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Quantization of Lie Bialgebras, I

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Abstract. In the paper [Dr3] V. Drinfeld formulated a number of problems in quantum group theory. In particular, he raised the question about the existence of a quantization for Lie bialgebras, which arose from the problem of quantization of Poisson Lie groups. When the paper [KL] appeared Drinfeld asked whether the methods of [KL] could be useful for the problem of quantization of Lie bialgebras. This paper gives a positive answer to a number of Drinfeld's questions, using the methods and ideas of [KL]. In particular, we show the existence of a quantization for Lie bialgebras. The universality and functoriality properties of this quantization will be discussed in the second paper of this series. We plan to provide positive answers to most of the remaining questions in [Dr3] in the following papers of this series.

Introduction

The main result of this paper is a construction of a quantization for Lie bialgebras (see [Dr3] Section 1).

The paper consists of two parts. In the first part we construct the quantization of a finite-dimensional Lie bialgebra. In the second part we generalize this result to the infinite-dimensional case. The construction in the first part consists of three steps.

1) Given a finite-dimensional Lie bialgebra $\mathfrak{a}$ over a field k of characteristic zero, we construct the double $\mathfrak{g}$ of $\mathfrak{a}$. Our definition of the double coincides with the one in [Dr1]. We consider the category $\mathcal{M}$ whose objects are $\mathfrak{g}$-modules and $\mathrm{Hom}_{\mathcal{M}}(U,W) = \mathrm{Hom}_{\mathfrak{g}}(U,W)[[h]]$. For any associator Φ ([Dr2, Dr4]) we define a structure of a braided monoidal category on $\mathcal{M}$, as in [Dr2].

2) We construct Verma modules M_+, M_- over $\mathfrak{g}$, and use them to construct a fiber functor from $\mathcal{M}$ to the tensor category of topologically free $k[[h]]$ modules: $F(V) = \mathrm{Hom}_{\mathcal{M}}(M_+ \otimes M_-, V)$. According to the categorical yoga, the existence of such a functor implies the existence of a (topological) Hopf algebra H isomorphic to

$U(\mathfrak{g})[[h]]$ such that the tensor category $\mathcal{M}$ is equivalent to the category of representations of H. We show that H is isomorphic, as a topological algebra, to $U(\mathfrak{g})[[h]]$, where $U(\mathfrak{g})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}$.

3) We construct Hopf subalgebras $H_{\pm}$ of H and show that H_+ is a quantization of $\mathfrak{a}$ and that the algebra H is the quantum double of the Hopf algebra H_+.

Remark. We do not expect the existence of a quantization of any Lie bialgebra $\mathfrak{a}$ which is isomorphic to $U(\mathfrak{a})[[h]]$ as a topological algebra.

As an application of our techniques, we prove that any classical r-matrix r over an associative algebra A ($r \in A\otimes A$) can be quantized. In other words, there exists a quantum R-matrix $R \in A\otimes A[[h]]$ such that $R = 1 + hr + O(h^2)$. We also show that R is unitary ($R^{21}R = 1$) if r is unitary ($r^{21} = -r$). This answers questions in Section 3 of [Dr3]. As another application, we show the existence of the quantization of a quasitriangular Lie bialgebra $\mathfrak{a}$ (not necessarily finite-dimensional) such that the obtained quantized universal enveloping algebra has a quasitriangular structure and is isomorphic to $U(\mathfrak{a})[[h]]$ as a topological algebra, which solves questions in Section 4 of [Dr3].

The construction of quantization given in Part I has two drawbacks. First, it does not work literally for infinite-dimensional Lie bialgebras. Second, it does not allow to prove functoriality and universality of quantization. Therefore, in Part II we slightly modify the construction, which puts the results of the first part in a more general setting. Now we consider arbitrary Lie bialgebras, not necessarily finite-dimensional. In this case the double $\mathfrak{g}$ of $\mathfrak{a}$ can also be constructed, but it carries a nontrivial topology if $\dim \mathfrak{a} = \infty$. Instead of the category of all $\mathfrak{g}$-modules, we now consider the category $\mathcal{M}^e$ whose objects are equicontinuous $\mathfrak{g}$-modules, which are topological $\mathfrak{g}$-modules satisfying certain conditions. On this category, we define a braided monoidal structure analogously to the finite-dimensional case.

We construct Verma modules M_+, M_- over $\mathfrak{g}$ analogously to the finite-dimensional case. The module M_- is equicontinuous. The module M_+, in general, is not equicontinuous, but the module M_+^*, dual to M_+ in an appropriate topology, is an equicontinuous $\mathfrak{g}$-module. Using M_- and M_+^*, we define a fiber functor from $\mathcal{M}^e$ to the category of topological $k[[h]]$-modules, by $F(V) = \mathrm{Hom}_{\mathcal{M}^e}(M_-, M_+^*\otimes V)$. Since the module M_+ is not always equicontinuous, this functor is not always representable in $\mathcal{M}^e$. We define a tensor structure on F similarly to the finite-dimensional case, and show that if $\mathfrak{g}$ is finite-dimensional, the functors obtained in the first and second parts of the paper are isomorphic as tensor functors.

Next, we consider the algebra $H = \mathrm{End}F$. It is a topological algebra over $k[[h]]$ with a "coproduct" Δ, which maps H into a completion of $H\otimes H$, but not necessarily in $H\otimes H$.

Finally, we construct a subalgebra H_+ of H such that $\Delta(H_+)\subset H_+\otimes H_+$. This is a quantized universal enveloping algebra which is a quantization of $\mathfrak{a}$. For finite-dimensional $\mathfrak{a}$, this quantization is isomorphic to the one obtained in the first part.

In the second paper of this series we will settle Drinfeld's question of the existence of a universal quantization of Lie bialgebras by showing that the quantization obtained in Part II of this paper is universal. In Drinfeld's language this means that the product and coproduct in the quantized algebra are expressed in terms of acyclic tensor calculus via the commutator and cocommutator. This result implies that our quantization of Lie bialgebras is a functor from the category of Lie bialgebras to the category of topological Hopf algebras. It also shows that our quantizations of classical r-matrices, unitary r-matrices, and quasitriangular Lie bialgebras are universal and functorial. Thus we will answer positively the corresponding questions of Drinfeld [Dr3].

Remarks.

1. The material of Part I does not seem sufficient for proving universality and functoriality. In fact, during the computation of the h^2-term of multiplication in $U_h(\mathfrak{a})$, using the method of Part I, one gets non-acyclic expressions, which cancel at the end of computation. Thus, the generalization to the infinite-dimensional case is essential for the proof of functoriality, even for finite-dimensional Lie bialgebras.
2. Most of the results of the paper could be formulated and proved over the ring $k[h]/(h^N)$ rather than $k[[h]]$, and then the results over $k[[h]]$ could be obtained as a limit. The only problem arises with the notions of the dual quantized universal enveloping algebra and the quantum double, which collapse over $k[[h]]/(h^N)$. This is why we chose to work over $k[[h]]$.

In fact, it is easy to see that the main results of this paper hold in a more general setting than stated. Namely, one can take the Lie bialgebra $\mathfrak{a}$ to be "dependent on h", i.e. to be a Lie bialgebra over the ring $k[[h]]$, which is topologically free as a $k[[h]]$-module. The procedure of quantization described in Part II is well defined for this case, and, as will be shown in the second paper, defines a functor $\mathfrak{a} \to U_h(\mathfrak{a})$, from the category of Lie bialgebras over $k[[h]]$ which are topologically free as $k[[h]]$-modules to the category of quantized universal enveloping algebras (See Chapter 3). We will show that this functor is in fact an equivalence of categories and will construct the inverse functor.

The third paper of this series is not written yet. Therefore we will only indicate the topics which we are planning to present in this part. First of all, we plan to consider the case of graded bialgebras with finite-dimensional homogeneous components and to show that in this case our formal quantization defines a family of Hopf algebras H_h, depending on a parameter $h \in k$. Our second goal is to prove that, for Kac–Moody bialgebras, our quantization coincides with the quantum Kac–Moody algebra. As another application of our techniques we plan to show how to define a quantum analog of the Kac–Moody algebra for arbitrary symmetrizable Cartan matrix (not necessarily integral) and show that for generic values of q the "size" of the quantized algebra is the same as of the usual Kac–Moody algebra. This would settle the questions in Section 8 of [Dr3].

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PART I

1. Drinfeld category

The definitions and statements of Sections 1.1, 1.2 can be found in [Dr1].

1.1. Lie algebras

Throughout this paper, k denotes a field of characteristic zero. Let $\mathfrak{a}$ be a Lie algebra over k, and δ be a linear map $\delta : \mathfrak{a} \to \mathfrak{a} \otimes \mathfrak{a}$.

Definition. One says that the map δ defines a Lie bialgebra structure on $\mathfrak{a}$ if it satisfies two conditions:

(i) δ is a 1-cocycle of $\mathfrak{a}$ with coefficients in $\mathfrak{a}\otimes\mathfrak{a}$, i.e.

$$\delta([ab]) = [1\otimes a + a\otimes 1, \delta(b)] + [\delta(a), 1\otimes b + b\otimes 1];$$

(ii) The map $\delta^* : \mathfrak{a}^*\otimes\mathfrak{a}^* \to \mathfrak{a}^*$ dual to δ is a Lie bracket on $\mathfrak{a}^*$.

In this case δ is called the cocommutator of $\mathfrak{a}$.

If $\mathfrak{a}$ is a finite-dimensional Lie bialgebra then $\mathfrak{a}^*$ is a Lie bialgebra as well. Namely, the commutator in $\mathfrak{a}^*$ is dual to the cocommutator in $\mathfrak{a}$, and the cocommutator in $\mathfrak{a}^*$ is dual to the commutator in $\mathfrak{a}$. If $\mathfrak{a}$ is infinite-dimensional, then $\mathfrak{a}^*$ is not in general a Lie bialgebra but is a topological Lie bialgebra. That is, $\mathfrak{a}^*$ is a Lie algebra in the usual sense, but the cocommutator maps $\mathfrak{a}^*$ into the completed tensor product $\mathfrak{a}^*\hat{\otimes}\mathfrak{a}^*$ and not necessarily into the usual tensor product $\mathfrak{a}^*\otimes\mathfrak{a}^*$.

For any Lie bialgebra $\mathfrak{a}$, the vector space $\mathfrak{g} = a\oplus\mathfrak{a}^*$ has a natural structure of a Lie algebra. Namely, $\mathfrak{a}$, $\mathfrak{a}^*$ are Lie subalgebras in $\mathfrak{g}$ with the bracket defined above, and the commutator between elements of $\mathfrak{a}$, $\mathfrak{a}^*$ is given by

$$[a, b] = (\mathrm{ad}^* a)b - (1\otimes b)(\delta(a)), \quad a \in \mathfrak{a},\ b \in \mathfrak{a}^*, \tag{1.1}$$

where ad^* denotes the coadjoint action. There is an invariant nondegenerate inner product on $\mathfrak{g}$ given by $\langle a + a', b + b'\rangle = a'(b) + b'(a)$, $a, b \in \mathfrak{a}$, $a', b' \in \mathfrak{a}^*$. It is easy to show that (1.1) is the unique extension of the commutator from $\mathfrak{a}$, $\mathfrak{a}^*$ to $\mathfrak{g}$ for which the inner product $\langle\ ,\ \rangle$ is ad-invariant.

1.2. Main triples

Definition. A triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra with a nondegenerate invariant inner product $\langle\, ,\rangle$, and $\mathfrak{g}_+, \mathfrak{g}_-$ are isotropic Lie subalgebras, such that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as a vector space, is called a finite-dimensional Manin triple.

To every finite-dimensional Lie bialgebra $\mathfrak{a}$, one can associate the corresponding Manin triple $(\mathfrak{g} = \mathfrak{a}\oplus\mathfrak{a}^*, \mathfrak{a}, \mathfrak{a}^*)$, where the Lie structure on $\mathfrak{g}$ is as above. Conversely, if $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a finite-dimensional Manin triple then $\mathfrak{g}_+$ (and $\mathfrak{g}_-$) is naturally a Lie bialgebra. Namely the pairing $\langle\, ,\rangle$ identifies $\mathfrak{g}_+$ with $\mathfrak{g}_-^*$, so we can define $\delta : \mathfrak{g}_+ \to \mathfrak{g}_+\otimes\mathfrak{g}_+$ to be the dual map to the commutator of $\mathfrak{g}_-$. This map is a 1-cocycle of the Lie algebra $\mathfrak{g}_+$ with coefficients in the module $\mathfrak{g}_+\otimes\mathfrak{g}_+$, so it defines a structure of a Lie bialgebra on $\mathfrak{g}_+$.

Thus, there is a one-to-one correspondence between finite-dimensional Lie bialgebras and finite-dimensional Manin triples.

If $\mathfrak{a}$ is a Lie bialgebra then the Lie algebra $\mathfrak{g} = \mathfrak{a}\oplus\mathfrak{a}^*$ also has a natural structure of a Lie bialgebra. Namely, the cocommutator on $\mathfrak{g}$ is $\delta_{\mathfrak{g}} = \delta_{\mathfrak{a}}\oplus(-\delta_{\mathfrak{a}^*})$, where $\delta_{\mathfrak{a}}$, $\delta_{\mathfrak{a}^*}$ are the cocommutators of $\mathfrak{a}$, $\mathfrak{a}^*$. The 1-cocycle $\delta_{\mathfrak{g}}$ is the coboundary of an element in $\mathfrak{g}\otimes\mathfrak{g}$. Namely, if $r \in \mathfrak{a}\oplus\mathfrak{a}^* \subset \mathfrak{g}\otimes\mathfrak{g}$ is the canonical element corresponding to the identity operator $\mathfrak{a}\to\mathfrak{a}$, then $\delta_{\mathfrak{g}} = dr$, where r is regarded as a 0-cochain of $\mathfrak{g}$ with coefficients in $\mathfrak{g}\otimes\mathfrak{g}$, and d is the differential in the cochain complex; that is $\delta_{\mathfrak{g}}(x) = [x\otimes 1 + 1\otimes x, r]$.

The Lie bialgebra $\mathfrak{g}$ is called the double of $\mathfrak{a}$.

Let $\mathfrak{a}$ be a Lie algebra, and $r \in \mathfrak{a}\otimes\mathfrak{a}$. The equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \tag{1.2}$$

in $U(\mathfrak{a})^{\otimes 3}$ is called the classical Yang–Baxter equation. It is easy to check that the canonical element r satisfies this equation.

Definition. We say that a Lie bialgebra $\mathfrak{a}$ is quasitriangular if it is equipped with an element $r \in \mathfrak{a}\otimes\mathfrak{a}$ satisfying the classical Yang–Baxter equation, such that $\delta(a) = [a\otimes 1 + 1\otimes a, r]$ for any $a \in \mathfrak{a}$ (i.e. δ is a coboundary of r). For example, the double $\mathfrak{g}$ of any finite-dimensional Lie bialgebra $\mathfrak{a}$ equipped with the canonical element r is a quasitriangular Lie bialgebra.

1.3. Associators

Recall some notation and definitions from the theory of associators [Dr2, BN]. Let T_n be the algebra over k generated by elements t_{ij}, $1 \le i, j \le n$, $i \ne j$, with defining relations $t_{ij} = t_{ji}$, $[t_{ij}, t_{lm}] = 0$ if i, j, l, m are distinct, and $[t_{ij}, t_{ik} + t_{jk}] = 0$.

Let $P_1, \ldots, P_n$ be disjoint subsets of $\{1, \ldots, m\}$. There exists a unique homomorphism $\rho_{P_1,\ldots,P_n} : T_n \to T_m$ defined by

$$\rho_{P_1,\ldots,P_m}(t_{ij}) = \sum_{p\in P_i, q\in P_j} t_{pq}. \tag{1.3}$$

For any $X \in T_n$, we denote $\rho_{P_1,\ldots,P_n}(X)$ by $X_{P_1,\ldots,P_n}$.

Let $\Phi \in T_3[[h]]$. The relation

$$\Phi_{1,2,34}\Phi_{12,3,4} = \Phi_{2,3,4}\Phi_{1,23,4}\Phi_{1,2,3} \tag{1.4}$$

in $T_4[[h]]$ (=relation (1.2) in [Dr2]) is called the pentagon relation.

Let $B = e^{ht_{12}/2} \in T_2[[h]]$. The relations

$$\begin{aligned} B_{12,3} &= \Phi_{3,1,2}B_{1,3}\Phi_{1,3,2}^{-1}B_{2,3}\Phi_{1,2,3}, \\ B_{1,23} &= \Phi_{2,3,1}^{-1}B_{1,3}\Phi_{2,1,3}B_{1,2}\Phi_{1,2,3}^{-1} \end{aligned} \tag{1.5}$$

in $T_3[[h]]$ (=relations (3.9a), (3.9b) in [Dr2]) are called the hexagon relations.

Definition. An element $\Phi \in T_3[[h]]$ of the form $\Phi = e^{P(h\hat{t}_{12},ht_{23})}$, where $P(X,Y)$ is a Lie formal series with coefficients in k, is called a Lie associator over k if it satisfies the pentagon and hexagon relations.

For $k = \mathbb{C}$, an example of a Lie associator is the Drinfeld associator Φ_{KZ} obtained from the KZ equations, as explained in [Dr2].

The following theorem about Lie associators is due to Drinfeld ([Dr4], Theorem A").

Theorem 1.1. *There exists a Lie associator defined over* $\mathbb{Q}$.

This theorem implies that there exists a Lie associator defined over any field k of characteristic zero. From now on we will fix such a Lie associator Φ.

1.4. Drinfeld category

Let $\mathfrak{g}$ be a Lie algebra over k, and $\Omega \in S^2\mathfrak{g}$ be a $\mathfrak{g}$-invariant element.

We will be mostly interested in the case when $\mathfrak{g}$ belongs to a finite-dimensional Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, and $\Omega = \sum_i g_i \otimes g^i$, where $\{g_i\}$ is a basis of $\mathfrak{g}$, and $\{g^i\}$ is the dual basis to $\{g_i\}$ with respect to the invariant inner product on $\mathfrak{g}$. In this case the element Ω is called the Casimir element.

Let $\mathcal{M}$ denote the category whose objects are $\mathfrak{g}$-modules, and $\mathrm{Hom}_{\mathcal{M}}(U,W) = \mathrm{Hom}_{\mathfrak{g}}(U,W)[[h]]$. This is a $k[[h]]$-linear additive category. For brevity we will later write Hom for $\mathrm{Hom}_{\mathcal{M}}$.

Drinfeld [Dr2] defined a structure of a braided monoidal category on $\mathcal{M}$ as follows: For any $V_1, V_2, V_3 \in \mathcal{M}$, consider a homomorphism $\theta : T_3[[h]] \to \mathrm{End}(V_1 \otimes V_2 \otimes V_3)$ by $\theta(t_{ij}) = \Omega_{ij}$, and define $\Phi_{V_1V_2V_3} = \theta(\Phi)$.

For any $V_1, V_2 \in \mathcal{M}$, define $V_1 \otimes V_2 \in \mathcal{M}$ to be the usual tensor product of V_1, V_2 and the associativity morphism to be $\Phi_{V_1V_2V_3}$, regarded as an element of $\mathrm{Hom}((V_1 \otimes V_2) \otimes V_3, V_1 \otimes (V_2 \otimes V_3))$. For any $V_1, V_2 \in \mathcal{M}$, introduce the braiding $\beta_{V_1V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1$ by the formula $\beta = s \circ e^{h\Omega/2}$, where s is the permutation. It follows from relations (1.4), (1.5) that the morphisms $\Phi_{V_1V_2V_3}$ and $\beta_{V_1V_2}$ define the structure of a braided monoidal category on $\mathcal{M}$ (see [Dr2]).

2. The fiber functor

2.1. The category of topologically free $k([h]]$-modules

Let V be a vector space over k. Then the space $V[[h]]$ of formal power series in h with coefficients in V has a natural structure of a topological $k[[h]]$-module. We call a topological $k[[h]]$-module topologically free if it is isomorphic to $V[[h]]$ for some V.

Let $\mathcal{A}$ be the category of topologically free $k[[h]]$-modules where morphisms are continuous $k[[h]]$-linear maps. It is an additive category. Define the tensor structure on $\mathcal{A}$ as follows: for $V, W \in A$ define $V \otimes W$ to be the projective limit of the $k[h]/h^n$-modules $(V/h^nV) \otimes_{k[h]/h^n} (W/h^nW)$ as $n \to \infty$.

Let Vect be the category of vector spaces. We have the functor of extension of scalars, $V \mapsto V[[h]]$, from Vect to $\mathcal{A}$. This functor respects the tensor product, i.e. $(V \otimes W)[[h]]$ is naturally isomorphic to $V[[h]] \otimes W[[h]]$. The category $\mathcal{A}$ equipped with the functor $\otimes$ is a symmetric monoidal category.

If $X \in \mathcal{A}$ then $X^* = \mathrm{Hom}_{\mathcal{A}}(X, k[[h]])$ is a topologically free $k[[h]]$-module. The assignment $X \to X^*$ is a contravariant functor from $\mathcal{A}$ to itself.

2.2. The forgetful functor

Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a finite-dimensional Manin triple, $\Omega \in S^2\mathfrak{g}$ be the Casimir element associated to the inner product $\langle\, , \rangle$ on $\mathfrak{g}$, and $\mathcal{M}$ be the Drinfeld category associated to $\mathfrak{g}$.

Let $F : \mathcal{M} \to \mathcal{A}$ be the functor given by $F(M) = \mathrm{Hom}(U(\mathfrak{g}), M)$, where $U(\mathfrak{g})$ is regarded as a left $\mathfrak{g}$-module. This functor is naturally isomorphic to the "forgetful" functor which assigns to every $\mathfrak{g}$-module M the $k[[h]]$-module $M[[h]]$. The isomorphism between these two functors is given by the assignment $f \in F(M) \to f(1) \in M[[h]]$.

2.3. Verma modules

Consider Verma modules $M_+ = \mathrm{Ind}_{\mathfrak{g}_+}^{\mathfrak{g}} \mathbf{1}$, $M_- = \mathrm{Ind}_{\mathfrak{g}_-}^{\mathfrak{g}} \mathbf{1}$ (here $\mathbf{1}$ denotes the trivial 1-dimensional representation). By the Poincaré–Birkhoff–Witt theorem, the product in $U(\mathfrak{g})$ defines linear isomorphisms $U(\mathfrak{g}_+) \otimes U(\mathfrak{g}_-) \to U(\mathfrak{g})$, and $U(\mathfrak{g}_-) \otimes U(\mathfrak{g}_+) \to U(\mathfrak{g})$. This shows that the modules $M_\pm$ are freely generated over $U(\mathfrak{g}_\mp)$ by vectors $1_\pm$ such that $\mathfrak{g}_\pm 1_\pm = 0$, and are identified (as vector spaces) with $U(\mathfrak{g}_\mp)$ via $x1_\pm \to x$. Since the vectors $1_\pm \otimes 1_\pm \in M_\pm \otimes M_\pm$ are $\mathfrak{g}_\pm$-invariant, there exist unique $\mathfrak{g}$-module morphisms $i_\pm : M_\pm \to M_\pm \otimes M_\pm$ such that $i_\pm(1_\pm) = 1_\pm \otimes 1_\pm$. These morphisms in the category $\mathcal{M}$ will play a crucial role in our constructions below.

Lemma 2.1. *The assignment* $1 \to 1_+ \otimes 1_-$ *extends to an isomorphism of* $\mathfrak{g}$*-modules*

$\phi : U(\mathfrak{g}) \to M_+ \otimes M_-$.

Proof. Since $M_\pm$ has been identified with $U(\mathfrak{g}_\mp)$, we can regard the map ϕ as a linear map $U(\mathfrak{g}) \to U(\mathfrak{g}_-) \otimes U(\mathfrak{g}_+)$. It is clear that this map preserves the standard filtration, so it defines a map of the associated graded objects: $S\mathfrak{g} \to S\mathfrak{g}_- \otimes S\mathfrak{g}_+$. This map is the isomorphism induced by the isomorphism $\mathfrak{g} \to \mathfrak{g}_- \oplus \mathfrak{g}_+$. Therefore, ϕ is an isomorphism. □

Lemma 2.1 implies that the functor F can be identified with the functor $V \to \mathrm{Hom}(M_+ \otimes M_-, V)$. This definition of F will be used from now on.

2.4. Tensor structure on the functor F

Let $(\mathcal{C}, \otimes)$ be a monoidal category, Φ be the associativity constraint in $\mathcal{C}$, and $\mathbf{1}$ be the identity object in $\mathcal{C}$. For simplicity we assume that $\mathbf{1} \otimes X = X \otimes \mathbf{1} = X$ for any object $X \in \mathcal{C}$, and the functorial isomorphisms $X \to X \otimes \mathbf{1}$, $X \to \mathbf{1} \otimes X$ the are identity morphisms.

Let $F : \mathcal{C} \to \mathcal{A}$ be a functor such that $F(\mathbf{1}) = k[[h]]$.

Definition. By a tensor structure on the functor F one means a functorial isomorphism $J_{VW} : F(V) \otimes F(W) \to F(V \otimes W)$ satisfying the associativity identity $F(\Phi_{VWU}) J_{V \otimes W, U} \circ (J_{VW} \otimes 1) = J_{V, W \otimes U} \circ (1 \otimes J_{WU})$, such that for any object V $J_{V1} = J_{1V} = 1$. A functor equipped with a tensor structure is called a tensor functor.

Now we describe a tensor structure on the functor F constructed in Section 2.2.

For any $v \in F(V)$, $w \in F(W)$ define $J_{VW}(v \otimes w)$ to be the composition of morphisms:

$$
\begin{aligned}
M_+ \otimes M_- &\xrightarrow{i_+ \otimes i_-} (M_+ \otimes M_+) \otimes (M_- \otimes M_-) \xrightarrow{\text{associativity morphism}} \\
&(M_+ \otimes (M_+ \otimes M_-)) \otimes M_- \xrightarrow{(1 \otimes \beta_{23}) \otimes 1} \\
&(M_+ \otimes (M_- \otimes M_+)) \otimes M_- \xrightarrow{\text{associativity morphism}} \\
&(M_+ \otimes M_-) \otimes (M_+ \otimes M_-) \xrightarrow{v \otimes w} V \otimes W, \qquad (2.1)
\end{aligned}
$$

where β_{23} denotes the braiding β acting in the second and third components of the tensor product.

It is clear from this definition that all combinatorial complexity of the morphism J comes from the arrows "associativity morphism" which involve associators.

The arrows "associativity morphism" make the problem of checking various identities for J (for example, the associativity identity) rather tedious. To avoid this, we can use MacLane's theorem, which says that any monoidal category is equivalent to a strict one. Namely, when we check identities between morphisms in the category, we will assume that the category $\mathcal{M}$ is replaced with an equivalent strict monoidal category and ignore associativity morphisms. For example, the definition of J will look as follows:

$$J_{VW}(v\otimes w) = (v\otimes w)\circ(1\otimes\beta_{23}\otimes 1)\circ(i_+\otimes i_-).$$

However, when we do computations with vectors in modules from $\mathcal{M}$, it is important to pay attention to brackets, since different positions of brackets are related with each other by the associator.

Proposition 2.2. *The maps J_{VW} are isomorphisms and define a tensor structure on the functor F.*

Proof. It is obvious that J_{VW} is an isomorphism since it is an isomorphism modulo h. It is also clear that $J_{V1} = J_{1V} = 1$. Thus the only thing we need to check is the associativity identity $J_{V\otimes W,U}\circ(J_{VW}\otimes 1) = J_{V,W\otimes U}\circ(1\otimes J_{WU})$. To prove this equality, we need the following result.

Lemma 2.3. $(i_\pm\otimes 1)\circ i_\pm = (1\otimes i_\pm)\circ i_\pm$ *in* $\mathrm{Hom}(M_\pm, M_\pm^{\otimes 3})$.

Proof. We prove the identity for i_+. The identity for i_- is proved in the same way.

We need to show that for any vector $x \in M_+$

$$\Phi\cdot(i_+\otimes 1)i_+x = (1\otimes i_+)i_+x. \tag{2.2}$$

Since comultiplication in $U(\mathfrak{g}_-)$ is coassociative, i.e. $(i_+\otimes 1)i_+x = (1\otimes i_+)i_+x$, it is sufficient to show that the associator Φ is the identity on the image of $(i_+\otimes 1)i_+$. Because Φ is $\mathfrak{g}$-invariant, it is enough to show that $\Phi\cdot(i_+\otimes 1)i_+1_+ = (i_+\otimes 1)i_+1_+$, i.e.

$$\Phi\cdot(1_+\otimes 1_+\otimes 1_+) = 1_+\otimes 1_+\otimes 1_+. \tag{2.3}$$

Since the subalgebras $\mathfrak{g}_+$, $\mathfrak{g}_-$ are isotropic, the operators Ω_{12}, Ω_{23} annihilate the vector $1_+\otimes 1_+\otimes 1_+$. Thus, equation (2.3) follows from the definition of Φ. □

Now we can finish the proof of the proposition. Let $\psi_1, \psi_2 : M_+\otimes M_-\to (M_+\otimes M_-)^{\otimes 3}$ be the morphisms defined by

$$\begin{aligned}\psi_1 &= (1\otimes\beta_{23}\otimes 1\otimes 1\otimes 1)\circ(i_+\otimes i_-\otimes 1\otimes 1)\circ(1\otimes\beta_{23}\otimes 1)\circ(i_+\otimes i_-),\\ \psi_2 &= (1\otimes 1\otimes 1\otimes\beta_{45}\otimes 1)\circ(1\otimes 1\otimes i_+\otimes i_-)\circ(1\otimes\beta_{23}\otimes 1)\circ(i_+\otimes i_-).\end{aligned} \tag{2.4}$$

Then for any $v\in F(V)$, $w\in F(W)$, $u\in F(U)$ we have

$$\begin{aligned}J_{V\otimes W,U}(J_{VW}\otimes 1)(v\otimes w\otimes u) &= (v\otimes w\otimes u)\circ\psi_1,\\ J_{V,W\otimes U}(1\otimes J_{WV})(v\otimes w\otimes u) &= (v\otimes w\otimes u)\circ\psi_2.\end{aligned}$$

Therefore, to prove the proposition, it is sufficient to show that $\psi_1 = \psi_2$.

To prove this equality, we observe that the functoriality of the braiding implies the identities

$$(i_+ \otimes i_- \otimes 1 \otimes 1) \circ (1 \otimes \beta_{23} \otimes 1) = (1 \otimes \beta_{3,45} \otimes 1) \circ (i_+ \otimes 1 \otimes i_- \otimes 1),$$
$$(1 \otimes 1 \otimes i_+ \otimes i_-) \circ (1 \otimes \beta_{23} \otimes 1) = (1 \otimes \beta_{23,4} \otimes 1) \circ (1 \otimes i_+ \otimes 1 \otimes i_-) \tag{2.5}$$

(here $\beta_{3,45}$ means the braiding applied to the third factor and to the product of the fourth and the fifth factors). Using Lemma 2.3 and identities (2.5), we reduce the statement $\psi_1 = \psi_2$ to the identity

$$(1 \otimes \beta_{23} \otimes 1 \otimes 1 \otimes 1) \circ (1 \otimes \beta_{3,45} \otimes 1) = (1 \otimes 1 \otimes 1 \otimes \beta_{45} \otimes 1) \circ (1 \otimes \beta_{23,4} \otimes 1), \tag{2.6}$$

which follows directly from the braiding axioms. □

We call the functor F equipped with the tensor structure J the fiber functor.

3. Quantization of the double of a Lie bialgebra

3.1. Topological Hopf algebras

Let A be an algebra over k with unit. Let I be a proper two-sided ideal in A. This ideal gives rise to a translation invariant topology on A such that $\{I^n,\ n \geq 0\}$ is a basis of neighborhoods of 0. We will call A a topological algebra if $A = \varprojlim A/I^n$.

Let A_0 be a topological algebra over k, and $A = A_0[[h]]$ as a topological $k[[h]]$-module. Suppose that A is equipped with a continuous, $k[[h]]$-linear, associative product, which coincides with the product in A_0 modulo h. In this case we say that A is a topological algebra over $k[[h]]$, which is a deformation of A_0.

Let A, B be two topological algebras over k, I, J be the the corresponding ideals. Define the completed tensor product $A \otimes B$ to be the projective limit of algebras $A/I^n \otimes B/J^n$ as $n \to \infty$. Then $A \otimes B$ is also a topological algebra, with topology defined by the ideal $I \otimes B + A \otimes J$.

The completed tensor product of topological algebras over $k[[h]]$ is defined similarly.

We say that a topological algebra A over k is a topological Hopf algebra if it is equipped with comultiplication $\Delta : A \to A \otimes A$, the counit $\varepsilon : A \to k$, and the antipode $S : A \to A$, which are linear, continuous, and satisfy the standard axioms of a Hopf algebra. Note that an infinite-dimensional topological Hopf algebra may not be literally a Hopf algebra because the image of comultiplication may not belong to the algebraic tensor square of A.

Topological Hopf algebras over $k[[h]]$ are defined similarly. If A is a topological Hopf algebra over $k[[h]]$ then $B = A/hA$ is a topological Hopf algebra over k. In such a case we say that A is a formal deformation of B over $k[[h]]$. In particular, if $B = U(\mathfrak{g})$ with the discrete topology, where $\mathfrak{g}$ is a Lie algebra, then A is called a quantized universal enveloping algebra [Dr1].

The following definition is due to Drinfeld [Dr1].

Definition. Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra. We say that a quantized universal enveloping algebra A is a guantization of $(\mathfrak{g}, \delta)$, or that $(\mathfrak{g}, \delta)$ is the quasiclassical limit of A, if

(i) An isomorphism of Hopf algebras $A/hA \to U(\mathfrak{g})$ has been fixed, and

(ii) For any $x_0 \in \mathfrak{g}$ and any $x \in A$ equal to $x_0 \mod h$ one has

$$h^{-1}(\Delta(x) - \Delta^{op}(x)) \equiv \delta(x_0) \mod h,$$

where Δ^{op} is the opposite comultiplication ($\Delta^{op} = s\Delta$).

3.2. The algebra of endomorphisms of the fiber functor

Let $H = \mathrm{End}(F)$ be the algebra of endomorphisms of the functor F. This algebra is naturally isomorphic to $U(\mathfrak{g})[[h]]$. Namely, the map $\alpha : U(\mathfrak{g})[[h]] \to H$ is defined on $x \in U(\mathfrak{g})$ by the formula $(\alpha(x)f)(y) = f(yx)$, where $f \in \mathrm{Hom}(U(\mathfrak{g}), M)$, and is extended by linearity and continuity to $U(\mathfrak{g})[[h]]$. This map is an isomorphism of algebras. From now on we will make no distinction between $U(\mathfrak{g})[[h]]$ and H, identifying them by α.

Let $F^2 : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ be the bifunctor defined by $F^2(V, W) = F(V) \otimes F(W)$. It is clear that $\mathrm{End}(F^2) = H \otimes H$.

The algebra H has a natural comultiplication $\Delta : H \to H \otimes H$ defined by $\Delta(a)_{V,W}(v \otimes w) = J_{VW}^{-1} a_{V \otimes W} J_{VW}(v \otimes w)$, $a \in H$, $v \in F(V)$, $w \in F(W)$, where a_V denotes the action of a in $F(V)$. We can also define the counit on H by $\varepsilon(a) = a_{\mathbf{1}} \in k[[h]]$, where $\mathbf{1}$ is the neutral object.

For any $V \in \mathcal{M}$, let V^* be the dual space to V (regarded as an object of $\mathcal{M}$), and let $\sigma_V : V^* \otimes V \to \mathbf{1}$ be the canonical pairing. We have a functorial isomorphism $\xi_V : F(V^*) \to F(V)^*$ defined by $\xi_V(v^*)(v) = F(\sigma_V) J_{V^*V}(v^* \otimes v)$, $v \in F(V)$, $v^* \in F(V^*)$. For any $a \in H$, let $\widetilde{S(a)}_V = (\xi_V^*)^{-1} a_{V^*}^* \xi_V^*$ be a morphism $F(V)^{**} \to F(V)^{**}$. It is easy to show that the subspace $F(V) \subset F(V)^{**}$ is invariant under this morphism. The antipode $S : H \to H$ is defined by $S(a)_V = \widetilde{S(a)}_V|_{F(V)}$.

Proposition 3.1. *The algebra H equipped with Δ, ε, S is a topological Hopf algebra.*

The proof is straightforward.

3.3. Explicit representation of complication and antipode

Let $\Delta_0 : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ be the standard coproduct. For any $V, W \in \mathcal{M}$, let $J_{VW}^0 : F(V) \otimes F(W) \to F(V \otimes W)$ be the morphism defined by the formula

$J^0_{VW}(v\otimes w)(x) = (v\otimes w)(\Delta_0(x))$, $x \in U(\mathfrak{g})$, $v \in F(V)$, $w \in F(W)$. It is clear that $J_{VW} \equiv J^0_{VW} \mod h$.

Let $J \in U(\mathfrak{g})^{\otimes 2}[[h]]$ be defined by the formula

$$J = (\phi^{-1}\otimes\phi^{-1})\left(\Phi^{-1}_{1,2,34}(1\otimes\Phi_{2,3,4})se^{h\Omega_{23}/2}(1\otimes\Phi_{2,3,4})\Phi_{1,2,34}(1_+\otimes 1_+\otimes 1_-\otimes 1_-)\right), \tag{3.1}$$

where ϕ is the isomorphism of Lemma 2.1.

Proposition 3.2. *For any $V, W \in \mathcal{A}$, $v \in F(V)$, $w \in F(W)$ one has $J_{VW}(v\otimes w) = J^0_{VW}J(v\otimes w)$.*

Proof. The statement follows from the definition (2.1) of J_{VW}. □

Lemma 3.3. *Let $a \in H$. Then*

$$\Delta(a) = J^{-1}\Delta_0(a)J. \tag{3.2}$$

Proof. The lemma follows from Proposition 3.2 and the identities $\Delta_0(a)_{V,W} = (J^0_{VW})^{-1}a_{V\otimes W}J^0_{VW}$, $\Delta(a)_{V,W} = J^{-1}_{VW}a_{V\otimes W}J_{VW}$, $a \in U(\mathfrak{g})$. □

Now consider the explicit expression for the antipode. For any $V \in \mathcal{M}$ define the morphism $\xi^*_V : F(V^*)\to F(V)^*$ by $\xi^0_V(v^*)(v) = F(\sigma_V)J^0_{V^*V}(v^*\otimes v)$, $v \in V$, $v^* \in V^*$. It is clear that $\xi_V \equiv \xi^0_V \mod h$.

Let $S_0 : U(\mathfrak{g})\to U(\mathfrak{g})$ be the usual antipode. Let $J = \sum_j x_j\otimes y_j$, $x_j, y_j \in U(\mathfrak{g})[[h]]$ (the sum is finite modulo h^n for any n). Define an element $Q \in U(\mathfrak{g})[[h]]$ by $Q = \sum_j S_0(x_j)y_j$.

Lemma 3.4. *Let $a \in H$. Then*

$$S(a) = Q^{-1}S_0(a)Q. \tag{3.3}$$

Proof. It follows from the definitions of ξ_V, ξ^0_V, and Q that $\xi_V = \xi^0_V S_0(Q)_{V^*}$. Thus the Lemma follows from the formulas $S(a)_V = (\xi^*_V)^{-1}a^*_{V^*}\xi^*_V|_{F(V)}$, $S_0(a)_V = (\xi^{0*}_V)^{-1}a^*_{V^*}\xi^{0*}_V|_{F(V)}$. □

Thus, we have proved the following result.

Corollary 3.5. *Introduce a new comultiplication and antipode on the topological Hopf algebra $U(\mathfrak{g})[[h]]$ by*

$$\Delta(x) = J^{-1}\Delta_0(x)J, \quad S(x) = Q^{-1}S_0(x)Q, \tag{3.4}$$

where Δ_0, S_0 are the usual comultiplication and antipode. Then $(U(\mathfrak{g})[[h]], \Delta, S)$ is a topological Hopf algebra isomorphic to H.

We will denote the topological Hopf algebra $(U(\mathfrak{g})[[h]], \Delta, S)$ by $U_h(\mathfrak{g})$.

Remark. It is easy to see that according to the terminology of [Dr2], the element J^{-1} is a twist that realizes an equivalence between the quasi-Hopf algebra $(U(\mathfrak{g})[[h]], \Phi)$ and the Hopf algebra $U_h(\mathfrak{g})$.

3.4. The quasiclassical limit of $U_h(\mathfrak{g})$

Proposition 3.6. *The topological Hopf algebra $U_h(\mathfrak{g})$ is a quantization of the Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$.*

Proof. Take $a \in \mathfrak{g} \subset U_h(\mathfrak{g})$. Let $\delta(a) \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ be defined by the formula $\delta(a) = h^{-1}(\Delta(a) - \Delta^{op}(a)) \mod h$. To prove the proposition, we need to show that for any $a \in \mathfrak{g}$ one has $\delta(a) = \delta_{\mathfrak{g}}(a)$, where $\delta_{\mathfrak{g}}(a)$ is defined in Chapter 1.

It is easy to check the following identities:

$$e^{h\Omega/2} \equiv 1 + h\Omega/2 \mod h^2, \quad \Phi \equiv 1 \mod h^2. \tag{3.5}$$

Let $\{g_j^+\}$ be a basis of $\mathfrak{g}_+$, $\{g_j^-\}$ be the dual basis of $\mathfrak{g}_-$, and $r = \sum_j g_j^+ \otimes g_j^-$. Identities (3.1) and (3.5) imply that

$$J \equiv 1 + hr/2 \mod h^2. \tag{3.6}$$

Therefore, by Lemma 3.3,

$$\Delta(a) \equiv \Delta_0(a) + \frac{h}{2}[\Delta_0(a), r] \mod h^2. \tag{3.7}$$

Thus,

$$\Delta(a) - \Delta^{op}(a) \equiv \frac{h}{2}[\Delta_0(a), r - sr] \mod h^2. \tag{3.8}$$

Since $r + sr (= \Omega)$ is $\mathfrak{g}$-invariant, we obtain

$$\delta = dr = \delta_{\mathfrak{g}}, \tag{3.9}$$

Q.E.D. □

3.5. The quasi triangular structure on $U_h(\mathfrak{g})$

Define the element

$$R = (J^{op})^{-1} e^{h\Omega/2} J \in U_h(\mathfrak{g})^{\otimes 2}, \tag{3.10}$$

where J^{op} is obtained from J by permuting components. We call this element the universal R-matrix of $U_h(\mathfrak{g})$.

Proposition 3.7. *R defines a quasitriangular structure on $U_h(\mathfrak{g})$. That is, R is invertible and*

$$R\Delta = \Delta^{op} R, \tag{3.11}$$

$$(\Delta \otimes 1)(R) = R_{13} R_{23}, (1 \otimes \Delta)(R) = R_{13} R_{12}. \tag{3.12}$$

Moreover, R is a quantization of r, i.e.

$$R \equiv 1 + hr \mod h^2. \tag{3.13}$$

Proof. Identity (3.13) follows from (3.5), (3.6) and the definition of R. This identity implies that R is invertible.

One has

$$\begin{aligned} R\Delta(a) &= (J^{op})^{-1}e^{h\Omega/2}J\Delta(a) = (J^{op})^{-1}e^{h\Omega/2}\Delta_0(a)J = (J^{op})^{-1}\Delta_0(a)e^{h\Omega/2}J \\ &= \Delta^{op}(a)(J^{op})^{-1}e^{h\Omega/2}J = \Delta^{op}(a)R, \end{aligned} \tag{3.14}$$

which proves (3.11).

Now let us prove the first identity of (3.12). The second identity is proved analogously.

According to the definition of R, for any $V, W \in \mathcal{M}$, $v \in F(V)$, $w \in F(W)$, one has $R(v\otimes w) = sJ_{WV}^{-1}F(\beta_{VW})J_{VW}$. Thus, for any $U \in \mathcal{M}$, $u \in F(U)$ one has

$$\begin{aligned} (\Delta\otimes 1)(R)(v\otimes w\otimes u) &= (J_{VW}^{-1}\otimes 1)R(J_{VW}\otimes 1)(v\otimes w\otimes u) \\ &= s_{12,3}(1\otimes J_{VW}^{-1})J_{U,V\otimes W}^{-1}F(\beta_{V\otimes W,U})J_{V\otimes W,U}(J_{VW}\otimes 1)(v\otimes w\otimes u), \end{aligned} \tag{3.15}$$

where $s_{12,3}$ is the permutation of the first two components with the third one. Using the braiding property $\beta_{V\otimes W,U} = (\beta_{VU}\otimes 1)\circ(1\otimes\beta_{WU})$, the associativity of J_{VW}, and the obvious identities $J_{U\otimes V,W}^{-1}F(\beta_{VU}\otimes 1)J_{V\otimes U,W} = F(\beta_{VU})\otimes 1$, $J_{V,U\otimes W}^{-1}F(1\otimes\beta_{WU})$ $J_{V,W\otimes U} = 1\otimes F(\beta_{WU})$, one finds that the right-hand side of (3.15) equals to $R_{13}R_{23}(v\otimes w\otimes u)$, as desired. □

4. Quantization of finite-dimensional Lie bialgebras

Our purpose in this section is to represent the quasitriangular topological Hopf algebra $U_h(\mathfrak{g})$ as a quantum double of another topological Hopf algebra, $U_h(\mathfrak{g}_+)$. The topological Hopf algebra $U_h(\mathfrak{g}_+)$ will be a quantization of the Lie bialgebra $\mathfrak{g}_+$.

4.1. The algebras $U_h(\mathfrak{g}_\pm)$

As we have seen, the fiber functor F which we used to construct the quantum group $U_h(\mathfrak{g})$, is represented by the object $M_+\otimes M_-$ of $\mathcal{M}$. Therefore, we have a homomorphism $\theta : \mathrm{End}(M_+\otimes M_-)\to\mathrm{End}(F) = U_h(\mathfrak{g})$ defined by $\theta(a)v = v\circ S(a)$, $v \in F(V)$, $V \in \mathcal{M}$, $a \in \mathrm{End}(M_+\otimes M_-)$.

Lemma 4.1. *The map θ is an isomorphism.*

Proof. The Lemma follows from Lemma 2.1. $\square$

Thus, we can identify $\mathrm{End}(M_+\otimes M_-)$ with $U_h(\mathfrak{g})$. From now on we make no distinction between them.

Now let us define the subalgebras $U_h(\mathfrak{g}_\pm)\subset U_h(\mathfrak{g})$.

Let $x \in F(M_+)$. Define the endomorphism $m_-(x)$ of $M_+\otimes M_-$ to be the composition of the following morphisms in $\mathcal{M}$: $m_-(x) = (x\otimes 1)\circ(1\otimes i_-)$. This defines a linear map $m_- : F(M_+)\to U_h(\mathfrak{g})$. Denote the image of this map by $U_h(\mathfrak{g}_-)$.

Let $m_-^0(x) \in U(\mathfrak{g}_-)$ be defined by the equation $x(1_+\otimes 1_-) = m_-^0(x)1_+$. It is easy to show that $m_-(x) \equiv m_-^0(x) \mod h$, which implies that m_- is an embedding.

A similar definition can be made for $x \in F(M_-)$. Define the endomorphism $m_+(x)$ of $M_+\otimes M_-$ to be the composition of the following morphisms in $\mathcal{M}$: $m_+(x) = (1\otimes x)\circ(i_+\otimes 1)$. This defines an injective linear map $m_+ : F(M_-)\to U_h(\mathfrak{g})$. Denote the image of this map by $U_h(\mathfrak{g}_+)$.

Proposition 4.2. *$U_h(\mathfrak{g}_\pm)$ are subalgebras in $U_h(\mathfrak{g})$.*

Proof. Let us give a proof for $U_h(\mathfrak{g}_-)$. The proof for $U_h(\mathfrak{g}_+)$ is analogous.

Using Lemma 2.3, we obtain

$$\begin{aligned} m_-(x)\circ m_-(y) &= (x\otimes 1)\circ(1\otimes i_-)\circ(y\otimes 1)\circ(1\otimes i_-) \\ &= (x\otimes 1)\circ(y\otimes 1\otimes 1)\circ(1\otimes 1\otimes i_-)\circ(1\otimes i_-) \\ &= (x\otimes 1)\circ(y\otimes 1\otimes 1)\circ(1\otimes i_-\otimes 1)\circ(1\otimes i_-) = (z\otimes 1)\circ(1\otimes i_-), \end{aligned} \tag{4.1}$$

where $z = x\circ(y\otimes 1)\circ(1\otimes i_-) \in F(M_+)$.

So by the definition we get $m_-(x)\circ m_-(y) = m_-(z)$. $\square$

Note that the algebra $U_h(\mathfrak{g}_-)$ is a deformation of the algebra $U(\mathfrak{g}_-)$. Indeed, we can define a linear isomorphism $\mu : U(\mathfrak{g}_-)[[h]]\to U_h(\mathfrak{g}_-)$ by $\mu(a)(1_+\otimes 1_-) = S(a)1_+$. This isomorphism has the property $\mu(ab) = \mu(a)\circ\mu(b) \mod h^2$, which follows from (3.5), but in general $\mu(ab) \neq \mu(a)\circ\mu(b)$.

The subalgebra $U_h(\mathfrak{g}_-)$ has a unit since it is a deformation of the algebra with unit $U(\mathfrak{g}_-)$. In fact, one can show that the unit equals to $\mu(1)$, $1 \in U(\mathfrak{g}_-)$.

Similar statements apply to the algebra $U(\mathfrak{g}_+)$.

Proposition 4.3. *The map $U_h(\mathfrak{g}_+)\otimes U_h(\mathfrak{g}_-)\to U_h(\mathfrak{g})$ given by $a\otimes b\to ab$ is an isomorphism.*

Proof. The statement is true because it holds modulo h. $\square$

4.2. Polarization of the R-matrix

Define the element $\tilde{R} \in U_h(\mathfrak{g}_+)\otimes U_h(\mathfrak{g}_-)$ by the identity

$$\tilde{R}\circ\beta^{-1}\circ(i_+\otimes i_-) = \beta \tag{4.2}$$

in $\mathrm{Hom}(M_+\otimes M_-, M_-\otimes M_+)$. It is obvious that such an element is unique. It can be computed as follows.

Let $\nu : M_\pm[[h]]\to U_h(\mathfrak{g}_\mp)$ be the linear isomorphism defined by the equation $\nu(x(1_+\otimes 1_-)) = m_\mp(x)$ for any $x \in F(M_\pm)$. Let $K \in U(\mathfrak{g})^{\otimes 2}[[h]]$ be given by

$$K = (\phi^{-1}\otimes\phi^{-1})\Big(\Phi^{-1}_{1,2,34}(1\otimes\Phi_{2,3,4})se^{-h\Omega_{23}/2}(1\otimes\Phi^{-1}_{2,3,4})\Phi_{1,2,34}(1_+\otimes 1_+\otimes 1_-\otimes 1_-)\Big). \tag{4.3}$$

Then it is easy to check, using (4.2), that

$$\tilde{R} = (\nu\otimes\nu)(K^{-1}e^{h\Omega/2}(1_-\otimes 1_+)). \tag{4.4}$$

Proposition 4.4. $\tilde{R} = R$.

Proof. According to (3.10), the R-matrix $R \in U_h(\mathfrak{g})\otimes U_h(\mathfrak{g})$ is defined by the condition that for any $V, W \in \mathcal{M}$ and $v \in F(V)$, $w \in F(W)$ one has the equality

$$\begin{aligned} &R^{op}(v\otimes w)\circ\beta_{23}\circ(i_+\otimes i_-) \\ &= \beta\circ(w\otimes v)\circ\beta_{23}\circ(i_+\otimes i_-) \end{aligned} \tag{4.5}$$

in $\mathrm{Hom}(M_+\otimes M_-, V\otimes W)$.

By the functoriality of the braiding, $R^{op}(v\otimes w) = \beta\circ R(w\otimes v)\circ\beta^{-1}_{12,34}$. Besides, $\beta_{12,34} = \beta_{23}\circ\beta_{12}\circ\beta_{34}\circ\beta_{23}$. Substituting this into (4.5) and taking into account that $\beta\circ i_\pm = i_\pm$, we get

$$\begin{aligned} &R(w\otimes v)\circ\beta^{-1}_{23}\circ(i_+\otimes i_-) \\ &= (w\otimes v)\circ\beta_{23}\circ(i_+\otimes i_-) \end{aligned} \tag{4.6}$$

in $\mathrm{Hom}(M_+\otimes M_-, W\otimes V)$.

To show that $R = \tilde{R}$ we have to prove the identity

$$\begin{aligned} &(1\otimes\tilde{R}\otimes 1)\circ(i_+\otimes 1\otimes 1\otimes i_-)\circ\beta^{-1}_{23}\circ(i_+\otimes i_-) \\ &= \beta_{23}\circ(i_+\otimes i_-) \end{aligned} \tag{4.7}$$

in $\mathrm{Hom}(M_+\otimes M_-, M_+\otimes M_-\otimes M_+\otimes M_-)$.

Interchanging the order of factors on the left-hand side of (4.7) and using Lemma 2.3, we can rewrite (4.7) in the form

$$\begin{aligned} &(1\otimes\tilde{R}\otimes 1)\circ\beta^{-1}_{34}\circ(1\otimes i_+\otimes i_-\otimes 1)\circ(i_+\otimes i_-) \\ &= \beta_{23}\circ(i_+\otimes i_-) \end{aligned} \tag{4.8}$$

in $\mathrm{Hom}(M_+\otimes M_-, M_+\otimes M_-\otimes M_+\otimes M_-)$.

It is obvious that identity (4.8) follows from the definition of $\tilde{R}$. The proposition is proved. □

4.3. Subalgebras $U_h(\mathfrak{g}_\pm)$ in terms of the R-matrix

Let $U_h(\mathfrak{g}_\pm)^* = \mathrm{Hom}_{\mathcal{A}}(U_h(\mathfrak{g}_\pm), k[[h]])$. Define $k[[h]]$-linear maps $\rho_\pm : U_h(\mathfrak{g}_\mp)^* \to U_h(\mathfrak{g}_\pm)$, by $\rho_+(f) = (1\otimes f)(R)$, $\rho_-(f) = (f\otimes 1)(R)$. Let $U_\pm$ be the images of the maps $\rho_\pm$, and $\tilde{U}_\pm$ be the closures of the $k[[h]]$-subalgebras generated by $U_\pm$.

Proposition 4.5.

$$U_h(\mathfrak{g}_\pm)\otimes_{k[[h]]}k((h)) \text{ is the } h\text{-adic completion of } \tilde{U}_\pm\otimes_{k[[h]]}k((h)).$$

Proof. We prove the statement for $\tilde{U}_+$. The proof for $\tilde{U}_-$ is similar.

We start with the following statement.

Lemma 4.6. *For any $x \in U(\mathfrak{g}_+)$ there exists an element $t_x \in \tilde{U}_+\otimes k((h))$ such that $t_x = x + O(h)$. If x has degree $\leq m$ with respect to the standard filtration in $U(\mathfrak{g}_+)$, then t_x can be chosen in such a way that $h^m t_x \in \tilde{U}_+$.*

Proof of the lemma. It is clear that $1 \in \tilde{U}_+$ since $1 = \rho_+(\varepsilon)$. So we can set $t_1 = 1$.

Now consider the case $x \in \mathfrak{g}_+$. Let $f \in U_h(\mathfrak{g}_-)^*$ be any element such that $f(1) = 0$ and $f(\tilde{a}) = \langle x, a\rangle$ for any $a \in \mathfrak{g}_-$ and $\tilde{a} \in U_h(\mathfrak{g}_-)$ such that $\tilde{a} = a \mod h$. Then it follows from (3.13) that $\rho_+(f) = hx + O(h^2)$. So we can let $t_x = h^{-1}\rho_+(f)$. Thus, the Lemma is true for $x \in \mathfrak{g}_+$. Since $\tilde{U}_+$ is an algebra, the validity of the Lemma for $x \in \mathfrak{g}_+$ implies its validity for any $x \in U(\mathfrak{g}_+)$. □

Now we can prove the proposition. Let $T_0 \in U_h(\mathfrak{g}_+)$. Let $x_0 \in U(\mathfrak{g}_+)$ be the reduction of $T_0 \mod h$. Then $T_0 - t_{x_0}$ is divisible by h, so we can consider $T_1 = h^{-1}(T_0 - t_{x_0})$ and repeat our procedure. This gives us a sequence $x_i \in U(\mathfrak{g}_+)$, and $T_0 = \sum_{m\geq 0} t_{x_m} h^m$. This shows that T_0 belongs to the h-adic completion of $\tilde{U}_+\otimes k((h))$, as desired. □

Theorem 4.7. *The subalgebras $U_h\mathfrak{g}_\pm$ are Hopf subalgebras in $U_h(\mathfrak{g})$.*

Proof. The fact that $U_h(\mathfrak{g}_\pm)$ are closed under the comultiplication Δ follows from Proposition 4.5 and identities (3.12). The fact that $U_h(\mathfrak{g}_\pm)$ are closed under the antipode S follows from Proposition 4.5 and the identity $(S\otimes 1)(R) = R^{-1}$, which holds in any quasitriangular Hopf algebra. □

Remark. In fact, it is possible to prove the following explicit formula for coproduct in $U_h(\mathfrak{g}_\mp)$: for any $x \in F(M_\pm)$

$$\Delta(m_\mp(x)) = (m_\mp\otimes m_\mp)(J^{-1}_{M_\pm M_\pm}(i_\pm \circ x)). \tag{4.9}$$

The proof is a direct verification. A similar formula is contained in Proposition 9.3.

It is obvious that $U_h(\mathfrak{g}_+)/hU_h(\mathfrak{g}_+)$ is isomorphic to $U(\mathfrak{g}_+)$ as a Hopf algebra. Therefore, $U_h(\mathfrak{g}_+)$ is a quantized universal enveloping algebra. It follows from Proposition 3.6 that its quasiclassical limit is the Lie bialgebra $\mathfrak{g}_+$. Similar statements apply to $U_h(\mathfrak{g}_-)$.

4.4. Duality of quantized universal enveloping algebras and the quantum double

The following general constructions can be found in [Dr1].

If A is a quantized universal enveloping algebra then the dual $A^* = \mathrm{Hom}_{\mathcal{A}}(A, k[[h]])$ carries a natural structure of a topological algebra. Namely, for any $x, y \in A$, $f, g \in A^*$ $fg(x) = (f \otimes g)(\Delta(x))$, and the unit is ε. It can be shown that A^* has a unique maximal ideal I^*, which is the kernel of the linear map $A \to k$ given by $f \to f(1) \mod h$. The topology on A^* is defined by the condition that $\{(I^*)^n, n \geq 0\}$ is a basis of neighborhoods of zero. This implies that the topological algebras $(A \otimes A)^*$ and $A^* \otimes A^*$ are isomorphic.

The algebra A^* has a natural structure of a topological Hopf algebra. Namely, the coproduct is defined by $\Delta(f)(x \otimes y) = f(xy)$, the counit is 1, and the antipode is S^*. (The definition of coproduct makes sense since the algebra $A^* \otimes A^*$ is isomorphic to $(A \otimes A)^*$.)

As a topological $k[[h]]$-module, A^* is isomorphic to $k[[X_1, \ldots, X_N]][[h]]$.

Let A be any quantized universal enveloping algebra. Let A^* be the dual algebra, and let I^* be the maximal ideal in A^*. Consider the h-adic completion $A^\vee$ of the subalgebra $\sum_{n \geq 0} h^{-n}(I^*)^n$ in the algebra $A^* \otimes_{k[[h]]} k((h))$. Then $A^\vee$ is a new quantized universal enveloping algebra [Dr1]. This algebra is called the dual quantized universal enveloping algebra to A.

The algebra A^* can be identified with a subalgebra in $A^\vee$ which is constructed as follows:

Let $\Delta^n : A \to A^{\otimes n}$ be the iterated coproduct maps: $\Delta^0(a) = \varepsilon(a)$, $\Delta^1(a) = a$, $\Delta^2(a) = \Delta(a)$,$\Delta^n(a) = (\Delta \otimes 1^{\otimes(n-2)})(\Delta^{n-1}(a))$, $n > 2$.

Let $\Sigma = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$, and $i_1 < \cdots < i_k$. Let $j_\Sigma : A^{\otimes k} \to A^{\otimes n}$ be the homomorphism defined by $j_\Sigma(a_1 \otimes \ldots \otimes a_k) = b_1 \otimes \ldots \otimes b_n$, $a_1, \ldots, a_k \in A$, where $b_i = 1$ if $i \notin \Sigma$, and $b_{i_m} = a_m$, $m = 1, \ldots, k$.

Let $\Delta_\Sigma(a) = j_\Sigma(\Delta^k(a))$, $a \in A$.

Define linear mappings $\delta_n : A \to A^{\otimes n}$ for all $n \geq 1$ by

$$\delta_n(a) = \sum_{\Sigma \subset \{1, \ldots, n\}} (-1)^{n - |\Sigma|} \Delta_\Sigma(a)$$

and a Hopf subalgebra $A' = \{a \in A : \delta_n(a) \in h^n A^{\otimes n}\}$ in A.

It is easy to check that $A^* = (A^\vee)'$. If A is any Hopf algebra, let A^{op} denote the Hopf algebra A with the comultiplication Δ replaced by Δ^{op}, and the antipode S replaced with S^{-1}. A^{op} is also a Hopf algebra.

Now we can define the notion of the quantum double. Let A be a quantized universal enveloping algebra. Consider the $k[[h]]$-module $D(A) = A \otimes (A^\vee)^{op}$. Let $R \in A \otimes A^* \subset A \otimes (A^\vee)^{op}$ be the canonical element. We can regard R as an element of $D(A) \otimes D(A)$ using the embedding $A \otimes (A^\vee)^{op} \to D(A) \otimes D(A)$ given by

$x\otimes y \to x\otimes 1\otimes 1\otimes y$. Drinfeld [Dr1] showed that there exists a unique structure of a topological Hopf algebra on $D(A)$ such that

(1) $A\otimes 1$, $1\otimes(A^\vee)^{op}$ are Hopf subalgebras in $D(A)$,
(2) R defines a quasitriangular structure on $D(A)$, i.e. is invertible and satisfies (3.12), (3.13), and
(3) The linear mapping $A\otimes(A^\vee)^{op}\to D(A)$ given by $a\otimes b\to ab$ is bijective.

$D(A)$, equipped with this structure, is a quasitriangular quantized universal enveloping algebra. It is called the quantum double of A.

4.5. The quantum double of $U_h(\mathfrak{g}_+)$

Proposition 4.8. *ρ_+ is a homomorphism of topological Hopf algebras $(U_h(\mathfrak{g}_-)^{op})^* \to U_h(\mathfrak{g}_+)$. ρ_- is a homomorphism of topological Hopf algebras $U_h(\mathfrak{g}_-)^*\to U_h(\mathfrak{g}_+)^{op}$.*

Proof. We only prove the first statement. The second one is proved analogously.

It is clear that ρ_+ is continuous. Also, for any $f,g\in (U_h(\mathfrak{g}_-)^{op})^*$ one has

$$\begin{aligned}\rho_+(fg) &= (1\otimes fg)(R) = (1\otimes f\otimes g)((1\otimes\Delta^{op})(R)) = (1\otimes f\otimes g)(R_{12}R_{23})\\ &= (1\otimes f)(R)\cdot(1\otimes g)(R) = \rho_+(f)\rho_+(g);\\ \Delta(\rho_+(f)) &= \Delta((1\otimes f)(R)) = (1\otimes 1\otimes f)((\Delta\otimes 1)(R))\\ &= (1\otimes 1\otimes f)(R_{13}R_{23}) = (1\otimes 1\otimes\Delta(f))(R_{13}R_{24}) = (\rho_+\otimes\rho_+)(\Delta(f)).\end{aligned}$$

It is obvious that $\rho_+(1)=1$ and $\varepsilon(\rho_+(f))=\varepsilon(f)$ for any f. Also, it is easy to check that $\rho_+((S^{-1})^*f)=S(\rho_+(f))$. The proposition is proved. □

Corollary 4.9. *$U_\pm$ are Hopf subalgebras in $U_h(\mathfrak{g}_\pm)$. In particular, $\tilde U_\pm = U_\pm$.*

Proof. The first statement is clear. The second statement follows from the first one and the fact that $U_\pm$ is closed in $U_h(\mathfrak{g}_\pm)$, which is easy to check. □

Proposition 4.10. *The maps ρ_+, ρ_- are injective.*

Proof. We show the injectivity of ρ_+ (the case of ρ_- is similar). Fix an element $f\in U_h(\mathfrak{g}_-)^*$, $f\neq 0$. We can always assume that $f\neq 0 \mod h$. Let $x\in U(\mathfrak{g}_-)$ be such that $f(t_x)\neq 0 \mod h$ (where t_x was defined in Lemma 4.6), $n\geq 0$ be such that $h^nt_x\in U_-$, and $g\in U_h(\mathfrak{g}_+)^*$ be such that $\rho_-(g)=h^nt_x$. Such a g exists by the definition of n. Then $g(\rho_+(f))=(g\otimes f)(R)=f(\rho_-(g))=h^nf(t_x)\neq 0$. Therefore, $\rho_+(f)\neq 0$. □

Proposition 4.11. *$U_\pm = U_h(\mathfrak{g}_\pm)'$.*

Proof. We give the proof for U_+. The proof for U_- is similar.

First we need the following statement.

Lemma 4.12. *Let $t \in U_h(\mathfrak{g}_+)'$ be an element such that $h^{-n}t \in U_h(\mathfrak{g}_+)$ and $h^{-n}t = x + O(h)$, $x \in U(\mathfrak{g}_+)$, $x \neq 0$. Then x has degree $\leq n$.*

Proof of the lemma. By the definition, $\delta_{n+1}(h^{-n}t)$ is divisible by h. On the other hand, $\delta_{n+1}(h^{-n}t) = \delta_{n+1}(x) + O(h)$. Thus, $\delta_{n+1}(x) = 0$, which implies that the degree of x is $\leq n$, since the kernel of δ_{n+1} on $U(\mathfrak{g}_+)$ is the set of all elements of $U(\mathfrak{g}_+)$ whose degree is $\leq n$. □

Now we can prove the proposition. By Lemma 4.6, for any $x \in U(\mathfrak{g}_+)$ of degree $\leq n$, an element t_x can be chosen in such a way that $h^n t_x \in U_+$. This implies the inclusion $U_+ \supset U_h(\mathfrak{g}_+)'$. Indeed, let $T_0 \in U_h(\mathfrak{g}_+)'$, and $T_0 \equiv h^m x_0 \mod h^{m+1}$, where $x_0 \in U(\mathfrak{g}_+)$. Then, according to Lemma 4.12, the degree of x_0 is $\leq m$. Therefore, $h^m t_{x_0} \in U_+$. Thus, $T_1 = T_0 - h^m t_{x_0} \in U_+$ and is divisible by h^{m+1}, so we can repeat our procedure. This gives us a sequence of elements $x_i \in U(\mathfrak{g}_+)$ of degrees m_i ($m_0 = m$), such that $m_0 < m_1 < \cdots < m_i < \dots$, and $T_0 = \sum_{i\geq 0} t_{x_i} h^{m_i}$. This shows that T_0 belongs to U_+, as desired.

To demonstrate the inclusion $U_+ \subset U_h(\mathfrak{g}_+)'$, observe that according to (3.12),

$$(\Delta^n \otimes 1)(R) = R_{1n+1} \dots R_{nn+1}.$$

This implies that

$$(\delta_n \otimes 1)(R) = (R_{1n+1} - 1) \dots (R_{nn+1} - 1) = O(h^n).$$

Therefore, $\delta_n(\rho_+(f))$ is divisible by h^n for any $f \in U_h(\mathfrak{g}_-)^*$. □

Comparing our results with the definitions of the previous section, we see that we have obtained the following result.

Theorem 4.13. *Let $\mathfrak{g}_+$ be a finite-dimensional Lie bialgebra and $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ the associated Manin triple. Then*

(i) *There exist quantized universal enveloping algebras $U_h(\mathfrak{g})$ and $U_h(\mathfrak{g}_\pm) \subset U_h(\mathfrak{g})$, which are quantizations of the Lie bialgebras $\mathfrak{g}$, $\mathfrak{g}_\pm \subset \mathfrak{g}$ respectively;*
(ii) *The multiplication map $U_h(\mathfrak{g}_+) \otimes U_h(\mathfrak{g}_-) \to U_h(\mathfrak{g})$ is a linear isomorphism;*
(iii) *The algebras $U_h(\mathfrak{g}_+)$, $U_h(\mathfrak{g}_-)^{op}$ are dual each other as quantized universaal enveloping algebras, in the sense of Drinfeld [Dr1];*
(iv) *The factorization $U_h(\mathfrak{g}) = U_h(\mathfrak{g}_+) U_h(\mathfrak{g}_-)$ defines an isomorphism of $U_h(\mathfrak{g})$ with the quantum double of $U_h(\mathfrak{g}_+)$;*
(v) *$U_h(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g})[[h]]$ as a topological algebra.*

5. Quantization of solutions of the classical Yang–Baxter equation

Let A be an associative algebra over k with unit, and $r \in A \otimes A$. The element r is called a classical r-matrix if it satisies the classical Yang–Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{5.1}$$

We say that r is unitary if $r^{op} = -r$. An algebra A equipped with a classical r-matrix r is called a classical Yang–Baxter algebra. A is called unitary if r is unitary.

Let A be a topological algebra over $k[[h]]$. Let $R \in A\otimes A$. We say that R is a quantum R-matrix if it satisfies the quantum Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{5.2}$$

We say that R is unitary if $R^{op} = R^{-1}$. A topological algebra A equipped with a quantum R-matrix R is called a quantum Yang–Baxter algebra. A is called unitary if R is unitary.

The following theorem answers question 3.1 in [Dr3]. It shows that any classical Yang–Baxter algebra can be quantized.

Theorem 5.1. *Let A be an associative algebra with unit over k, and $r \in A\otimes A$ be a classical r-matrix. Then there exists a quantum R-matrix $R \in A\otimes A[[h]]$ such that $R = 1 + hr \mod h^2$. If in addition r is unitary then R is also unitary.*

Proof. We start with a construction of Reshetikhin and Semenov-Tian-Shansky [RS]. Let $\mathfrak{g}_+ = \{(1\otimes f)(r),\ f \in A^*\}$, $\mathfrak{g}_- = \{(f\otimes 1)(r),\ f \in A^*\}$ be vector subspaces in A. It is clear that $\mathfrak{g}_+$, $\mathfrak{g}_-$ are finite-dimensional, $r \in \mathfrak{g}_+\otimes\mathfrak{g}_-$, and the map $\chi_r : \mathfrak{g}_+^* \to \mathfrak{g}_-$ defined by $\chi_r(f) = (f\otimes 1)(r)$, is an isomorphism of vector spaces.

Remark. Note that the spaces $\mathfrak{g}_+$ and $\mathfrak{g}_-$ may intersect nontrivially and even coincide.

Lemma 5.2. $\mathfrak{g}_+$, $\mathfrak{g}_-$ *are Lie subalgebras in A.*

Proof. Let $x, y \in \mathfrak{g}_+$, $x = (1\otimes f)(r)$, $y = (1\otimes g)(r)$. Using (5.1), we have

$$[xy] = (1\otimes f\otimes g)([r_{12}r_{23}]) = -(1\otimes f\otimes g)([r_{12} + r_{13}, r_{23}]) = (1\otimes h)(r), \tag{5.3}$$

where $h \in A^*$, $h(a) = (f\otimes g)([r, a\otimes 1 + 1\otimes a])$. Thus, $[xy] \in \mathfrak{g}_+$, i.e. $\mathfrak{g}_+$ is a Lie algebra. The proof for $\mathfrak{g}_-$ is similar. □

Let $\mathfrak{g} = \mathfrak{g}_+\otimes\mathfrak{g}_-$ be a vector space. Define the skew-symmetric bracket $[\,,] : \mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ as follows. If $x, y \in \mathfrak{g}_+$ or $x, y \in \mathfrak{g}_-$ then the bracket $[xy]$ is the Lie bracket in $\mathfrak{g}_+$ or $\mathfrak{g}_-$, respectively. If $x \in \mathfrak{g}_+$, $y \in \mathfrak{g}_-$, then $[xy]$ is defined by

$$[xy] = (\mathrm{ad}^*x)y - (\mathrm{ad}^*y)x. \tag{5.4}$$

Let $\pi : \mathfrak{g}\to A$ be the linear map whose restrictions to $\mathfrak{g}_+$, $\mathfrak{g}_-$ are the corresponding embeddings. The restrictions of π to $\mathfrak{g}_+$, $\mathfrak{g}_-$ are injective but in general π itself is not an embedding.

Lemma 5.3. $\pi([xy]) = [\pi(x), \pi(y)]$, $x, y \in \mathfrak{g}$.

Proof. The Lemma is a direct consequence of the classical Yan–Baxter equation. □

Lemma 5.4. $(\mathfrak{g}, [\,,\,])$ *is a Lie algebra.*

Proof. We have to check the Jacobi identity in $\mathfrak{g}$. It is enough to check it for three elements a, x, y such that $a \in \mathfrak{g}_+$, $x, y \in \mathfrak{g}_-$. For brevity we write $a(x)$ for $(\mathrm{ad}^* a)x$. We have

$$\begin{aligned} [a[xy]] &= a([xy]) - [xy](a), \\ [y[ax]] &= [y, a(x) - x(a)] = [y, a(x)] - y(x(a)) + y(a(x)), \\ [x[ya]] &= [x, y(a) - a(y)] = -[x, a(y)] + x(y(a)) - x(a(y)). \end{aligned} \tag{5.5}$$

Adding these three identities, and using the fact that $[xy](a) = x(y(a)) - y(x(a))$, we get

$$[a[xy]] + [y[ax]] + [x[ya]] = a([xy]) + [y, a(x)] - [x, a(y)] + y(a(x)) - x(a(y)). \tag{5.6}$$

Denote the right-hand side of (5.6) by X. Applying π to both sides of (5.6), and using Lemma 5.3 and the Jacobi identity in A, we get

$$\pi(X) = 0. \tag{5.7}$$

Since $X \in \mathfrak{g}_+$, and π is injective on $\mathfrak{g}_+$, we get $X = 0$, which implies the Jacobi identity in $\mathfrak{g}$. □

Let $\langle\,,\,\rangle$ be the inner product on $\mathfrak{g}$ such that $\langle x_+ + x_-, y_+ + y_- \rangle = x_- \cdot y_+ + y_- \cdot x_+$, where $x_+, y_+ \in \mathfrak{g}_+$, $x_-, y_- \in \mathfrak{g}_-$, and the dot denotes the natural pairing $\mathfrak{g}_- \otimes \mathfrak{g}_+ \to k$ defined by the map χ_r. This inner product is ad-invariant. Thus, $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple.

Now we can finish the proof of the theorem. Lemma 5.3 implies that $\pi : \mathfrak{g} \to A$ is a homomorphism of Lie algebras. Therefore, it extends to a homomorphism of associative algebras $\pi : U(\mathfrak{g}) \to A$. Furthermore, $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple. The Lie bialgebra $\mathfrak{g}$ is quasitriangular, and its quasitriangular structure is defined by the classical r-matrix $\tilde{r} = \sum x_+^i \otimes x_-^i$, where x_+^i is a basis of $\mathfrak{g}_+$, and x_-^i is a dual basis of $\mathfrak{g}_-$. Note that $(\pi \otimes \pi)(\tilde{r}) = r$.

By Theorem 4.13, there exists a quasitriangular topological Hopf algebra $U_h(\mathfrak{g})$, with a quasitriangular structure $\tilde{R} \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$. Moreover, the associative algebra $U_h(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g})[[h]]$, and the isomorphism can be chosen to be the identity modulo h. Thus, we can assume that $\tilde{R} \in (U(\mathfrak{g}) \otimes U(\mathfrak{g}))[[h]]$.

Set $R = (\pi \otimes \pi)(\tilde{R})$. From what we said above it follows that R satisfies (5.2) and $R = 1 + hr$ modulo h^2.

Assume now that $r^{op} = -r$. Let $\tilde{\Omega} = \tilde{r} + \tilde{r}^{op}$. It follows immediately from the construction of $\tilde{R}$ that $\tilde{R}^{op}\tilde{R}$ is conjugate to $e^{h\tilde{\Omega}}$. But $(\pi \oplus \pi)(\tilde{\Omega}) = r + r^{op} = 0$. This implies that $R^{op}R = 1$, as desired. The theorem is proved. □

Let $\mathcal{R}$ be the ring of algebraic functions of a variable h with coefficients in k which are regular at $h = 0$.

Theorem 5.5. *Let A be a finite-dimensional associative algebra with unit over k and $r \in A\otimes A$ be a classical r-matrix. Then there exists a family of quantum R-matrices $R(h) \in A\otimes A\otimes\mathcal{R}$ such that $R = 1 + hr + O(h^2)$, $h\to 0$. If in addition r is unitary then $R(h)$ is also unitary.*

Proof. The theorem follows immediately from Theorem 5.1 and the following result of M. Artin [Ar].

Theorem. *Any system of polynomial equations in indeterminates $x_1, \ldots, x_n$ with coefficients in $k[h]$ which has solutions over $k[[h]]$ also has solutions over $\mathcal{R}$.*

Indeed, let us write R in the form $R = 1 + hr + h^2X(h)$, and look for a series $X(h)$ such that R satisfies the quantum Yang–Baxter equation, and the unitarity condition in the case when r is unitary. This is a system of polynomial equations on the components of $X(h)$ with coefficients in $k[h]$. By Theorem 5.1, it has solutions over $k[[h]]$. Therefore, by Artin's theorem, it has solutions over $\mathcal{R}$.

6. Quantization of quasitriangular Lie bialgebras

6.1. Quasitriangular quantization of quasitriangular Lie bialgebras

In this section we give a recipe of quantization of a quasitriangular Lie bialgebra $\mathfrak{a}$ (not necessarily finite-dimensional), which produces a quantized universal enveloping algebra isomorphic to $U(\mathfrak{a})[[h]]$ as a topological algebra. This answers questions from Section 4 of [Dr3].

Let $\mathfrak{g}_+ = \{(1\otimes f)(r),\ f \in \mathfrak{a}^*\}$, $\mathfrak{g}_- = \{(f\otimes 1)(r),\ f \in \mathfrak{a}^*\}$ be subspaces in $\mathfrak{a}$. By Lemma 5.2, applied to $A = U(\mathfrak{a})$, these subspaces are finite-dimensional Lie subalgebras in $\mathfrak{a}$. Moreover, let $\mathfrak{g}$ be the vector space $\mathfrak{g}_+\oplus\mathfrak{g}_-$. This space is a Lie algebra with bracket defined by (5.4) and an invariant inner product. By Lemma 5.3, we have a natural homomorphism of Lie algebras $\pi : \mathfrak{g}\to\mathfrak{a}$, and it is easy to see that this homomorphism is a morphism of quasitriangular Lie bialgebras.

Let $\mathcal{M}_\mathfrak{a}$ be the category whose objects are $\mathfrak{a}$-modules, and morphisms are defined by $\mathrm{Hom}_{\mathcal{M}_\mathfrak{a}}(V, W) = \mathrm{Hom}_\mathfrak{a}(V, W)[[h]]$. Let $\mathcal{M}_\mathfrak{a}$ be the Drinfeld category associated to $\mathfrak{g}$. We have the pullback functor $\pi^* : \mathcal{M}_\mathfrak{a}\to\mathcal{M}_\mathfrak{g}$. Define the braided monoidal structure on $\mathcal{M}_\mathfrak{a}$ to be the pullback of the braided monoidal structure on $\mathcal{M}_\mathfrak{g}$. This definition makes sense, since the element $\Omega = r + r^{op} \in \mathfrak{g}\otimes\mathfrak{g}$ is $\mathfrak{g}$-invariant by the definition of a quasitriangular Lie bialgebra.

Let M_+, M_- be Verma modules over $\mathfrak{g}$. Define a functor $F : \mathcal{M}_\mathfrak{a}\to\mathcal{A}$ by $F(V) = \mathrm{Hom}_{\mathcal{M}_\mathfrak{g}}(M_+\otimes M_-, \pi^*(V))$. The tensor structure on F is introduced in the same way as in Section 1.8. Let $H = \mathrm{End}F$. Since the functor F is isomorphic to the "forgetful" functor $V\to$ "the $k[[h]]$-module $V[[h]]$", the algebra H is isomorphic to $U(\mathfrak{a})[[h]]$ as a topological algebra over $k[[h]]$. On the other hand, H has a natural coproduct and antipode defined analogously to Section 3.2, and a quasitriangular structure $R \in H\otimes H$ defined analogously to Section 3.5. It is easy to check that the quasiclassical limit of H is the Lie bialgebra $\mathfrak{a}$, and $R = 1 + hr + O(h^2)$, so r is the quasiclassical limit of R.

Furthermore, suppose that the original Lie bialgebra $\mathfrak{a}$ is triangular, i.e. r is a unitary r-matrix. Then $\Omega = r + r^{op} = 0$, and hence $R^{op}R = J^{-1}e^{h\Omega}J = 1$ so the Hopf algebra H is triangular, too.

Thus, we have the following theorem:

Theorem 6.1. *Any quasitriangular Lie bialgebra $\mathfrak{a}$ admits a quantization $U_h^{qt}(\mathfrak{a})$ which is a quasitriangular quantized universal enveloping algebra isomorphic to $U(\mathfrak{a})[[h]]$ as a topological algebra. If $\mathfrak{a}$ is trianglar, so is $U_h^{qt}(\mathfrak{a})$.*

Remark. In the second paper of this series, we will show that as a topological Hopf algebra, $U_h^{qt}(\mathfrak{a})$ is isomorphic to $U_h(\mathfrak{a})$.

6.2. Representations of $U_h(\mathfrak{a})$

Let $\mathfrak{a}$ be a quasitriangular Lie bialgebra (not necessarily finite-dimensional). By a representation of $U_h^{qt}(\mathfrak{a})$ we mean a topologically free $k[[h]]$-module V together with a homomorphism $\pi_V : U_h^{qt}(\mathfrak{a}) \to \mathrm{End}_{k[[h]]}V$. Representations of $U_h(\mathfrak{a})$ form a braided tensor category, with the trivial associativity morphism and braiding defined by the R-matrix. Denote this category by $\mathcal{R}$.

The functor $F : \mathcal{M}_\mathfrak{a} \to \mathcal{A}$ can be regarded as a functor from $\mathcal{M}_\mathfrak{a}$ to $\mathcal{R}$, since for any $W \in \mathcal{M}_\mathfrak{a}$ the $k[[h]]$-module $F(W)$ is equipped with a natural action of $U_h^{qt}(\mathfrak{a})$. We denote this new functor also by F. This functor inherits the tensor structure defined by the maps J_{VW}.

Theorem 6.2. *The functor F defines an equivalence of braided tensor categories $\mathcal{M}_\mathfrak{a} \to \mathcal{R}$.*

Proof. The theorem follows from the definition of the functor F, the algebra $U_h^{qt}(\mathfrak{a})$ and the R-matrix R. □

Part II

7. Drinfeld category for an arbitrary Lie bialgebra

7.1. Topological vector spaces

Recall the definition of the product topology. Let S be a set, T a topological space, and T^S the space of functions from S to T. This space has a natural weak topology, which is the weakest of the topologies in which all the evaluation maps

$T^S \to T$, $f \to f(s)$, are continuous. Namely, let B be a basis of the topology on T. For any integer $n \geq 1$, elements $s_1, \ldots, s_n \in S$, and open sets $U_1, \ldots, U_n \in B$, define $V(s_1, \ldots, s_n, U_1, \ldots, U_n) = \{f \in T^S : f(s_i) \in U_i,\ i = 1, \ldots, n\}$. Let $\mathcal{B}$ be the collection of all such sets V. This is a basis of a topology on T^S which is called the weak topology. The obtained topological space is the product of copies of T corresponding to elements of S. If X is any subset in T^S, the weak topology on T^S induces a topology on X. We will call it the weak topology as well.

Let k be a field of characteristic zero with the discrete topology. Let V be a topological vector space over k. The topology on V is called linear if open subspaces of V form a basis of neighborhoods of 0.

Remark. It is clear that in any topological vector space, an open subspace is also closed.

Let V be a topological vector space over k with linear topology. V is called separated if the map $V \to \varprojlim(V/U)$ is a monomorphism, where U runs over open subspaces of V.

Topology on all vector spaces we consider in this paper will be linear and separated, so we will say "topological vector space" for "separated topological vector space with linear topology".

Let M, N be topological vector spaces over k. We denote by $\mathrm{Hom}_k(M, N)$ the space of continuous linear operators from M to N, equipped with the weak topology. A basis of neighborhoods of zero in $\mathrm{Hom}_k(M, N)$ is generated by sets of the form $\{A \in \mathrm{Hom}_k(M, N) : Av \in U\}$, where $v \in M$, and $U \subset N$ is an open set.

In particular, if $N = k$ with the discrete topology, the space $\mathrm{Hom}_k(M, N)$ is the space of all continuous linear functionals on M, which we denote by M^*. It is clear that a basis of neighborhoods of zero in M^* consists of orthogonal complements of finite-dimensional subspaces in M. In particular, if M is discrete then the canonical embedding $M \to (M^*)^*$ is an isomorphism of linear spaces. However, if M is infinite-dimensional, this embedding is not an isomorphism of topological vector spaces since the space $(M^*)^*$ is not discrete.

7.2. Complete vector spaces

Let V be a topological vector space over k. V is called complete if the map $V \to \varprojlim(V/U)$ is a epimorphism, where U runs over open subspaces of V.

In particular, if a complete space M has a countable basis of neighborhoods of 0, then there exists a filtration $M = M_0 \supset M_1 \supset \ldots$, such that $\cap_{n \geq 0} M_n = 0$, and $\{M_n\}$ is a basis of neighborhoods of zero in M. In this case $M = \lim_{n \to \infty} M/M_n$.

Examples.

1. Any discrete vector space is complete.

2. If V is a discrete vector space then the topological space $M = V[[h]]$ of formal power series in h with coefficients in V is a complete vector space.

Let V be a complete vector space, $U \subset V$ an open subspace. Then U is complete and V/U is discrete.

Let V, W be complete vector spaces. Consider the space $V \hat{\otimes} W = \varprojlim V/V_1 \otimes W/W_1$, where the projective limit is taken over open subspaces $V_1 \subset V$, $W_1 \subset W$. It is easy to see that $V \hat{\otimes} W$ is a complete vector space. We call the operation $\hat{\otimes}$ the completed tensor product.

A basis of neighborhoods of 0 in $V \hat{\otimes} W$ is the collection of subspaces $V \hat{\otimes} W_1 + V_1 \hat{\otimes} W$, where V_1, W_1 are open subspaces in V, W.

Example. Let V be a discrete space. Then $V \hat{\otimes} k[[h]] = V[[h]]$.

Complete vector spaces form an additive category in which morphisms are continuous linear operators. This category, equipped with tensor product $\hat{\otimes}$, is a symmetric tensor category.

7.3. Equicontinuous $\mathfrak{g}$-modules

Let M be a topological vector space over k, and $\{A_x, x \in X\}$ be a family of elements of $\mathrm{End}M$. We say that the family $\{A_x\}$ is equicontinuous if for every neighborhood of the origin $U \subset M$ there exists another neighborhood of the origin $U' \subset M$ such that $A_x U' \subseteq U$ for all $x \in X$. For example, if M is complete and $A \in \mathrm{End}M$ is any continuous linear operator, then $\{\lambda A, \lambda \in k\}$ is equicontinuous.

Fix a topological Lie algebra $\mathfrak{g}$.

Definition. Let M be a complete vector space. We say that M is an equicontinuous $\mathfrak{g}$-module if one is given a continuous homomorphism of topological Lie algebras $\pi : \mathfrak{g} \to \mathrm{End}M$, such that the family of operators $\pi(g), g \in \mathfrak{g}$, is equicontinuous.

Example. If M is a complete vector space with a trivial $\mathfrak{g}$-module structure then M is an equicontinuous $\mathfrak{g}$-module.

Let V, W be equicontinuous $\mathfrak{g}$-modules. It is easy to check that $V \hat{\otimes} W$ has a natural structure of an equicontinuous $\mathfrak{g}$-module. Moreover, $(V \hat{\otimes} W) \hat{\otimes} U$ is naturally identified with $V \hat{\otimes} (W \hat{\otimes} U)$ for any equicontinuous $\mathfrak{g}$-modules V, W, U. This means that the category of equicontinuous $\mathfrak{g}$-modules, where morphisms are continuous homomorphisms, is a monoidal category. This category is symmetric since the objects $V \hat{\otimes} W$ and $W \hat{\otimes} V$ are identified by the permutation of components. We denote this category by $\mathcal{M}_0^{\mathfrak{e}}$.

7.4. Lie bialgebras and Manin triples

Let $\mathfrak{a}$ be a Lie bialgebra over k. We will regard $\mathfrak{a}$ as a topological Lie algebra with the discrete topology. Let $\mathfrak{a}^*$ be the dual space to $\mathfrak{a}$. The cocommutator defines a Lie bracket on $\mathfrak{a}^*$ which is continuous in the weak topology, so $\mathfrak{a}^*$ has a natural structure of a topological Lie algebra.

Furthermore, we equip the space $\mathfrak{g} = \mathfrak{a}\oplus\mathfrak{a}^*$ with the product topology. The Lie bracket on $\mathfrak{g}$ defined by (1.1) is continuous in this topology.

Let $\mathfrak{g}$ be a Lie algebra with a nondegenerate invariant inner product $\langle,\rangle$. So far we have no topology on $\mathfrak{g}$. Let $\mathfrak{g}_+$,$\mathfrak{g}_-$ be isotropic Lie subalgebras in $\mathfrak{g}$, such that $\mathfrak{g} = \mathfrak{g}_+\oplus\mathfrak{g}_-$ as a vector space. The inner product $\langle,\rangle$ defines an embedding $\mathfrak{g}_-\to\mathfrak{g}_+{}^*$. If this embedding is an isomorphism then we equip $\mathfrak{g}$ with a topology, by putting the discrete topology on $\mathfrak{g}_+$ and the weak topology on $\mathfrak{g}_-$. If in addition the commutator in $\mathfrak{g}$ is continuous in this topology then the triple $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-)$ is called a Manin triple.

To every Lie bialgebra $\mathfrak{a}$ one can associate the corresponding Manin triple ($\mathfrak{g} = \mathfrak{a}\oplus\mathfrak{a}^*, \mathfrak{a}, \mathfrak{a}^*$), where the Lie structure on $\mathfrak{g}$ is as above. Conversely, if $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-)$ is a Manin triple then $\mathfrak{g}_+$ is naturally a Lie bialgebra: the pairing $\langle,\rangle$ identifies $\mathfrak{g}_+{}^*$ with $\mathfrak{g}_-$, which defines a commutator on $\mathfrak{g}_+{}^*$. This commutator turns out to be dual to a 1-cocycle (cf. [Dr1]). Thus, there is a one-to-one correspondence between Lie bialgebras and Manin triples.

Let $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-)$ be a Manin triple. Let $\{a_i, i\in I\}$ be a basis of $\mathfrak{g}_+$, and $b^i\in\mathfrak{g}_-$ be the linear functions on $\mathfrak{a}$ defined by $b^i(a_j) = \delta_{ij}$.

Lemma 7.1. *Let M be a complete vector space with a continuous homomorphism $\mathfrak{g}\to\mathrm{End}M$. Then for any $v\in M$ and any neighborhood of zero $U\subset M$ one has $b^iv\in U$ for all but finitely many $i\in I$.*

Proof. Let $\{i_m\in I : m\geq 1\}$ be any sequence of distinct elements. The $b^{i_m}\to 0$, $m\to\infty$, so $b^{i_m}v\to 0$, $m\to\infty$, for any $v\in M$. This means that $b^iv\in U$ for almost all i. □

7.5. Examples of equicontinuous $\mathfrak{g}$-modules

In this section we will construct examples of equicontinuous $\mathfrak{g}$-modules in the case when $\mathfrak{g}$ belongs to a Manin triple $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-)$.

Consider the Verma modules $M_+ = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_+}\mathbf{1}$ $M_- = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_-}\mathbf{1}$, (here $\mathbf{1}$ denotes the trivial 1-dimensional representation). The modules $M_\pm$ are freely generated over $U(\mathfrak{g}_\mp)$ by a vector $1_\pm$ such that $\mathfrak{g}_\pm 1_\pm = 0$, and thus are identified (as vector spaces) with $U(\mathfrak{g}_\mp)$ via $x1_\pm\to x$.

Below we show that the module M_- and the module M_+^* dual to M_+ in an appropriate sense are equicontinuous $\mathfrak{g}$-modules.

Lemma 7.2. *The module M_-, equipped with the discrete topology, is an equicontinuous $\mathfrak{g}$-module.*

Proof. In order to prove the continuity of $\pi_{M_-}(g)$ as a function on $\mathfrak{g}$, we have to check that for any $v\in M_-$ the subspace $Y_v = \{b\in\mathfrak{g}_- : bv = 0\}\subset\mathfrak{g}_-$ is open in $\mathfrak{g}_-$.

One may assume that $v = a_{i_1} a_{i_2} \dots a_{i_n} 1_-$. We show that Y_v is open by induction in n. The base of induction is clear since $\mathfrak{g}_- v = 0$ if $n = 0$. Now assume that $v = a_j w$, where $w = a_{i_1} \dots a_{i_{n-1}} 1_-$. By the induction assumption, we know that Y_w is open. For any $b \in \mathfrak{g}_-$ we have $bv = ba_j w = [ba_j]w + a_j bw$. For any $j \in I$ we denote by $W_j \subset \mathfrak{g}_-$ the space of all $b \in \mathfrak{g}_-$ such that $(1 \otimes b)(\delta(a_j)) = 0$. It is clear that W_j is open. For any $b \in W_j$, we have $[ba_j] \in \mathfrak{g}_-$, since $\mathrm{ad}^* b(a_j) = 0$ by the definition of W_j. Let $Z = \{b \in \mathfrak{g}_-, [ba_j] \in Y_w\} \subset W_j$. By continuity of $[,]$, Z is open. From the formula $bv = [ba_j]w + a_j bw$ we get $Z \cap Y_w \subset Y_v$, i.e. Y_v is open, as desired. This implies the continuity of the homomorphism $\pi_{M_-} : \mathfrak{g} \to \mathrm{End} M_-$. The equicontinuity condition is trivial. □

Let us now introduce a topology on the space M_+. This topology comes from the identification of M_+ with $U(\mathfrak{g}_-)$. The space $U(\mathfrak{g}_-)$ can be represented as a union of $U_n(\mathfrak{g}_-)$, $n \geq 0$, where $U_n(\mathfrak{g}_-)$ is the set of all elements of $U(\mathfrak{g}_-)$ of degree $\leq n$. Furthermore, for any $n \geq 0$, we have a linear map $\mathfrak{g}_-^{\otimes n} \to U_n(\mathfrak{g}_-)$ given by $x_1 \otimes \dots \otimes x_n \to x_1 \dots x_n$. This map induces a linear isomorphism $\xi_n : \oplus_{j=0}^n S^j \mathfrak{g}_- \to U_n(\mathfrak{g}_-)$, where $S^j \mathfrak{g}_-$ is the j-th symmetric power of $\mathfrak{g}_-$ (as usual we set $\mathfrak{g}_-^{\otimes 0} = S^0 \mathfrak{g}_- = k$). Since $S^j \mathfrak{g}_-$ has a natural weak topology, coming from its embedding to $(\mathfrak{g}_+^{\otimes j})^*$, the isomorphism ξ_j defines a topology on $U_n(\mathfrak{g}_-)$. Moreover, by the definition, if $m < n$ then $U_m(\mathfrak{g}_-)$ is a closed subspace in $U_n(\mathfrak{g}_-)$. This allows us to equip $U(\mathfrak{g}_-)$, i.e. M_+, with the topology of inductive limit. By the definition, a set $U \subset U(\mathfrak{g}_-)$ is open in this topology if and only if $U \cap U_n(\mathfrak{g}_-)$ is open for all n.

Lemma 7.3. *Let $g \in \mathfrak{g}$. Then $\pi_{M_+}(g)$ is a continuous operator $M_+ \to M_+$.*

Proof. Let $g \in \mathfrak{g}$. We need to show that for any neighborhood of the origin $U \subset M_+$ there exists a neighborhood of the origin $U' \subset M_+$ such that $\pi_{M_+}(g) U' \subset U$.

Let $U \in U(\mathfrak{g}_-)$ be a neighborhood of zero, and $U_n = U \cap U_n(\mathfrak{g}_-)$. To construct U', we need to construct $U'_n = U' \cap U_n(\mathfrak{g}_-)$ such that $U'_n = U'_{n+1} \cap U_n(\mathfrak{g}_-)$. Before giving the construction of U'_n, we make some definitions.

For any neighborhood U of zero, there exists an increasing sequence of finite subsets $T_n \subset I$, $n \geq 1$, such that for any $f \in S^m \mathfrak{g}_-$, $m \leq n$ satisfying the equation $f(a_{i_1}, \dots, a_{i_m}) = 0$ for any $i_1, \dots, i_m \in T_n$, one has $\xi_n(f) \in U$. Fix such a sequence $\{T_n, n \geq 1\}$.

Let I be as in Section 7.4. For any finite subset $J \subset I$ denote by $S(J)$ the set of all $i \in I$ such that there exists $b \in \mathfrak{g}_-$ and $j \in J$ with the property $[bb^i](a_j) \neq 0$. Since $[bb^i](a_j) = b \otimes b^i(\delta(a_j))$, the set $S(J)$ is finite. Let the sets $S_n(J) \subset I$ be defined recursively by $S_0(J) = J$, $S_n(J) = S(S_{n-1}(J))$.

To construct U', we consider separately the cases $g \in \mathfrak{g}_+$ and $g \in \mathfrak{g}_-$. First consider the case $g \in \mathfrak{g}_-$.

For any elements $x_1, \dots, x_n \in \mathfrak{g}_-$ $(n \geq 1)$ consider the element

$$X = \sum_{\sigma \in S_n} x_{\sigma(1)} \dots x_{\sigma(n)}$$

in $U_n(\mathfrak{g}_-)$, where S_n is the symmetric group. Consider the element $gX \in U_{n+1}(\mathfrak{g}_-)$. It is easy to see that it is possible to write gX as a linear combination of elements of the form $\sum_{\sigma\in S_m} y_{\sigma(1)}\cdots y_{\sigma(m)}$, $y_p \in \mathfrak{g}_-$, $0 \le m \le n+1$, in such a way that y_p are iterated commutators of g and $x_1,\dots,x_n$, and the number of commutators involved in each term y_p does not exceed n.

Now we make a crucial observation.

Claim. *Let $J\subset I$ be a finite subset. If for some m, $1\le m\le n$, we have $x_m(a_i)=0$, for all $i\in S_n(J)$, then every monomial $y_1\dots y_m$, in the symmetrized expression of gX contains a factor y_p such that $y_p(a_i)=0$, $i\in J$.*

Proof. Clear.

The construction of U' is as follows. For $n\ge 1$, let $U_n'\subset U_n(\mathfrak{g}_-)$ be the span of all elements $\xi_m(f)$, $0\le m\le n$, where $f\in S^m\mathfrak{g}_-$ are such that $f(a_{i_1},\dots,a_{i_m})=0$ whenever $i_1,\dots,i_m\in S_n(T_{n+1})$. Also, set $U_0'=0$ (recall that $\{0\}\subset k$ is a neighborhood of zero since k is discrete). Our observation shows that for any $X\in U_n'$, $gX\in U_{n+1}$, as desired.

Now consider the case $g\in\mathfrak{g}_+$. Let $R_0(g)\subset I$ be the set of all $i\in I$ such that $b^i(g)\ne 0$. This is a finite set. Define inductively the sets $R_n(g)$ by $R_n(g)=S(R_{n-1}(g))$.

For any finite subsets $K,J\subset I$ denote by $P(K,J)$ the set of all $i\in I$ such that there exists $j\in J$ and $k\in K$ with $[a_kb^i](a_j)\ne 0$. It is clear that if K,J are finite then $P(K,J)$ is finite. Let $P_n(K,J)$ be defined inductively by $P_n(K,J)=P(K,P_{n-1}(K,J))$.

Let $n\ge 1$ be an integer, $X\in U_n(\mathfrak{g}_-)$ be as above, and $K=R_n(g)$. Consider the vector $gX1_+\in M_+$. Using the relations in M_+, we can reduce this vector to a linear combination of vectors of the form $\sum_{\sigma\in S_m} y_{\sigma_1}\cdots y_{\sigma_m}$, $y_p\in\mathfrak{g}_-$, $0\le m\le n+1$, in such a way that y_p are obtained by iterated commutation of g, $x_1,\dots,x_n$. As before, it is easy to see that the resulting symmetrized expression will contain no more than n commutators.

Now let us make a crucial observation.

Claim. *Let $J\subset I$ be any finite subset. If for some m, $1\le m\le n$, we have $x_m(a_i)=0$, for all $i\in S_n(P(K,S_n(J))$, then every monomial $y_1\dots y_m$ in the symmetrized expression of $gX1_+$ contains a factor y_p such that $y_p(a_i)=0$, $i\in J$.*

Proof. Clear.

The construction of U' is as follows. For $n\ge 1$, let $U_n'\subset U_n(\mathfrak{g}_-)$ be the span of all elements $\xi_m(f)$, $f\in S^m\mathfrak{g}_-$, $0\le m\le n$ such that $f(a_{i_1},\dots,a_{i_m})=0$ whenever $i_1,\dots,i_m\in S_n(P(K,S_n(T_{n+1})))$. Also, set $U_0'=0$. Our observation shows that for any $X\in U_n'$, $gX\in U_{n+1}$, as desired. □

Consider the vector space M_+^* of continuous linear functionals on M_+. By definition, M_+^* is naturally isomorphic to the projective limit of $U_n(\mathfrak{g}_-)^*$ as $n\to\infty$. As vector spaces, $U_n(\mathfrak{g}_-)^*=(S^j\mathfrak{g}_-)^*=S^j\mathfrak{g}_+$. Therefore, it is natural to put the discrete topology on $U_n(\mathfrak{g}_-)^*$. This equips the module M_+^* with a natural structure of a complete vector space. It is also equipped with a filtration by subspaces $(M_+^*)_n = U_{n-1} + (\mathfrak{g}_-)^\perp$, $n \geq 1$ such that $M_+ = \varprojlim M_+^*/(M_+^*)_n$.

Remark. The topology of projective limit on M_+^* does not, in general, coincide with the weak topology of the dual. In fact, it is stronger than the weak topology.

By Lemma 7.3, M_+^* has a natural structure of a $\mathfrak{g}$-module. Namely, the action of $\mathfrak{g}$ on M_+^* is defined to be the dual to the action of $\mathfrak{g}$ on M_+.

Lemma 7.4. *M_+^* is an equicontinuous $\mathfrak{g}$-module.*

Proof. It is easy to see that $a(M_+^*)_n \subset (M_+^*)_n$, $a \in \mathfrak{g}_+$, and $b(M_+^*)_n \subset (M_+^*)_{n-1}$, $b \in \mathfrak{g}_-$. This means that the operators $\pi_{M_+^*}(g)$ are continuous for any $g \in \mathfrak{g}$, and $\pi_{M_+^*}(\mathfrak{g}) \subset \mathrm{End} M_+^*$ is an equicontinuous family of operators. It remains to show that the assignment $g \to \pi_{M_+^*}(g)$ is continuous for $g \in \mathfrak{g}$. Since $\mathfrak{g}_+$ is discrete, it is enough to check this statement for $g \in \mathfrak{g}_-$.

Let $f \in M_+^*$. Let f_n be the reduction of f modulo $(M_+^*)_n$. We can regard f as an element of $\oplus_{j=0}^n S^j \mathfrak{g}_+$. Let us write f_n in terms of the basis $\{a_i\}$, and let $T_n(f)$ be the set of all $i \in I$ such that a_i is involved in this expression.

Let $S_n(J)$ be as in the proof of Lemma 7.3, and $i \in I \setminus S_n(T_{n+1}(f))$. Then it is easy to see that $b^i f \in (M_+^*)_n$. This shows that for any $n \geq 0$ and any $f \in M_+^*$ $b^i f \in (M_+^*)_n$ for almost all $i \in I$.

Thus, M_+^* is an equicontinuous $\mathfrak{g}$-module. □

Remark. If $\mathfrak{g}_+$ is infinite-dimensional then M_+ is not, in general, an equicontinuous $\mathfrak{g}$-module, since the family of operators $\{\pi_{M_+}(g),\ g \in \mathfrak{g}_+\}$ may fail to be equicontinuous.

7.6. The Casimir element

Consider the tensor product $\mathfrak{a} \otimes \mathfrak{a}^*$. This space can be embedded into $\mathrm{End}\mathfrak{a}$, by $(x \otimes f)(y) = f(y)x$, $x, y \in \mathfrak{a}$, $f \in \mathfrak{a}^*$. This embedding defines a topology on $\mathfrak{a} \otimes \mathfrak{a}^*$, obtained by restriction of the weak topology on $\mathrm{End}\mathfrak{a}$. Let $\mathfrak{a} \hat{\otimes} \mathfrak{a}^*$ be the completion of a $\mathfrak{a} \otimes \mathfrak{a}^*$ in this topology. Since the image of $\mathfrak{a} \otimes \mathfrak{a}^*$ is dense in $\mathrm{End}\mathfrak{a}$, this completion is identified with $\mathrm{End}\mathfrak{a}$.

Lemma 7.5. *Let $V, W \in \mathcal{M}_0^e$. The map $\pi_V \otimes \pi_W : \mathfrak{a} \otimes \mathfrak{a}^* \to \mathrm{End}(V \hat{\otimes} W)$ extends to a continuous map $\mathfrak{a} \hat{\otimes} \mathfrak{a}^* \to \mathrm{End}(V \hat{\otimes} W)$.*

Proof. Let $x \in V \hat{\otimes} W$ be a vector. It is easy to see that the map $\pi_V \otimes \pi_W(\cdot)x$: $\mathfrak{a} \otimes \mathfrak{a}^* \to V \hat{\otimes} W$ is continuous. Since the space $V \hat{\otimes} W$ is complete, this map extends to a continuous map $\mathfrak{a} \hat{\otimes} \mathfrak{a}^* \to V \hat{\otimes} W$. This allows us to define a linear map $\pi_V \otimes \pi_W$: $\mathfrak{a} \hat{\otimes} \mathfrak{a}^* \to \mathrm{End}(V \hat{\otimes} W)$. We would like to show that this map is continuous.

Let $x \in V \hat{\otimes} W$ be a vector, and $n \geq 0$ be an integer. Let $P \subset V \hat{\otimes} W$ be an open subspace, and $U = \{A \in \mathrm{End}(V \hat{\otimes} W) : Ax \in P\}$. Since open sets of this form generate the topology on $\mathrm{End}(V\hat{\otimes}W)$, it is enough to show that there exists a neighborhood of zero $Y\subset\mathfrak{a}\hat{\otimes}\mathfrak{a}^*$ such that $(\pi_V\otimes\pi_W)(Y)\subset U$, i.e. $(\pi_V\otimes\pi_W)(Y)x\subset P$.

We can assume that $P = V_1\hat{\otimes}W + W_1\hat{\otimes}V$, where V_1, W_1 are open subspaces of V, W. By the equicontinuity of $\pi_V(g)$, $\pi_W(g)$, $g \in \mathfrak{g}$, there exist open subspaces $V_2\subset V$, $W_2\subset W$ such that $\pi_V(\mathfrak{g})V_2\subset V_1$, $\pi_W(\mathfrak{g})W_2\subset W_1$. Let $y \in V\otimes W$ be a vector in the usual tensor product of V and W such that $y - x \in V_2\hat{\otimes}W + V\hat{\otimes}W_2$. Then for any $t \in \mathfrak{a}\hat{\otimes}\mathfrak{a}^*$ $(\pi_V\otimes\pi_W)(t)(y - x) \in P$, so it is enough to find Y satisfying the condition $(\pi_V\otimes\pi_W)(Y)y\subset P$.

We have $y = \sum_{j=1}^{m} v_j\otimes w_j$, $v_j \in V$, $w_j \in W$. Let $X\subset\mathfrak{a}$ be a finite-dimensional subspace such that for any $b \in X^\perp\subset\mathfrak{a}^*$ $bw_j \in W_1$ for $j = 1,\dots,m$. Such a subspace exists by Lemma 7.1. The set $Y = \mathfrak{a}\hat{\otimes}X^\perp$ (the completion of $\mathfrak{a}\otimes X^\perp$ in $\mathfrak{a}\hat{\otimes}\mathfrak{a}^*$) is open in $\mathfrak{a}\hat{\otimes}\mathfrak{a}^*$, and $(\pi_V\otimes\pi_W)(Y)y\subset P$, as desired. This shows the continuity of $\pi_V\otimes\pi_W$ on $\mathfrak{a}\hat{\otimes}\mathfrak{a}^*$. $\square$

Let $r \in \mathfrak{a}\hat{\otimes}\mathfrak{a}^*$ be the vector corresponding to the identity operator under the identification $\mathfrak{a}\hat{\otimes}\mathfrak{a}^*$ with $\mathrm{End}\mathfrak{a}$. Let $r^{op} \in \mathfrak{a}^*\hat{\otimes}\mathfrak{a}$ be the element obtained from r by permutation of the components. We define the Casimir element $\Omega \in \mathfrak{a}\hat{\otimes}\mathfrak{a}^*\oplus\mathfrak{a}^*\hat{\otimes}\mathfrak{a}$ to be the sum $r + r^{op}$. It is easy to see that $r = \sum a_i\otimes b^i$, $r^{op} = \sum b^i\otimes a_i$, $\Omega = \sum(a_i\otimes b^i + b^i\otimes a_i)$.

Let V, W be equicontinuous $\mathfrak{g}$-modules, and denote by $\pi_V : \mathfrak{g}\to\mathrm{End}V$, $\pi_W : \mathfrak{g}\to\mathrm{End}W$ the corresponding linear maps. Let $\Omega_{VW} = \pi_V\otimes\pi_W(\Omega)$. This endomorphism of $V\hat{\otimes}W$ is well defined and continuous by Lemma 7.5. Moreover, it is easy to see that Ω_{VW} commutes with $\mathfrak{g}$, so it is an endomorphism of $V\hat{\otimes}W$ as an equicontinuous $\mathfrak{g}$-module.

Remark. Although the Casimir operator $\Omega = \sum(a_i\otimes b^i + b^i\otimes a_i)$ is defined in the product of any two equicontinuous $\mathfrak{g}$-modules $V\hat{\otimes}W$, the Casimir element $C = \sum(a_ib^i + b^ia_i)$ in general (for $\dim\mathfrak{a} = \infty$) has no meaning as an operator in an equicontinuous $\mathfrak{g}$-module V.

7.7. Drinfeld category

Let $\mathcal{M}^e$ denote the category whose objects are equicontinuous $\mathfrak{g}$-modules, and $\mathrm{Hom}_{\mathcal{M}^e}(U, W) = \mathrm{Hom}_{\mathfrak{g}}(U, W)[[h]]$. This is an additive category. For brevity we will later write Hom for $\mathrm{Hom}_{\mathcal{M}^e}$.

Define a structure of a braided monoidal category on $\mathcal{M}^e$ analogously to Section 1.4, using an associator Φ and the functor $\hat{\otimes}$. As before, we identify $\mathcal{M}^e$ with a strict category and forget about positions of brackets.

Let γ be the functorial isomorphism defined by $\gamma_{XY} = \beta_{YX}^{-1} \in \mathrm{Hom}(X\otimes Y, Y\otimes X)$, $X, Y \in \mathcal{M}^e$. It is easy to check that γ is a braiding on $\mathcal{M}^e$. We will need the braiding γ in our construction below.

8. The fiber functor

8.1. The category of complete $k[[h]]$-modules

Let V be a complete vector space over k. Then the space $V[[h]] = V\hat{\otimes}k[[h]]$ of formal power series in h with coefficients in V is also a complete vector space. Moreover, $V[[h]]$ has a natural structure of a topological $k[[h]]$-module. We call a topological $k[[h]]$-module complete if it is isomorphic to $V[[h]]$ for some complete V.

Let $\mathcal{A}^c$ be the category of complete $k[[h]]$-modules, where morphisms are continuous $k[[h]]$-linear maps. It is an additive category. Define the tensor structure on $\mathcal{A}^c$ as follows. For $V, W \in \mathcal{A}^c$ define $V\tilde{\otimes}W$ to be the quotient of the completed tensor product $V\hat{\otimes}W$ by the image of the operator $h\otimes 1 - 1\otimes h$. It is clear that for $V, W \in \mathcal{A}^c$, $V\tilde{\otimes}W$ is also in $\mathcal{A}^c$. The category $\mathcal{A}^c$ equipped with the functor $\tilde{\otimes}$ is a symmetric monoidal category.

Let CVect be the category of complete vector spaces. We have the functor of extension of scalars, $V \mapsto V[[h]]$, acting from CVect to $\mathcal{A}^c$. This functor respects the tensor product, i.e. $(V\hat{\otimes}W)[[h]]$ is naturally isomorphic to $V[[h]]\tilde{\otimes}W[[h]]$.

8.2. Properties of the Verma modules

Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a Manin triple, and $\mathcal{M}^e$ be the Drinfeld category associated to $\mathfrak{g}$. Let M_+, M_- be the Verma modules over $\mathfrak{g}$ defined in Section 7.5.

Recall that the modules $M_\pm$ are identified with $U(\mathfrak{g}_\mp)$. Thus, we can define the maps $i_\pm : M_\pm \to M_\pm \otimes M_\pm$ given by comultiplication in the universal enveloping algebras $U(\mathfrak{g}_\mp)$. These maps are $U(\mathfrak{g})$-intertwiners, since they are $U(\mathfrak{g}_\pm)$-intertwiners and map the vector $1_\pm$ to the $\mathfrak{g}_\mp$-invariant vector $1_\pm \otimes 1_\pm$.

Let M_+^* be as in Section 7.5, and $f, g \in M_+^*$. Consider the linear functional $M_+ \to k$ defined by $v \to (f\otimes g)(i_+(v))$. It is easy to check that this functional is continuous, so it belongs to M_+^*. Define the map $i_+^* : M_+^* \otimes M_+^* \to M_+^*$ by $i_+^*(f\otimes g)(v) = (f\otimes g)(i_+(v))$, $v \in M_+$. It is clear that i_+^* is continuous, so it extends to a morphism in $\mathcal{M}^e : i_+^* : M_+^* \hat{\otimes} M_+^* \to M_+^*$.

Let $V \in \mathcal{M}$. Consider the space $\mathrm{Hom}_{\mathfrak{g}}(M_-, M_+^* \hat{\otimes} V)$, where $\mathrm{Hom}_{\mathfrak{g}}$ denotes the set of continuous homomorphisms. Equip this space with the weak topology (see Section 7.1).

Lemma 8.1. *The complete vector space* $\mathrm{Hom}_{\mathfrak{g}}(M_-, M_+^* \hat{\otimes} V)$ *is isomorphic to* V*. The isomorphism is given by* $f \to (1_+ \otimes 1)(f(1_-))$, $f \in \mathrm{Hom}_{\mathfrak{g}}(M_-, M_+^* \hat{\otimes} V)$.

Proof. By Frobenius reciprocity, $\mathrm{Hom}_{\mathfrak{g}}(M_-, M_+^* \hat{\otimes} V)$ is isomorphic, as a topological vector space, to the space of invariants $(M_+^* \hat{\otimes} V)^{\mathfrak{g}_-}$, via $f \to f(1_-)$. Consider the space $\mathrm{Hom}_k(M_+, V)$ of continuous homomorphisms from M_+ to V, equipped with the weak topology, and the map $\phi : (M_+^* \hat{\otimes} V) \to \mathrm{Hom}_k(M_+, V)$, given by $\phi(f\otimes v)(x) = f(x)v$, $u \in M_+^*$, $x \in M_+$, $v \in V$. It is clear that ϕ is injective and continuous.

Claim. *The map ϕ restricts to an isomorphism $(M_+^*\hat{\otimes}V)^{\mathfrak{g}_-}\to\mathrm{Hom}_{\mathfrak{g}}(M_+,V)$.*

Proof. It is clear that $\phi((M_+^*\hat{\otimes}V)^{\mathfrak{g}_-})\subset\mathrm{Hom}_{\mathfrak{g}}(M_+,V)$. So it is enough to show that any continuous $\mathfrak{g}_-$-intertwiner $g : M_+\to V$ is of the form $\phi(g')$, $g'\in(M_+^*\hat{\otimes}V)^{\mathfrak{g}_-}$, where g' continuously depends on $\mathfrak{g}$.

Let $X\subset V$ be an open subspace. Then for any $\mathfrak{g}_-$-intertwiner $g : M_+\to V$ and $n\geq 1$ the image of $g(U_n(\mathfrak{g}_-)1_+)$ in V/V_m is finite-dimensional. This shows that $g=\phi(g')$ for some $g'\in(V\otimes M_+^*)^{\mathfrak{g}_-}$. It is clear that g' is continuous in g. The claim is proved.

By Frobenius reciprocity, the space $\mathrm{Hom}_{\mathfrak{g}_-}(M_+,V)$ is isomorphic to V as a topological vector space, via $f\to f(1_+)$. The lemma is proved. □

8.3. The forgetful functor

Let $F : \mathcal{M}^e\to\mathcal{A}^c$ be a functor given by $F(V)=\mathrm{Hom}(M_-,M_+^*\hat{\otimes}V)$. Lemma 8.1 implies that this functor is naturally isomorphic to the "forgetful" functor which associates to every equicontinuous $\mathfrak{g}$-module M the complete $k[[h]]$-module $M[[h]]$. The isomorphism between these two functors is given by $f\to(1_+\otimes 1)(f(1_-))$, for any $f\in F(M)$. Denote this isomorphism by τ.

8.4. Tensor structure on the functor F

From now on, when no confusion is possible, we will denote the tensor product in the categories $\mathcal{M}^e$ and $\mathcal{A}^c$ by $\otimes$, instead of $\hat{\otimes}$ and $\tilde{\otimes}$.

Define a tensor structure on the functor F constructed in Section 8.3.

For any $v\in F(V)$, $w\in F(W)$ define $J_{VW}(v\otimes w)$ to be the composition of morphisms:

$$M_-\xrightarrow{i_-}M_-\otimes M_-\xrightarrow{v\otimes w}M_+^*\otimes V\otimes M_+^*\otimes W\xrightarrow{1\otimes\gamma_{23}\otimes 1}$$
$$M_+^*\otimes M_+^*\otimes V\otimes W\xrightarrow{i_+^*\otimes 1\otimes 1}M_+^*\otimes V\otimes W, \tag{8.1}$$

where γ_{23} denotes the braiding γ acting in the second and third components of the tensor product. That is,

$$J_{VW}(v\otimes w)=(i_+^*\otimes 1\otimes 1)\circ(1\otimes\gamma_{23}\otimes 1)\circ(v\otimes w)\circ i_-. \tag{8.2}$$

Proposition 8.2. *The maps J_{VW} are isomorphisms and define a tensor structure on the functor F.*

Proof. It is obvious that J_{VW} is an isomorphism since it is an isomorphism modulo h.

To prove the associativity of J_{VW}, we need the following result.

Lemma 8.3. *$(i_-\otimes 1)\circ i_- = (1\otimes i_-)\circ i_-$ in* $\mathrm{Hom}(M_-, M_-^{\otimes 3})$; $(i_+^*\otimes 1)\circ i_+^* = (1\otimes i_+^*)\circ i_+^*$ *in* $\mathrm{Hom}(M_+^*, (M_+^*)^{\otimes 3})$.

Proof. The proof of the first identity coincides with the proof of Lemma 2.3 in Part I. To prove the second identity, define $M_+\hat{\otimes}M_+\hat{\otimes}M_+$ to be space of continuous linear functionals on $M_+^*\hat{\otimes}M_+^*\hat{\otimes}M_+^*$. Since the operators $\Omega_{ij}\in \mathrm{End}_{\mathfrak{g}}(M_+^*\hat{\otimes}M_+^*\hat{\otimes}M_+^*)$ are continuous, one can define the dual operators $\Omega_{ij}^*\in\mathrm{End}_{\mathfrak{g}}(M_+\hat{\otimes}M_+\hat{\otimes}M_+)$, and hence the operator Φ^* dual to Φ. It is easy to show analogously to the proof of Lema 2.3 that $\Phi^*(1_+\otimes 1_+\otimes 1_+) = 1_+\otimes 1_+\otimes 1_+$, which implies the second identity of Lemma 8.3.

Now we can finish the proof of the proposition. We need to show that for any $v\in F(V)$, $w\in F(W)$, $u\in F(U)$ $J_{V\otimes W,U}\circ(J_{VW}\otimes 1)(v\otimes w\otimes u) = J_{V,W\otimes U}\circ(1\otimes J_{WU})(v\otimes w\otimes u)$, i.e.

$$\begin{aligned}&(i_+^*\otimes 1\otimes 1\otimes 1)\circ\gamma_{23,5}\circ(i_+^*\otimes 1\otimes 1\otimes 1\otimes 1)\circ\gamma_{23}\circ(v\otimes w\otimes u)\circ(i_-\otimes 1)\circ i_-\\ &= (i_+^*\otimes 1\otimes 1\otimes 1)\circ\gamma_{23}\circ(1\otimes 1\otimes i_+^*\otimes 1\otimes 1)\circ\gamma_{45}\circ(v\otimes w\otimes u)\circ(1\otimes i_-)\circ i_-\end{aligned}\tag{8.3}$$

in $F(V\otimes W\otimes U)$, where $\gamma_{23,4}$ means the braiding applied to the product of the second and the third factors and to the fourth factor. Because of Lemma 8.3 and commutation relation of $\gamma_{23,4}$ and $i_+^*\otimes 1\otimes 1\otimes 1\otimes 1$, identity (8.3) is equivalent to the identity

$$\begin{aligned}&(i_+^*\otimes 1\otimes 1\otimes 1)\circ(i_+^*\otimes 1\otimes 1\otimes 1\otimes 1)\circ\gamma_{34,5}\circ\gamma_{23}\\ &= (i_+^*\otimes 1\otimes 1\otimes 1)\circ\gamma_{23}\circ(1\otimes 1\otimes i_+^*\otimes 1\otimes 1)\circ\gamma_{45}\end{aligned}\tag{8.4}$$

in $\mathrm{Hom}(M_+^*\otimes V\otimes M_+^*\otimes W\otimes M_+^*\otimes U, M_+^*\otimes V\otimes W\otimes U)$.

To prove this equality, we observe that the functoriality of the braiding implies the identity

$$\gamma_{23}\circ(1\otimes 1\otimes i_+^*\otimes 1\otimes 1) = (1\otimes i_+^*\otimes 1\otimes 1\otimes 1)\circ\gamma_{2,34}. \tag{8.5}$$

Using (8.5) and the identity $(i_+^*\otimes 1)\circ i_+^* = (1\otimes i_+^*)\circ i_+^*$, which follows from Lemma 8.3, we reduce (8.4) to the identity $\gamma_{34,5}\gamma_{23} = \gamma_{2,34}\gamma_{45}$ which follows directly from the braiding axioms. □

We will call the functor F equipped with the tensor structure defined above *the fiber functor*.

9. Quantization of Lie bialgebras

9.1. The algebra of endomorphisms of the fiber functor

Let $H = \mathrm{End}(F)$ be the algebra of endomorphisms of the functor F, with a topology defined by the ideal $hH\subset H$. It is clear that H is a topological algebra over $k[[h]]$ (see Part I, Section 3.1).

Let H_0 be the algebra of endomorphisms of the forgetful functor $\mathcal{M}_0^e \to C\mathrm{Vect}$. It follows from Lemma 8.1 that the algebra H is naturally isomorphic to $H_0[[h]]$.

Let $F^2 : \mathcal{M}^e \times \mathcal{M}^e \to \mathcal{A}^c$ be the bifunctor defined by $F^2(V,W) = F(V) \otimes F(W)$. Let $H^2 = \mathrm{End}(F^2)$. It is clear that $H^2 \supset H \otimes H$ but $H^2 \neq H \otimes H$ unless $\mathfrak{g}$ is finite-dimensional.

The algebra H has a natural "comultiplication" $\Delta : H \to H^2$ defined by $\Delta(a)_{V,W}$ $(v \otimes w) = J_{VW}^{-1} a_{V \otimes W} J_{VW}(v \otimes w)$, $a \in H$, $v \in F(V)$, $w \in F(W)$ where a_V denotes the action of a in $F(V)$. We can also define the counit on H by $\varepsilon(a) = a_{\mathbf{1}} \in k[[h]]$, where $\mathbf{1}$ is the neutral object.

A topological algebra A over $k[[h]]$ is said to be a topological bialgebra if it is equipped with a coproduct $\Delta : A \to A \otimes A$ (where $\otimes$ is the tensor product in $\mathcal{A}$) and a counit $\varepsilon : A \to k[[h]]$ which are $k[[h]]$-linear, continuous, and satisfy the standard axioms of a bialgebra.

We will need the following statement.

Proposition 9.1. *Let $A \subset H$ be a topological subalgebra such that $\Delta(A) \subset A \otimes A$. Then (A, Δ, ε) is a topological bialgebra over $k[[h]]$.*

The proof is straightforward.

Remark. For infinite-dimensional $\mathfrak{g}$, the algebra H equipped with the topology defined by the ideal hH is not a topological bialgebra since $\Delta(H)$ is not a subset of $H \otimes H$.

In the following sections we construct a quantum universal enveloping algebra $U_h(\mathfrak{g}_+)$, which is a quantization of the Lie bialgebra $\mathfrak{g}_+$, in the sense of Drinfeld (see [Dr1] and Part I, Section 3.1). Namely, the algebra $U_h(\mathfrak{g}_+)$ is obtained as a subalgebra of H such that $\Delta(A) \subset A \otimes A$.

9.2. The algebra $U_h(\mathfrak{g}_+)$

Let $x \in F(M_-)$. Define the endomorphism $m_+(x)$ of the functor F as follows. For any $V \in \mathcal{M}^e$, $v \in F(V)$, define the element $m_+(x)v \in F(V)$ to be the composition of the following morphisms in $\mathcal{M}^e$: $m_+(x)v = (i_+^* \otimes 1) \circ (1 \otimes v) \circ x$. This defines a linear map $m_+ : F(M_-) \to H$. Denote the image of this map by $U_h(\mathfrak{g}_+)$.

It is easy to see that for any $a \in U(\mathfrak{g}_+)$ $\tau(m_+(a1_-)v) \equiv a\tau(v) \mod h$, which implies that m_+ is an embedding.

Proposition 9.2. *$U_h(\mathfrak{g}_+)$ is a subalgebra in H.*

Proof. Using Lemma 8.3, for any $x, y \in F(M_-)$, $V \in \mathcal{M}^e$, $v \in F(V)$ we obtain

$$\begin{aligned} m_+(x)m_+(y)v &= (i_+^* \otimes 1) \circ (1 \otimes i_+^* \otimes 1) \circ (1 \otimes 1 \otimes v) \circ (1 \otimes y) \circ x \\ &= (i_+^* \otimes 1) \circ (i_+^* \otimes 1 \otimes 1) \circ (1 \otimes 1 \otimes v) \circ (1 \otimes y) \circ x \\ &= (i_+^* \otimes 1) \circ (1 \otimes v) \circ (i_+^* \otimes 1) \circ (1 \otimes y) \circ x \\ &= (i_+^* \otimes 1) \circ (1 \otimes v) \circ z, \end{aligned} \tag{9.1}$$

where $z=(i_+^*\otimes 1)\circ(y\otimes 1)\circ x\in F(M_-)$. So by the definition we get $m_+(x)\circ m_+(y)=m_+(z)$. $\square$

Note that the algebra $U_h(\mathfrak{g}_+)$ is a deformation of the algebra $U(\mathfrak{g}_+)$. Indeed, we can define a linear isomorphism $\mu: U(\mathfrak{g}_+)[[h]]\to U_h(\mathfrak{g}_+)$ by $\mu(a)=m_-(a1_-)$, $a\in U(\mathfrak{g}_+)[[h]]$. This isomorphism has the property $\mu(ab)=\mu(a)\circ\mu(b) \mod h^2$, which follows from the fact that $\Phi\equiv 1 \mod h$, but in general $\mu(ab)\neq\mu(a)\circ\mu(b)$.

The subalgebra $U_h(\mathfrak{g}_+)$ has a unit which is equal to $\mu(1)$, $1\in U(\mathfrak{g}_+)$. To check this, it is enough to observe that $\mu(1)$ is invertible and check the identity $\mu(1)^2=\mu(1)$.

9.3. The coproduct on $U_h(\mathfrak{g}_+)$

Proposition 9.3. *The algebra $U_h(\mathfrak{g}_+)$ is closed under the coproduct Δ, i.e. $\Delta(U_h(\mathfrak{g}_+))\subset U_h(\mathfrak{g}_+)\otimes U_h(\mathfrak{g}_+)$, and for any $x\in F(M_-)$ one has*

$$\Delta(m_+(x))=(m_+\otimes m_+)(J^{-1}_{M_-M_-}((1\otimes i_-)\circ x)). \tag{9.2}$$

Proof. Let $x\in F(M_-)$, $V,W\in\mathcal{M}^e$, $v\in V$, $w\in W$. By the definition of Δ and m_+, the element $\Delta(m_+(x))\in H^2$ is uniquely determined by the identity

$$\begin{aligned}(i_+^*\otimes 1\otimes 1)&\circ(1\otimes i_+^*\otimes 1\otimes 1)\gamma_{34}\circ(1\otimes v\otimes w)\circ(1\otimes i_-)\circ x\\ &=(i_+^*\otimes 1\otimes 1)\circ\gamma_{23}\circ\Delta(m_+(x))(v\otimes w)\circ i_-\end{aligned} \tag{9.3}$$

in $F(V\otimes W)$.

The element $X=J^{-1}_{M_-M_-}((1\otimes i_-)x)\in F(M_-)\otimes F(M_-)$ is, by the definition, uniquely determined by the identity

$$(1\otimes i_-)\circ x=(1\otimes i_+^*\otimes 1\otimes 1)\circ\gamma_{23}\circ X\circ i_- \tag{9.4}$$

in $F(M_-\otimes M_-)$. Therefore, to prove formula (9.2), it is enough to prove the equality obtained by substitution of $(i_-^*\otimes 1\otimes i_-^*\otimes 1)\circ(1\otimes v\otimes 1\otimes w)\circ X$ instead of $\Delta(m_+(x))(v\otimes w)$ in (9.3):

$$\begin{aligned}(i_+^*\otimes 1\otimes 1)&\circ(1\otimes i_+^*\otimes 1\otimes 1)\circ\gamma_{34}\circ(1\otimes v\otimes w)\circ(1\otimes i_-)\circ x\\ &=(i_+^*\otimes 1\otimes 1)\circ\gamma_{23}\circ(i_+^*\otimes 1\otimes i_+^*\otimes 1)\circ(1\otimes v\otimes 1\otimes w)\circ X\circ i_-\end{aligned} \tag{9.5}$$

in $F(V\otimes W)$.

Using the functoriality of the braiding and Lemma 8.3, we obtain

$$\begin{aligned}&(i_+^*\otimes 1\otimes 1)\circ\gamma_{23}\circ(i_+^*\otimes 1\otimes i_+^*\otimes 1)\circ(1\otimes v\otimes 1\otimes w)\\ &=(i_+^*\otimes 1\otimes 1)\circ\gamma_{23}\circ(i_+^*\otimes 1\otimes i_+^*\otimes 1)\circ\gamma^{-1}_{23,4}\circ(1\otimes 1\otimes v\otimes w)\circ\gamma_{23}\\ &=(i_+^*\otimes 1\otimes 1)\circ(i_+^*\otimes i_+^*\otimes 1\otimes 1)\circ\gamma_{3,45}\circ\gamma^{-1}_{23,4}\circ(1\otimes 1\otimes v\otimes w)\circ\gamma_{23}\\ &=(i_+^*\otimes 1\otimes 1)\circ(i_+^*\otimes i_+^*\otimes 1\otimes 1)\circ\gamma_{45}\circ\gamma^{-1}_{23}\circ(1\otimes 1\otimes v\otimes w)\circ\gamma_{23}\\ &=(i_+^*\otimes 1\otimes 1)\circ(i_+^*\otimes 1\otimes 1\otimes 1)\circ(1\otimes i_+^*\otimes 1\otimes 1\otimes 1)\circ\gamma_{45}\circ\gamma^{-1}_{23}\circ(1\otimes 1\otimes v\otimes w)\circ\gamma_{23}\end{aligned} \tag{9.6}$$

in $\mathrm{Hom}(M_+^*\otimes M_-\otimes M_+^*\otimes M_-, M_+^*\otimes V\otimes W)$. It is easy to see that $i_+^*\circ\gamma = i_+^*$, so using Lemma 8.3 again, we get from (9.6):

$$\begin{aligned}&(i_+^*\otimes 1\otimes 1)\circ\gamma_{23}\circ(i_+^*\otimes 1\otimes i_+^*\otimes 1)\circ(1\otimes v\otimes 1\otimes w)\\ &= (i_+^*\otimes 1\otimes 1)\circ(1\otimes i_+^*\otimes 1\otimes 1)\circ\gamma_{34}\circ(1\otimes v\otimes w)\circ(i_+^*\otimes 1\otimes 1)\circ\gamma_{23}.\end{aligned} \tag{9.7}$$

Substituting (9.7) into the right-hand side of (9.5) and using (9.4), we get

$$\begin{aligned}&(i_+^*\otimes 1\otimes 1)\circ\gamma_{23}\circ(i_+^*\otimes 1\otimes i_+^*\otimes 1)\circ(1\otimes v\otimes 1\otimes w)\circ X\circ i_-\\ &= (i_+^*\otimes 1\otimes 1)\circ(1\otimes i_+^*\otimes 1\otimes 1)\circ\gamma_{34}\circ(1\otimes v\otimes w)\circ(i_+^*\otimes 1\otimes 1)\circ\gamma_{23}\circ X\circ i_-\\ &. = (i_+^*\otimes 1\otimes 1)\circ(1\otimes i_+^*\otimes 1\otimes 1)\circ\gamma_{34}\circ(1\otimes v\otimes w)\circ(1\otimes i_-)\circ x\end{aligned} \tag{9.8}$$

in $F(V\otimes W)$, which proves (9.2). The proposition is proved. □

Corollary 9.4. *The algebra $U_h(\mathfrak{g}_+)$ equipped with the coproduct Δ, is a quantized universal enveloping algebra.*

Proof. It follows from Lemma 9.1 and Propositions 9.2, 9.3 that $U_h(\mathfrak{g}_+)$ is a topological bialgebra over $k[[h]]$ isomorphic to $U(\mathfrak{g}_-)[[h]]$ as a topological $k[[h]]$-module, and such that $U_h(\mathfrak{g}_+)/hU_h(\mathfrak{g}_+)$ is isomorphic to $U(\mathfrak{g}_+)$ as a bialgebra. This implies that $U_h(\mathfrak{g})$ has an antipode, because the antipode exists $\mod h$. Thus, $U_h(\mathfrak{g}_+)$ is a quantized universal enveloping algebra. □

9.4. The algebra $U_h(\mathfrak{g}_+)$ is a quantization of $\mathfrak{g}_+$

Proposition 9.5. *The algebra $U_h(\mathfrak{g}_+)$ is a quantization of the Lie bialgebra $\mathfrak{g}_+$.*

Proof. Let $x \in U_h(\mathfrak{g}_+)$ be such that there exists $x_0 \in \mathfrak{g}_+\subset U(\mathfrak{g}_+)$ satisfying the condition $x \equiv x_0 \mod h$. It is easy to show that for any $V, W \in \mathcal{M}^e$

$$\tau_{V\otimes W}^{-1}\circ J_{VW}\circ(\tau_V\otimes\tau_W) = 1 + hr/2 + O(h^2) \tag{9.9}$$

in $\mathrm{End}(V\otimes W)$. From (9.9) and the definition of coproduct, analogously to the proof of Proposition 3.6 in Part I, it is easy to obtain the congruence

$$h^{-1}(\Delta(x) - \Delta^{op}(x)) \equiv \delta(x_0) \mod h. \tag{9.10}$$

which means that $U_h(\mathfrak{g}_+)$ is the quantization of $\mathfrak{g}_+$. □

Thus we have proved the following theorem, which answers question 1.1 in [Dr3].

Theorem 9.6. *Let $\mathfrak{a}$ be a Lie bialgebra over k. Then there exists a quantized universal enveloping algebra $U_h(\mathfrak{a})$ over k which is a quantization of $\mathfrak{a}$.*

9.5. The isomorphism between two constructions of the quantization

Let us compare the results of the previous sections to the results of Part I.

In Part I, we showed the existence of quantization for any finite-dimensional Lie bialgebra. Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a finite-dimensional Manin triple. Let $U_h(\mathfrak{g}_+)$ denote the quantization of $\mathfrak{g}_+$ constructed in this section, and by $\tilde{U}_h(\mathfrak{g}_+)$ the quantization constructed in Part I.

Proposition 9.7. *The quantized universal enveloping algebras $U_h(\mathfrak{g}_+)$, $\tilde{U}_h(\mathfrak{g}_+)$ are isomorphic.*

Proof. If $\mathfrak{g}$ is finite-dimensional, then M_+ is an equicontinuous $\mathfrak{g}$-module. Let $\tilde{F} : \mathcal{M}^e \to \mathcal{A}^c$ be the functor defined by $\tilde{F}(V) = \mathrm{Hom}(M_+ \otimes M_-, V)$, $V \in \mathcal{M}^e$. The tensor structure on $\tilde{F}$ can be defined as in Part I.

Let $\sigma \in \mathrm{Hom}(\mathbf{1}, M_+^* \otimes M_+)$ be the canonical element. Consider the morphism $\chi : \tilde{F} \to F$, defined as follows. For any $V \in M$, $v \in \tilde{F}(V)$, define $\chi_V(v) \in F(V)$ as the composition $\chi_V(v) = (1 \otimes v) \circ (\sigma \otimes 1)$. It is obvious that χ is an isomorphism of additive functors.

Claim. *χ is an isomorphism of tensor functors.*

Proof. The statement is equivalent to the identity

$$\begin{aligned}(1\otimes v\otimes w)\circ\beta_{34}\circ(1\otimes i_+\otimes i_-)\circ(\sigma\otimes 1)\\ = (i_+^*\otimes 1\otimes 1)\circ\gamma_{23}\circ(1\otimes v\otimes 1\otimes w)\circ(\sigma\otimes 1\otimes\sigma\otimes 1)\circ i_-,\end{aligned} \tag{9.11}$$

which should be satisfied in $\mathrm{Hom}(M_-, M_+^* \otimes V \otimes W)$ for any $V, W \in \mathcal{M}^e$, $v \in \tilde{F}(V)$, $w \in \tilde{F}(W)$. Using the identity $(1\otimes v\otimes 1\otimes w)\circ\gamma_{23} = \gamma_{23,4}\circ(1\otimes 1\otimes v\otimes w)$, we reduce (9.11) to the identity

$$\begin{aligned}\beta_{34}\circ(1\otimes i_+\otimes i_-)\circ(\sigma\otimes 1)\\ = (i_+^*\otimes 1\otimes 1\otimes 1\otimes 1)\circ\gamma_{23,4}\circ(\sigma\otimes 1\otimes\sigma\otimes 1)\circ i_-\end{aligned} \tag{9.12}$$

in $\mathrm{Hom}(M_-, M_+^* \otimes M_+ \otimes M_- \otimes M_+ \otimes M_-)$. Moving β_{34} from left to right and interchanging β_{34}^{-1} with $i_+^* \otimes 1 \otimes 1 \otimes 1 \otimes 1$, we see that (9.12) is equivalent to the identity:

$$\begin{aligned}(1\otimes i_+\otimes i_-)\circ(\sigma\otimes 1)\\ = (i_+^*\otimes 1\otimes 1\otimes 1\otimes 1)\circ\beta_{45}^{-1}\gamma_{23,4}\circ(\sigma\otimes 1\otimes\sigma\otimes 1)\circ i_-\end{aligned} \tag{9.13}$$

in $\mathrm{Hom}(M_-, M_+^* \otimes M_+ \otimes M_+ \otimes M_- \otimes M_-)$. It is clear that $\gamma_{1,23}\circ(1\otimes\sigma) = \sigma\otimes 1$ in $\mathrm{Hom}(M_-, M_- \otimes M_+^* \otimes M_+)$. Therefore, using the relations $\gamma_{23,4}\gamma_{3,45}^{-1} = \gamma_{23}\gamma_{45}^{-1}$, and $\beta\gamma = 1$, we reduce (9.13) to

$$(1\otimes i_+)\circ\sigma = (i_+^*\otimes 1\otimes 1)\circ\gamma_{23}\circ(\sigma\otimes\sigma) \tag{9.14}$$

in $\operatorname{Hom}(\mathbf{1}, M_+^* \otimes M_+ \otimes M_+)$. Since $i_+^* \circ \gamma = i_+^*$, we can rewrite (9.14) as

$$(1 \otimes i_+) \circ \sigma = (i_+^* \otimes 1 \otimes 1) \circ \gamma_{12,3} \circ (\sigma \otimes \sigma). \tag{9.15}$$

Using the equality $\gamma_{12,3} \circ (\sigma \otimes 1) = 1 \otimes \sigma$, we reduce (9.15) to

$$(1 \otimes i_+) \sigma = (i_+^* \otimes 1 \otimes 1) \circ (1 \otimes \sigma \otimes 1) \circ \sigma. \tag{9.16}$$

To prove this equality, we compute the image of $1 \in \mathbf{1}$ under right-hand side of (9.16). In this calculation, we can ignore the action of the associator because for any representations V_1, V_2, V_3 of $\mathfrak{g}$ the associator acts trivially on the $\mathfrak{g}$-invariants in $V_1 \otimes V_2 \otimes V_3$. The calculation yields that 1 goes to $(1 \otimes i_+)(\sigma(1))$, which proves (9.16). The claim is proved.

Let $\mathcal{M} \subset \mathcal{M}^e$ be the full subcategory of discrete $\mathfrak{g}$-modules, and $\tilde{U}_h(\mathfrak{g}) = \operatorname{End}(\tilde{F}|_{\mathcal{M}})$ be the quantization of $\mathfrak{g}$ constructed in Part I. It is easy to show that the homomorphism of topological Hopf algebras $\operatorname{End}\tilde{F} \to \tilde{U}_h(\mathfrak{g})$ defined by restriction from $\mathcal{M}^e$ to $\mathcal{M}$ is an isomorphism, since both algebras are canonically isomorphic to $U(\mathfrak{g})[[h]]$. This means that the morphism χ defined above induces an isomorphism of topological Hopf algebras $\tilde{U}_h(\mathfrak{g})$, $U_h(\mathfrak{g})$. It is easy to check that this isomorphism maps $\tilde{U}_h(\mathfrak{g}_+)$ onto $U_h(\mathfrak{g}_+)$, which proves the proposition. □

Appendix: computation of the product in $U_h(\mathfrak{a})$ modulo h^3

To illustrate the construction of quantization of Lie bialgebras, here we compute the product in the quantization $U_h(\mathfrak{a})$ of a Lie bialgebra $\mathfrak{a}$ modulo h^3. In the text below we always assume summation over repeated indices.

Let $\{a_i, i \in I\}$ be a basis of $\mathfrak{a}$, and $\{b^i\}$ be the topological basis of $\mathfrak{a}^*$ dual to $\{a_i\}$. Let us write down the commutation relations for the Lie algebra $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*$:

$$[a_i a_j] = c_{ij}^k a_k, [b^i b^j] = f_k^{ij} b^k, [a_i b^j] = f_i^{jk} a_k - c_{ik}^j b^k. \tag{A1}$$

Let $1_+^* \in M_+^*$ be the functional on M_+^* defined by $1_+^*(x 1_+) = \varepsilon(x)$, $x \in U(\mathfrak{a})$.

Let $\{(M_+^*)_n\}$ be the filtration of M_+^* which was defined in Chapter 7.

For $x \in U(\mathfrak{a})$, let $\psi_x : M_- \to M_+^* \hat{\otimes} M_-$ be the $\mathfrak{g}$-intertwiner such that

$$\psi_x 1_- \equiv 1_+^* \otimes x 1_- \mod (M_+^*)_1.$$

For $x, y \in U(\mathfrak{a})$, we defined the quantized product $z = y \circ x$ to be the element of $U(\mathfrak{a})[[h]]$ such that the operator ψ_z is the composition

$$M_- \xrightarrow{\psi_x} M_+^* \hat{\otimes} M_- \xrightarrow{1 \otimes \psi_y} M_+^* \hat{\otimes} (M_+^* \hat{\otimes} M_-) \xrightarrow{\Phi^{-1}}$$
$$(M_+^* \hat{\otimes} M_+^*) \hat{\otimes} M_- \xrightarrow{i_+^* \otimes 1} M_+^* \hat{\otimes} M_-. \tag{A2}$$

We want to compute the product $a_q \circ a_p$ modulo h^3. We fix elements $\rho_i \in M_+^*$, $i \in I$, such that $\rho_i(1_+) = 0$, $\rho_i(b^j 1_+) = \delta_i^j$. These elements are uniquely defined modulo $(M_+^*)_2$.

Let $w^i \in M_-$ be the vectors such that

$$\psi_{a_p} 1_- \equiv 1_+^* \otimes a_p 1_- + \rho_i \otimes w^i \mod (M_+^*)_2 \otimes M_-. \tag{A3}$$

We must have $b^j \psi_{a_p} 1_- = 0$ for all j, so $1_+^* \otimes b^j a_p 1_- + b^j \rho_i \otimes w_i = 0$. But $b^j \rho_i(1_+) = \rho_i(-b^j 1_+) = -\delta_i^j$, so we get $w^i = b^i a_p 1_- = -f_p^{ik} a_k 1_-$.

Thus we get

$$\psi_{a_p} 1_- \equiv 1_+^* \otimes a_p 1_- - f_p^{ik} \rho_i \otimes a_k 1_- \mod (M_+^*)_2 \hat{\otimes} M_-. \tag{A4}$$

Using (A4), we get

$$\psi_{a_q} a_r 1_- \equiv (a_r \otimes 1 + 1 \otimes a_r) \psi_{a_q} 1_- \equiv$$
$$1_+^* \otimes a_r a_q 1_- - f_q^{ik} a_r \rho_i \otimes a_k - f_q^{ik} \rho_i \otimes a_r a_k 1_- \mod (M_+^*)_2 \hat{\otimes} M_-. \tag{A5}$$

We have

$$a_r \rho_i(b^j 1_+) = -\rho_i(a_r b^j 1_+) = \rho_i(c_{rk}^j b^k 1_+) = c_{ri}^j, \tag{A6}$$

Thus, substituting (A6) into (A5), we get

$$\psi_{a_q} a_r 1_- \equiv 1_+^* \otimes a_r a_q 1_- - f_q^{ik} c_{ri}^j \rho_j \otimes a_k 1_- - f_q^{ik} \rho_i \otimes a_r a_k 1_- \mod (M_+^*)_2 \hat{\otimes} M_-. \tag{A7}$$

In particular, we have

$$\begin{gathered}(1 \otimes \psi_{a_q}) \psi_{a_p} 1_- \equiv 1_+^* \otimes 1_+^* \otimes a_q a_p 1_- \\ -c_{pi}^j f_q^{ik} 1_+^* \otimes \rho_j \otimes a_k 1_- - f_q^{ik} 1_+^* \otimes \rho_i \otimes a_p a_k 1_- - f_p^{ik} \rho_i \otimes 1_+^* \otimes a_k a_q 1_- + \\ f_p^{ik} c_{kl}^j f_q^{ls} \rho_i \otimes \rho_j \otimes a_s 1_- + f_p^{ik} f_q^{ls} \rho_i \otimes \rho_l \otimes a_k a_s 1_- \\ \mod (M_+^*)_2 \hat{\otimes} M_+^* \hat{\otimes} M_- + M_+^* \hat{\otimes} (M_+^*)_2 \hat{\otimes} M_-.\end{gathered} \tag{A8}$$

The definition of an associator implies

$$\Psi = 1 + \frac{h^2}{24}[t_{12}, t_{23}] + O(h^3) \tag{A9}$$

(see [Dr2], [Dr4]). This means that the part of the h^2-coefficient of $\Phi^{-1}_{V_1 V_2 V_3}$ which belongs to $\mathfrak{a}^* \otimes \mathfrak{a}^* \otimes \mathfrak{a}$ a is $\frac{1}{24} c_{ij}^k b^i \otimes b^j \otimes a_k$.

Now let us apply Φ^{-1} to both sides of (A8). We want to compute the answer in the form $1_+^* \otimes 1_+^* \otimes u + \dots$, $u \in M_-[[h]]$. To do this, we only need to use the last two terms on the r.h.s. of (A8) and the $\mathfrak{a}^*\otimes\mathfrak{a}$-part of the quadratic term of Φ. The calculation gives

$$\Phi^{-1}(1\otimes\psi_{a_q})\psi_{a_p}1_- \equiv 1_+^*\otimes 1_+^*\otimes u \mod (M_+^*)_1\hat{\otimes}M_+^*\hat{\otimes}M_- + M_+^*\hat{\otimes}(M_+^*)_1\hat{\otimes}M_-,$$

$$u = a_q a_p 1_- + \frac{h^2}{24}(f_p^{ik}f_q^{ls}c_{kl}^j c_{ij}^m a_m a_s + f_p^{in}f_q^{ls}c_{il}^r a_r a_n a_s)1_-. \tag{A10}$$

This shows that

$$a_q \circ a_p = a_q a_p + \frac{h^2}{24}(f_p^{ik}f_q^{ls}c_{kl}^j c_{ij}^m a_m a_s + f_p^{in}f_q^{ls}c_{il}^r a_r a_n a_s) + O(h^3). \tag{A11}$$

This formula is analogous to the formula deduced by Drinfeld [Dr3] (equation 1.1).

Remark. It is easy to see that this formula contains only acyclic monomials. In the second paper we will show that this is true for all coefficients of the quantization.

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