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**Zusammenfassung**

Es wird die Aufgabe der freien Flüssigkeitsschwingungen in Behältern geringer Tiefe untersucht, und zwar für ebene und axial-symmetrische Schwingungen. Insbesondere wird die Frage beantwortet, welchen Wert die zweite harmonische Schwingungsfrequenz höchstens annehmen kann, wenn zwar der Inhalt des Behälters vorgegeben ist, dagegen nicht seine Gestalt. Im Gegensatz zu einer früheren Untersuchung werden jedoch für dieses isoperimetrische Problem nur konvexe Behälter zugelassen. Mathematisch lässt sich das Ergebnis etwas weiter fassen: Es wird eine obere Schranke für den niedrigsten (nicht trivialen) Eigenwert einer Klasse von Sturm-Liouville Aufgaben ermittelt, wobei sich herausstellt, dass zur Abgrenzung dieser Klasse zwei feste Punkte im Integrationsintervall eine ausschlaggebende Rolle spielen.

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# Optimal Structural Design for Given Deflection[1])

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Dedicated to Professor Ziegler on his sixtieth birthday

## 1. Introduction

Design of structural elements for minimum weight was one of the early applications of variational calculus (see, for instance, [1]). The customary procedure uses only the Euler equation of the problem, which is a *necessary* condition for the structural weight to be stationary, but does not guaranty a local or global minimum. As Prager and Taylor [2] have shown, this procedure can, in many cases, be supplemented by an energetic approach that yields a *sufficient* condition for a *global* minimum of structural weight. This approach, however, is only feasible if the constraint imposed on the design concerns a structural property that can be characterized by a *global* minimum or maximum principle. For example, let the *compliance* of a linearly elastic structure under given loads be defined as the work that these loads do on the displace-

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ments they produce. It then follows from the principle of minimum potential energy that, for a structure of prescribed compliance, this compliance can be characterized as the global minimum of the strain energy for all kinematically admissible displacement fields (i.e. fields that satisfy the kinematical boundary conditions and certain continuity and differentiability requirements).

The energetic approach has been used in minimum-weight design for static elastic compliance [3–7], dynamic elastic compliance [8], compliance in stationary creep [9], elastic buckling load [10, 11], fundamental natural frequency [12–14], and plastic load-carrying capacity [15]. In the last-named field, the energetic approach had already been introduced by Drucker and Shield [16].

The design constraints treated in these references concern structural properties that are characterized by classical extremum principles of global type. A practically important constraint that cannot be cast in this mold specifies an upper bound on the deflection at a certain point of the structure. The present paper discusses the extent to which the energetic approach can be applied to design problems of this kind. For the sake of brevity, the general discussion is restricted to sandwich beams, but one of the examples concerns the optimal design of a rod.

## 2. Principle of Stationary Mutual Potential Energy

Consider a statically determinate or indeterminate beam that is simply supported, built in, or free, at the ends $x = 0$ and $x = l$, and may have intermediate hinges or supports at specified cross sections. Let $s(x)$ be the variable bending stiffness of the beam and consider two states of loading, denoting their distributed loads by $q(x)$, $\bar{q}(x)$, and typical concentrated loads and couples by $Q$, $\bar{Q}$ and $C$, $\bar{C}$. Finally, let $u^*(x)$, $\bar{u}^*(x)$ be any *kinematically admissible* deflections of the beam and $\theta^*(x) = u^{*\prime}(x)$, $\bar{\theta}^*(x) = \bar{u}^{*\prime}(x)$ and $\varkappa^*(x) = u^{*\prime\prime}(x)$, $\bar{\varkappa}^*(x) = \bar{u}^{*\prime\prime}(x)$ the corresponding rotations and curvatures. Kinematically admissible deflections will be defined in the usual manner as continuous deflections that satisfy the kinematic conditions at the supports and have continuous rotations except, possibly, at hinges. This means that beams with other intermediate cross sections of vanishing bending stiffness are excluded from the discussion because additional discontinuities of rotation could develop at these sections. This restriction is meaningful because beams of the excluded type are not practical.

The *mutual potential energy* for these deflections and the given loads will be defined as the functional

$$U[u^*, \bar{u}^*; s] = \frac{1}{2} \left\{ \int s\, \varkappa^*\, \bar{\varkappa}^*\, dx - \int q\, \bar{u}^*\, dx - \int \bar{q}\, u^*\, dx \right.$$
$$\left. - \sum Q\, \bar{u}^* - \sum \bar{Q}\, u^* - \sum C\, \bar{\theta}^* - \sum \bar{C}\, \theta^* \right\}, \tag{2.1}$$

where the integrations extend over the entire beam and the sums include all concentrated loads and couples. Note that this definition reduces to the customary definition of potential energy when the barred quantities are identical with the unbarred quantities. Note further that the mutual potential energy for the true deflections $u(x)$, $\bar{u}(x)$, rotations $\theta(x)$, $\bar{\theta}(x)$, and curvatures $\varkappa(x)$, $\bar{\varkappa}(x)$ produced by the considered systems of loads may be written as

$$
\begin{aligned}
U[u, \bar{u};\ s] &= -\frac{1}{2}\int s\,\varkappa\,\bar{\varkappa}\,dx \\
&= -\frac{1}{2}\left\{\int q\,\bar{u}\,dx + \sum Q\,\bar{u} + \sum C\,\bar{\theta}\right\} \\
&= -\frac{1}{2}\left\{\int \bar{q}\,u\,dx + \sum \bar{Q}\,u + \sum \bar{C}\,\theta\right\}. \qquad (2.2)
\end{aligned}
$$

We shall now prove that the functional $U[u^*, \bar{u}^*; s]$ is stationary at $u^* = u$, $\bar{u}^* = \bar{u}$. Indeed, it follows from (2.1) and the first equality of (2.2) that

$$
\begin{aligned}
U[u^*, \bar{u}^*; s] - U[u, \bar{u}; s] &= \frac{1}{2}\left\{\int s\,\varkappa^*\,\bar{\varkappa}^*\,dx + \int s\,\varkappa\,\bar{\varkappa}\,dx\right. \\
&\left. -\int q\,\bar{u}^*\,dx - \int \bar{q}\,u^*\,dx - \sum Q\,\bar{u}^* - \sum \bar{Q}\,u^* - \sum C\,\bar{\theta}^* - \sum \bar{C}\,\theta^*\right\}. \qquad (2.3)
\end{aligned}
$$

On the other hand, using the principle of virtual work, one readily shows that

$$
\begin{aligned}
\int s\,(\varkappa^* - \varkappa)\,(\bar{\varkappa}^* - \bar{\varkappa})\,dx &= \int s\,\varkappa^*\,\bar{\varkappa}^*\,dx + \int s\,\varkappa\,\bar{\varkappa}\,dx \\
&-\int q\bar{u}^*\,dx - \int \bar{q}\,u^*\,dx - \sum Q\,\bar{u}^* - \sum \bar{Q}\,u^* - \sum C\,\bar{\theta}^* - \sum \bar{C}\,\theta^*. \qquad (2.4)
\end{aligned}
$$

Use of (2.4) in (2.3) furnishes the identity

$$
U[u^*, \bar{u}^*; s] - U[u, \bar{u}; s] = \frac{1}{2}\int s\,(\varkappa^* - \varkappa)\,(\bar{\varkappa}^* - \bar{\varkappa})\,dx\,, \qquad (2.5)
$$

which applies to any kinematically admissible deflections $u^*$, $\bar{u}^*$ with rotations $\theta^*$, $\bar{\theta}^*$ and curvatures $\varkappa^*$, $\bar{\varkappa}^*$, and the true deflections $u$, $\bar{u}$ with rotations $\theta$, $\bar{\theta}$ and curvatures $\varkappa$, $\bar{\varkappa}$.

Applied to the neighborhood $u^* = u + \delta\,u$, $\bar{u}^* = \bar{u} + \delta\,\bar{u}$ of the true deflections, (2.5) furnishes the following first and second-order relations

$$
\delta U = 0\,, \qquad (2.6)
$$

$$
\delta^2 U = \frac{1}{2}\int s\,\delta\varkappa\,\delta\bar{\varkappa}\,dx\,, \qquad (2.7)
$$

where $\delta\,\varkappa = (\delta\,u)''$, $\delta\,\bar{\varkappa} = (\delta\,\bar{u})''$. The first of these relations shows that $U[u^*, \bar{u}^*; s]$ is stationary at $u^* = u$, $\bar{u}^* = \bar{u}$. Because, however, the integral on the right of (2.7) is not, in general, restricted in sign, we cannot assert that $U[u^*, \bar{u}^*; s]$ has a minimum or maximum at $u^* = u$, $\bar{u}^* = \bar{u}$.

The principle of stationary mutual potential energy is readily extended to elastic plates and shells, and to three-dimensional elastic bodies.

## 3. Application to Optimal Design for Given Deflections or Rotations

According to the third equality in (2.2), the deflection $u(x_0)$ that the loads $q$, $Q$, $C$ produce at the cross section $x_0$ of the beam equals $-2\,U[u, \bar{u}; s]$, when the second system of loads is reduced to a concentrated unit force $\bar{Q}$ at $x = x_0$, and similar statements can be made for arbitrary linear combinations of deflections or rotations at specified cross sections. The principle of stationary mutual potential energy can therefore be used in optimal design for given static deflections or rotations in very much the same way in which the principle of minimum potential energy has been applied in [2] through [7] to optimal design for given static compliance. As a rule, however, the first principle furnishes only a sufficient condition for the structural weight to be *stationary*, whereas the second principle provides a sufficient condition for a *global minimum* of structural weight.

For brevity, most of the following discussion will be restricted to minimum-weight design of a sandwich beam of given constant core dimensions and continuously variable thickness of the identical cover sheets. Since the core weight is not subject to variation, minimizing the structural weight is then equivalent to minimizing the integral of the bending stiffness over the entire beam. The given loads $q$, $Q$, $C$ are to produce the prescribed deflection $u_0$ at the cross section $x_0$. In addition to this state of loading, we consider a second state that only involves the concentrated unit force $\bar{Q}$ at $x_0$.

Let $s$ and $s^*$ be the bending stiffnesses of two designs that satisfy the constraint on the deflection at $x_0$, and denote by $u$, $u^*$ and $\bar{u}$, $\bar{u}^*$ the deflections of these designs under the first and second states of loading, respectively, and by $\varkappa$, $\varkappa^*$ and $\bar{\varkappa}$, $\bar{\varkappa}^*$ the corresponding curvatures. Since both designs satisfy the constraint on deflection,

$$U[u^*, \bar{u}^*; s^*] - U[u, \bar{u}; s] = 0\,. \tag{3.1}$$

On the other hand, since the deflections $u$, $\bar{u}$ are kinematically admissible for the design $s^*$, it follows from (2.5) *applied to this design* that

$$U[u, \bar{u}; s^*] - U[u^*, \bar{u}^*; s^*] = \frac{1}{2} \int s^* \, (\varkappa^* - \varkappa)\, (\bar{\varkappa}^* - \varkappa)\, dx\,. \tag{3.2}$$

Substituting $U[u^*, \bar{u}^*; s]$ from (3.1) into (3.2) and using the definition (2.1), we obtain

$$\int (s^* - s)\, \varkappa\, \bar{\varkappa}\, dx = \int s^* \, (\varkappa^* - \varkappa)\, (\bar{\varkappa}^* - \bar{\varkappa})\, dx\,. \tag{3.3}$$

Restricting the design $s^*$ to the neighborhood of the design $s$, and writing $s^* = s + \delta s$, $u^* = u + \delta u$, etc., we deduce from (3.3) that

$$\varkappa\, \bar{\varkappa} = \text{const.}\ (= c^2, \text{say}) \tag{3.4}$$

is a *sufficient* condition for $\int \delta s \, dx$ to vanish to first order, i.e., for the structural weight to be stationary for all designs $s + \delta s$ that satisfy the constraint on the deflection at $x_0$. That this condition is also *necessary* for stationary weight is readily shown by variational calculus (see, for instance, [17, 18]).

If $M = s\,\varkappa$ and $\overline{M} = s\,\bar{\varkappa}$ are the bending moments for the optimal design $s(x)$ in the two states of loading, (3.4) is equivalent to $M\,\overline{M}/s^2 = c^2$ or

$$s = \frac{1}{c}\,(M\,\overline{M})^{1/2}\,. \tag{3.5}$$

The constant $c$ in (3.5) must be determined from the condition that the deflection at $x_0$ should have the value $u_0$. Using the principle of virtual work and (3.5), we have

$$\overline{Q}\,u_0 = \int [M\,\overline{M}/s]\,dx = c \int (M\,\overline{M})^{1/2}\,dx\,. \tag{3.6}$$

Substitution of $c$ from (3.6) into (3.5) finally yields

$$s = \frac{(M\,\overline{M})^{1/2}}{\overline{Q}\,u_0} \int (M\,\overline{M})^{1/2}\,dx\,. \tag{3.7}$$

In using (3.7), we must keep in mind that the load $\overline{Q}$ has unit intensity; the only reason why it has not been suppressed in (3.7) is the desire to bring out the dimensional correctness of this equation.

It is readily shown that, for a statically determinate beam, the design $s$ satisfying (3.4) in addition to the constraint on deflection corresponds to a *global minimum* of structural weight. Indeed, for a statically determinate beam, the bending moments do not depend on the choice of bending stiffness: $M = M^*$, $\overline{M} = \overline{M}^*$. Accordingly,

$$s\,\varkappa = s^*\,\varkappa^*\,, \quad s\,\bar{\varkappa} = s^*\,\bar{\varkappa}^*\,. \tag{3.8}$$

The first of these equations may be written as $(s^* - s)\,\varkappa + s^*\,(\varkappa^* - \varkappa) = 0$ or

$$\varkappa^* - \varkappa = -\frac{(s^* - s)\,\varkappa}{s^*}\,. \tag{3.9}$$

Substituting this and the analogous equation for $\bar{\varkappa}^* - \bar{\varkappa}$ into (3.3) and using (3.4), we obtain the inequality

$$\int (s^* - s)\,dx = \int \frac{(s^* - s)^2}{s^*}\,dx \geqslant 0\,, \tag{3.10}$$

which shows that, for a statically determinate beam, any design $s^*$ that satisfies the constraint on deflection cannot be lighter than the design $s$ that, in addition, satisfies (3.4).

While the preceding discussion has for brevity been restricted to beams, the general method applies equally well to other simple structures, for example, rods or plates.

## 4. Examples

A) A simply supported sandwich beam of the span $2\,l$ under the uniformly distributed load $q$ is to be designed for minimum weight subject to the constraint that the deflection at the center of the span ($x = 0$) has the value $u_0$. Here,

$$M = -\frac{1}{2} q (l^2 - x^2), \quad \overline{M} = -\frac{1}{2} \overline{Q} (l - |x|). \tag{4.1}$$

Substitution of (4.1) into (3.7) and performance of the integration furnishes the optimal design

$$s(x) = \frac{1}{15} \frac{q l^{5/2}}{u_0} (8\sqrt{2} - 7) (l - |x|) (l + |x|)^{1/2}, \tag{4.2}$$

which represents a global minimum of structural weight because the beam is statically determinate.

B) A sandwich beam of the length $4l$ that is built in at both ends carries the uniformly distributed load $q$; it is to be designed for minimum weight subject to the constraint that the deflection at the center of the span ($x = 0$) is to have the value $u_0$. The beam is to have the constant bending stiffnesses $s_1$ in $-l < x < l$ and $s_2$ in $-2l \leqslant x < -l$ and $l < x \leqslant 2l$.

In view of the symmetry with respect to the center of the span, we have $u'(0) = 0$ in addition to $u'(l) = 0$. Thus $u''(x)$, and hence $M(x)$, has a zero in $0 \leqslant x \leqslant 2l$, say at $x = a$. Similarly, $\overline{M}(x)$ has a zero in this interval, say at $x = b$. Accordingly,

$$M = -\frac{1}{2} q (a^2 - x^2), \quad \overline{M} = -\frac{1}{2} \overline{Q} (b - x) \text{ in } 0 \leqslant x \leqslant 2l. \tag{4.3}$$

The arguments that furnished the optimality condition (3.4) for a sandwich beam of continuously varying bending stiffness yield the optimality condition

$$\frac{1}{l_i} \int \varkappa_i \overline{\varkappa}_i \, dx_i = \text{independent of } i \ (= c^2, \text{ say}) \tag{4.4}$$

when the bending stiffness is segmentwise constant. In (4.4), $l_i$ denotes the length of the $i$-th segment, $x_i$ is the abscissa of the typical cross section of this segment, and $\varkappa_i = \varkappa(x_i)$, $\overline{\varkappa}_i = \overline{\varkappa}(x_i)$.

When the curvatures in (4.4) are expressed in terms of the bending moments $M_i = M(x_i)$ and $\overline{M}_i = \overline{M}(x_i)$, use of (4.3) furnishes the optimality conditions

$$\int M_1 \overline{M}_1 \, dx_1 = \frac{q \overline{Q} l^4}{48} (12 \alpha^2 \beta - 6 \alpha^2 - 4 \beta + 3) = c^2 l s_1^2, \tag{4.5}$$

$$\int M_2 \overline{M}_2 \, dx_2 = \frac{q \overline{Q} l^4}{48} (12 \alpha^2 \beta - 18 \alpha^2 - 28 \beta + 45) = c^2 l s_2^2,$$

where $\alpha = a/l$, $\beta = b/l$.

Because $u'(0) = u'(2l) = 0$, the value of $\alpha$ must be determined in such a manner that

$$\int_0^{2l} \varkappa \, dx = \frac{1}{s_1} \int M_1 \, dx_1 + \frac{1}{s_2} \int M_2 \, dx_2 = 0. \tag{4.6}$$

Substituting (4.3) into (4.6) and solving for $s_1/s_2$, we obtain

$$s_1/s_2 = (3\,\alpha^2 - 1)/(7 - 3\,\alpha^2)\,. \tag{4.7}$$

When $\varkappa$ and $M$ in (4.6) are replaced by $\bar{\varkappa}$ and $\overline{M}$, the same procedure yields

$$s_1/s_2 = (2\,\beta - 1)/(3 - 2\,\beta)\,. \tag{4.8}$$

Elimination of $s_1/s_2$ from (4.7) and (4.8) furnishes

$$\beta = \frac{1}{6}\,(3\,\alpha^2 + 2)\,. \tag{4.9}$$

Equating the values of $s_1^2/s_2^2$ that follow from (4.5) and (4.7) and using (4.9), we obtain

$$\left(\frac{(3\,\alpha^2 - 1)}{(7 - 3\,\alpha^2)}\right)^2 = \frac{18\,\alpha^4 - 12\,\alpha^2 + 5}{18\,\alpha^4 - 84\,\alpha^2 + 107}\,. \tag{4.10}$$

The real roots of (4.10) are found to correspond to $\alpha^2 = \sqrt{3} - (2/3)$, which yield $s_1/s_2 = \sqrt{3}/3$ and $\beta = \sqrt{3}/2$ by (4.7) and (4.9), respectively.

The value of $s_1$ may finally be obtained from the deflection constraint, which may be written as

$$\int M_1\,\overline{M}_1\,dx_1 + \frac{s_1}{s_2}\int M_2\,\overline{M}_2\,dx_2 = \frac{1}{2}\,\overline{Q}\,u_0\,s_1\,. \tag{4.11}$$

One finds $s_1 = 0.480\,q\,l^4/u_0$ and hence $s_2 = 0.831\,q\,l^4/u_0$ by (4.7).

## 5. Generalizations

In the following, some generalizations of the results of Section 3 will be illustrated by examples.

A) *Constraint on Rotation of Cross Section.* Haug, Streeter and Newell [19] have described a situation in which imposition of a constraint on the rotation of a cross section $x = x_0$ is meaningful. To make the discussion in Section 3 cover this type of constraint, one only replaces the unit load $\overline{Q}$ at $x_0$ by a unit couple $\overline{C}$ at $x_0$. Consider, for instance, a cantilever sandwich beam of length $l$ that is built in at $x = l$ and carries the uniformly distributed load $q$. If $|u'(0)|$ is to have the value $\theta_0$, the optimal design $s$ is found from (3.7) by replacing $\overline{Q}\,u_0$ by $\overline{C}\,\theta_0$ and using

$$M = \frac{1}{2}\,q\,x^2\,,\quad \overline{M} = \overline{C}\,. \tag{5.1}$$

Thus,

$$s(x) = q\,l^2\,x/(4\,\theta_0)\,. \tag{5.2}$$

B) *Other Types of Cross Section.* For the sandwich section considered throughout the preceding discussion, stationary structural weight requires $\int \delta\,s\,dx = 0$. On the other hand, for a solid beam with rectangular section of fixed width $b$ and variable height $2h$, the weight per unit length is proportional to $s^{1/3}$. Stationary weight therefore requires $\int s^{-2/3}\, \delta s\, dx = 0$, and it follows from (3.3) that

$$s^{2/3}\, \varkappa\, \bar{\varkappa} = c^2 \tag{5.3}$$

is a sufficient condition for stationary weight. Proceeding as in the derivation of (3.7), we obtain

$$s = \frac{(M\,\overline{M})^{3/4}}{\overline{Q}\, u_0} \int (M\,\overline{M})^{1/4}\, dx \tag{5.4}$$

as the design of stationary structural weight. For the example in Section 5A, this yields

$$s(x) = \frac{q(l\,x)^{3/2}}{3\,\theta_0}\,. \tag{5.5}$$

It can be shown that this design represents, in fact, a global minimum of structural weight.

The discussion is readily extended to cross sections with two variable dimensions. Consider, for example, a sandwich beam with a core of fixed width $b$ and variable height $2h$ and identical cover sheets of width $b$ and variable thickness $t \ll h$. As $t$ and $h$ vary independently, the variation of the weight per unit length is proportional to $\delta t + \beta\, \delta h$, where $\beta \ll 1$ is the ratio between the specific weights of core and cover sheets. The variation of the bending stiffness, however, is proportional to $h^2\, \delta t + 2\, t\, h\, \delta h$ and (3.3) furnishes

$$\int (h^2\, \delta t + 2\, t\, h\, \delta h)\, \varkappa\, \bar{\varkappa}\, dx = 0\,. \tag{5.6}$$

If this is to be equivalent to $\int \delta w\, dx = 0$, we must have

$$h^2\, \varkappa\, \bar{\varkappa} = c^2\,, \quad 2\, h\, t\, \varkappa\, \bar{\varkappa} = \beta\, c^2\,. \tag{5.7}$$

These are sufficient conditions for the structural weight to be stationary. Elimination of $\varkappa\, \bar{\varkappa}/c^2$ from these conditions yields

$$t = \beta\, h/2\,. \tag{5.8}$$

Since this means that the weight per unit length is proportional to $s^{1/3}$, the optimal design of the uniformly loaded cantilever beam with prescribed tip rotation is again given by (5.5).

C) To illustrate minimum weight design for given bound on the displacement in either one of two alternative states of loading, we consider a rod of the length $l$ that is fixed at $x = l$ and free at $x = 0$. The absolute value of the axial displacement at $x = 0$ is not to exceed the given value $u_0$ under either one of the following alternative loads: (1) a concentrated tensile load $Q$ at $x = 0$, and (2) a uniformly distributed tensile load of intensity $q$.

If only one of the two loads is relevant, the optimal design of this statically determinate rod is given by the formula

$$s\,(x) = \frac{(N\,\overline{N})^{1/2}}{\overline{Q}\, u_0} \int (N\,\overline{N})^{1/2}\, dx\,, \tag{5.9}$$

which corresponds to (3.7). Here, $N$ and $\overline{N}$ are the axial forces under the relevant load and under the unit axial load $\overline{Q}$ at $x = 0$, and $s(x) = E\,A(x)$, where $E$ is Young's modulus and $A(x)$, the cross-sectional area at $x$. For the given states of loading, the axial forces are

$$N_1 = Q\,, \quad N_2 = q\,x\,, \quad \overline{N} = \overline{Q}\,. \tag{5.10}$$

Accordingly, if only the concentrated load $Q$ is relevant for the optimal design, we have the optimal axial stiffness

$$s_1 = Ql/u_0\,. \tag{5.11}$$

For this design, the tip displacement caused by the load $q$ has the absolute value

$$|\,u_1(0)\,| = \int \frac{N_2}{s_1}\,dx = \frac{q\,l}{2\,Q}\,u_0\,, \tag{5.12}$$

and the assumption that only $Q$ is relevant requires that this value be smaller than $u_0$, that is, that $q\,l < 2\,Q$. If, on the other hand, only the distributed load is relevant, (5.9) yields the optimal design

$$s_2 = \frac{2}{3}\,\frac{q\,l^{3/2}}{u_0}\,x^{1/2}\,, \tag{5.13}$$

and the tip displacement of this design caused by $Q$ has the absolute value

$$|\,u_2(0)\,| = \int \frac{N_1}{s_2}\,dx = \frac{3\,Q}{q\,l}\,u_0\,. \tag{5.14}$$

The assumption that only $q$ is relevant requires that this value be smaller than $u_0$, that is, that $q\,l > 3\,Q$. Thus, both states of loading are relevant for the optimal design if

$$2\,Q < q\,l < 3\,Q\,. \tag{5.15}$$

For loadings in this range, it can be shown as in [3] that the optimality condition $N\,\overline{N} = c^2\,s^2$, which corresponds to (3.5), must be replaced by the condition

$$(\lambda\,N_1 + \mu\,N_2)\,\overline{N} = s^2\,, \tag{5.16}$$

where the nonnegative constants $\lambda$, $\mu$ must be determined from the condition that each loading produces a tip deflection of the absolute value $u_0$:

$$u_0 = \int \frac{N_1}{s}\,dx = \int \frac{N_2}{s}\,dx\,. \tag{5.17}$$

Setting $x/l = \xi$, $\mu/\lambda = \alpha$ and $q\,l = \beta\,Q$, where $2 < \beta < 3$ by (5.15), we write (5.16) in the form

$$s = (\lambda\,Q\,\overline{Q})^{1/2}\,(1 + \alpha\,\beta\,\xi)^{1/2}\,. \tag{5.18}$$

Substituting (5.18) into (5.17) and performing the integrations, one obtains

$$\frac{\alpha\,\beta\,u_0}{2\,Q\,l}\,(\lambda\,Q\,\overline{Q})^{1/2} = (1+\alpha\,\beta)^{1/2} - 1$$
$$= \frac{1}{\alpha}\left\{\frac{1}{3}\,[(1+\alpha\,\beta)^{3/2} - 1] - [(1+\alpha\,\beta)^{1/2} - 1]\right\}. \tag{5.19}$$

Taking, for instance, $1 + \alpha\,\beta = 9$, we obtain $\alpha = 10/3$ from the second part of this continued equation and hence $\beta = 8/\alpha = 12/5$, which is in the range (2.3). Using these values in the first part of (5.19), and substituting the resulting value of $\lambda\,Q\,\overline{Q}$ into (5.18), we find the optimal design

$$s = \frac{Q\,l}{2\,u_0}\left(1 + 8\,\frac{x}{l}\right)^{1/2}, \tag{5.20}$$

which corresponds to $q\,l = 12\,Q/5$.

D) Minimum-weight design for a single system of loads but two or more constraints on deflection or rotation can be treated in a similar manner. Consider, for instance, a cantilever sandwich beam of fixed core dimensions and variable thickness of the identical face sheets that is built in at $x = l$ and carries the transverse load $Q$ at $x = 0$. If $u_0$ and $\theta_0$ are given upper bounds on tip deflection and tip rotation, respectively, both constraints are found to be relevant for $5/3 < \theta_0\,l/u_0 < 2$. The optimal design has the bending stiffness

$$s = \{Q\,x\,[\lambda\,\overline{C} + \mu\,\overline{Q}\,x]\}^{1/2} \tag{5.21}$$

where $\overline{C}$ is a unit couple, $\overline{Q}$ a unit load, and the positive constants $\lambda$, $\mu$ must be chosen in such a manner that tip deflection and rotation have the prescribed values $u_0$ and $\theta_0$.

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**Zusammenfassung**

Es wird ein Prinzip der stationären gegenseitigen potentiellen Energie aufgestellt für zwei Belastungssysteme eines elastischen Balkens veränderlicher Biegesteifigkeit. Aus diesem Prinzip wird eine hinreichende Bedingung für stationäres Gewicht eines Sandwichbalkens abgeleitet, wenn die von einer Belastung an einem bestimmten Querschnitt erzeugte Durchbiegung vorgeschrieben ist. Für statisch bestimmte Balken wird gezeigt, dass diese Bedingung ein globales Minimum des Gewichts sicherstellt. Anwendungsbeispiele und Erweiterungen werden besprochen.

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# Der Einfluß verschiedener Kräftearten auf die Stabilität linearer Systeme

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Herrn Professor Dr. Hans Ziegler zum sechzigsten Geburtstag gewidmet

## 1. Problemstellung

Zur Bestimmung der Stabilität von Bewegungen linearer Systeme finden sich im Schrifttum überaus viele Verfahren. Es sei hier an die algebraischen Kriterien von Hermite, Routh, Hurwitz, Cremer und Bilharz sowie an die völlig äquivalenten geometrischen Kriterien von Nyquist, Leonhard, Michailow und Neumark erinnert. Diese Kriterien lassen bei geeigneter Art der Anwendung den Einfluss einzelner Systemparameter erkennen und haben sich deshalb recht gut bewährt. Jedoch wird eine physikalische Interpretation der Ergebnisse um so schwieriger, je höher die Ordnung der zu untersuchenden Systeme ist.