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SHORT COMMUNICATION

ON Q-MATRICES

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In a recent paper [1], Aganagic and Cottle have established a constructive characterization for a P_0-matrix to be a Q-matrix. Among the principal results in this paper, we show that the same characterization holds for an L-matrix as well, and that the symmetric copositive-plus Q-matrices are precisely those which are strictly copositive.

Key words: Characterization, Classes of Matrices, Linear Complementarity Problem.

1. Introduction

A *Q-matrix* is a real square matrix M for which the linear complementarity problem, (q, M), of finding a vector x such that

$$q + Mx \geq 0, \quad x \geq 0 \quad \text{and} \quad x^{\mathrm{T}}(q + Mx) = 0$$

has a solution for every vector q. The class of Q-matrices is denoted by Q. A *P_0-matrix* is a real square matrix with nonnegative principal minors. Recently, Aganagic and Cottle [1] established the characterization below.

Theorem. *Let M be a P_0-matrix. The following are equivalent*
(A) $M \in Q$,
(B) $M \in R$, *and*
(C) $M \in R_0$.

Here, R denotes Karamardian's class of regular matrices [5], i.e., the following system is inconsistent

$$\begin{matrix} M_i x + t = 0, & x_i > 0, \\ M_j x + t \geq 0, & x_j = 0, \end{matrix} \qquad 0 \neq x \geq 0, \quad t \geq 0 \tag{1}$$

where M_i is the i-th row of M; and R_0 denotes the class of matrices M for which the problem $(0, M)$ has a unique solution, or equivalently, system (1) is inconsistent for $t = 0$.

In this paper, we investigate the validity of the above theorem for some other classes of matrices.

2. Main results

We review some definitions. The class of *S-matrices*, denoted by S, consists of those real square matrices M for which there is a vector $x \geq 0$ such that $Mx > 0$. Obviously, $Q \subseteq S$. A real square matrix M is said to be *semi-monotone* if for every $0 \neq x \geq 0$, there exists an index k such that $x_k > 0$ and $(Mx)_k \geq 0$. The class of semi-monotone matrices is denoted by L_1 [3]. P_0-matrices are certainly semi-monotone [4]. So are *copositive matrices* (i.e., real square matrices M such that $x^T Mx \geq 0$ for all $x \geq 0$). The class of *L_2-matrices*, denoted by L_2, consists of those real square matrices M satisfying the condition: for every $0 \neq x \geq 0$ with $Mx \geq 0$ and $x^T Mx = 0$, there exist nonnegative diagonal matrices D_1 and D_2 such that $D_2 x \neq 0$ and $(D_1 M + M^T D_2)X = 0$. We define $L = L_1 \cap L_2$. This class L was introduced by Eaves [3] who showed that if M is an L-matrix, then it is a Q-matrix if and only if it is an S-matrix. A copositive matrix M is called *copositive-plus* if $x^T Mx = 0$ and $x \geq 0$ imply $(M + M^T)x = 0$ and *strictly copositive if* $x^T Mx = 0$ and $x \geq 0$ imply $x = 0$. Clearly, copositive-plus and strictly copositive matrices belong to L. A flat point of M [2] is a vector x such that $x^T Mx = 0$ and $(M + M^T)x = 0$.

We establish our first result which shows that Lemma 1 in [1] remains valid if P_0 is replaced by the larger class L_1. The proof is virtually a restatement of that for the cited Lemma, but with the substitution of P_0 by L_1.

Lemma 1. *Let $M \in L_1 \cap R_0$. Then $M \in R$, thus $M \in Q$.*

Proof. Suppose that the system (1) has a solution $\bar{x}$ for $t = 0$. We must have $t > 0$. Hence for $x_i > 0$, we have $M_i x < 0$. This contradicts the assumption that $M \in L_1$. The contradiction establishes the lemma.

Corollary 2. *Let $M \in R_0$ be copositive. Then $M \in R$, thus $M \in Q$.*

Lemma 3. *Let $M \in L_2 \cap Q$. Then $M \in R_0$.*

Proof. Suppose that the system (1) has a solution $\bar{x}$ for $t = 0$. We must have $\bar{x}^T M\bar{x} = 0$. By assumption, there exist nonnegative diagonal matrices D_1 and D_2 such that $D_2\bar{x} \neq 0$ and $(D_1 M + M^T D_2)\bar{x} = 0$. Hence

$$(\bar{x}^T D_2)M = -(M\bar{x})^T D_1 \leq 0.$$

Consequently, if we choose $q \leq 0$ with $q_i < 0$ for $(D_2\bar{x})_i \neq 0$, the problem (q, M) is infeasible. This contradiction establishes the lemma.

Combining Lemmas 1 and 3, we deduce:

Theorem 4. *Let $M \in L$. The following are equivalent*
(A) $M \in Q$,
(B) $M \in R$,
(C) $M \in R_0$, *and*
(D) $M \in S$.

It is quite natural to wonder whether Theorem 4 (without condition (D)) will remain valid if L is replaced by L_1. The difficulty lies in establishing (or providing a counterexample to) the inclusion ($L_1 \cap Q \subseteq R_0$). In the rest of the paper, we establish several results which we believe will be useful for the resolution of this conjecture.

Proposition 5. *Let $M \in L_1 \cap Q$. Then the system*

$$Mx = 0, \quad x > 0 \tag{2}$$

is inconsistent.

Proof. The semi-monotonicity of M implies that $Mx < 0$, $x \geq 0$ is inconsistent. By Tucker's theorem of the alternative [6, p. 29], there is a $0 \neq y \geq 0$ such that $M^{\mathrm{T}}y \geq 0$. If the system (2) is consistent, then we must have $M^{\mathrm{T}}y = 0$. Consequently, the problem (q, M) is infeasible if $q \leq 0$ with $q_i < 0$ for $y_i > 0$. This contradiction establishes the proposition.

Remark. The inconsistency of the system (2) is equivalent to the fact that any nonzero solution to $(0, M)$ must have some zero component(s).

Corollary 6. *Let $M \in Q$ be copositive. Then the system* (2) *is inconsistent.*

Proposition 7. *Let $M \in L_1 \cap Q$. If $\bar{x}$ is a nonzero solution to $(0, M)$, then $\bar{x}$ contains at least two nonzero components.*

Proof. Indeed, if $\bar{x}_i > 0$ and $\bar{x}_j = 0$ for all $j \neq i$, then we must have $m_{ii} = 0$ and $m_{ji} \geq 0$ for all $j \neq i$. Choose a vector q with $q_i < 0$ and $q_j > 0$ for $j \neq i$. Let $\bar{z}$ be a solution to (q, M). Then by the choice of q, we must have $\bar{z}_j > 0$ for some $j \neq i$. For such an index j, we have

$$(q + M\bar{z})_j = 0$$

which implies

$$\sum_{k \neq i} m_{jk}\bar{z}_k = -(q_j + m_{ji}\bar{z}_i) < 0.$$

Consequently, for the vector x defined by $x_i = 0$ and $x_k = \bar{z}_k$ for $k \neq i$, there is no index k such that $x_k > 0$ and $(Mx)_k \geq 0$. This contradicts the assumption that $M \in L_1$. The contradiction establishes the proposition.

Theorem 8. *Let $M \in Q$ be copositive. Then the only flat point of M which is also a solution to $(0, M)$ is the zero vector.*

Proof. Suppose that $x \geq 0$ is a nonzero flat point of M that is also a solution to $(0, M)$. Let

$$\alpha = \{i : x_i > 0\} \quad \text{and} \quad \bar{\alpha} = \{j : x_j = 0\}.$$

Since $Mx \geq 0$ and $x^T Mx = 0$, we must have $(Mx)_\alpha = 0$. It is easy to see that

$$(x + \theta u)^T M(x + \theta u) = \theta^2 u^T Mu.$$

If $u_{\bar{\alpha}} \geq 0$, the left side of the above equation is nonnegative for $\theta > 0$ sufficiently small. Thus $u^T Mu \geq 0$ provided that $u_{\bar{\alpha}} \geq 0$. This implies that for all θ,

$$(x + \theta u)^T M(x + \theta u) \geq 0 \quad \text{if } u_{\bar{\alpha}} \geq 0. \tag{3}$$

Choose a vector q such that $q_\alpha < 0$ and $q_{\bar{\alpha}} > 0$. Let z be a solution to (q, M). It is then easy to show that if $\lambda > 0$ is sufficiently small so that $x_\alpha - \lambda z_\alpha > 0$, then

$$(x - \lambda z)_i (M(x - \lambda z))_i < 0 \quad \text{for } (x - \lambda z)_i \neq 0.$$

Hence it follows that for such a λ, we have

$$(x - \lambda z)^T M(x - \lambda z) < 0$$

which contradicts (3). The contradiction establishes the theorem.

Corollary 9. *Let $M \in Q$ be symmetric and copositive. The following implication is valid:*

$$Mx = 0, \quad x \geq 0 \Rightarrow x = 0. \tag{4}$$

Proof. In fact, any vector $x \geq 0$ satisfying $Mx = 0$ is a flat point of M which is also a solution to $(0, M)$. By Theorem 8, the only such vector is zero.

Remark. The implication (4) is weaker than the statement that $(0, M)$ has a unique solution. Nevertheless, Corollary 9 does not follow from either Theorem 4 or Proposition 5.

Corollary 10. *Let $M \in Q$ be symmetric and copositive-plus. Then M is strictly copositive.*

Proof. Let $x \geq 0$ be such that $x^T Mx = 0$. Since M is symmetric and copositive-plus, it follows that $Mx = 0$. Hence by Corollary 9, we must have $x = 0$. Consequently, M is strictly copositive.

Combining Theorem 4 and Corollary 10, we deduce:

Theorem. 11. *Let M be copositive plus. The following are equivalent:*

(A) $M \in Q$,

(B) $M \in R$,

(C) $M \in R_0$, *and*

(D) $M \in S$.

If in addition, M is symmetric, then any one of the above is equivalent to:

(E) *M is strictly copositive, and*

(F) *the implication* (4) *holds.*

Remark. Corollary 10 (and thus the entire Theorem 11) also follows directly from Theorem 4.

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