclude their usefulness as a W–L matrix characterization. (Cf. author's foreword.)

In the theory of resistor networks, both linear and non-linear, either of Kirchoff's laws can be replaced by an appropriate minimization condition. (In the continuous case the corresponding theory applies to the Dirichlet problem.) Suppose that potential is applied between two vertices of a passive resistor network. Then the validity of the width–length inequality is a necessary and sufficient condition that the two minimizations yield the same result. Since the width–length inequality is automatic for graphs (and in Euclidean space [1, p. 214]) the preceding statement has meaning only for more generral path collections. The details are given in [4].

Either of the above minimizations can be reformulated as a maximization in such a way that the function being maximized does not exceed (but eventually equals) the function being minimized. It is the width–length inequality which keeps these functions from passing. A motivation in the study of minimal and W–L matrices was to gain insight into the corresponding connective structure. Unfortunately only the following network interpretation has been found: The fixing of the potential difference between two vertices of a graph determines the potential difference solely for those edges which directly connect the two vertices. This is trivial for graphs as the potentials at all internal vertices remain arbitrary. Yet this is the property which is abstracted in the notion of a W–L matrix. Basically the question of structure is still unsolved. A suitable answer might be expected to extent this theory to the continuous case.

Finally, it should be noted that the equivalence of the W–L property and the width–length inequality (Theorem 1) applies to certain infinite matrices: Assume that M is at most countably infinite and that A is a finite non-singular submatrix. The "min" (minimum) operation of the width–length inequality is replaced by the "inf" (infimum) operation. Also it is assumed that the sum of the components of l and w converge. (This assumption appears to be necessary.) Lemma 1, that the width–length inequality implies the W–L property, is valid for these infinite matrices. The reverse implication follows from Lemmas 2 and 3 provided that l and w have only a finite number of non-zero entries. It is also valid, by the continuity of the infimum operation, if the sum of the components of l and w converge.

References

[1] R.J. Duffin, "The extremal length of a network", *Journal of Mathematical Analysis and Applications* 5 (1962) 200–215.

[2] R.J. Duffin and A.J. Hoffman, The path-cut inequality and networks, mimeographed.

[3] L.R. Ford Jr. and D.R. Fulkerson, *Flows in networks* (Princeton Univ. Press, Princeton, NJ, 1962).
[4] A. Lehman, Some resistor network inequalities, manuscript.
[5] E.F. Moore and C.E. Shannon, Reliable circuits using less reliable relays, *Journal of the Franklin Institute* 262 (1956) 204–205.