Mathematical Programming 44 (1989) 293-295
North-Holland

OPEN QUESTIONS CONCERNING WEISZFELD'S ALGORITHM FOR THE FERMAT-WEBER LOCATION PROBLEM

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Received 9 March 1987
Revised manuscript received 29 February 1988

The Fermat-Weber location problem is to find a point in $\mathbb{R}^n$ that minimizes the sum of the weighted Euclidean distances from m given points in $\mathbb{R}^n$. A popular iterative solution method for this problem was first introduced by Weiszfeld in 1937. In 1973 Kuhn claimed that if the m given points are not collinear then for all but a denumerable number of starting points the sequence of iterates generated by Weiszfeld's scheme converges to the unique optimal solution. We demonstrate that Kuhn's convergence theorem is not always correct. We then conjecture that if this algorithm is initiated at the affine subspace spanned by the m given points, the convergence is ensured for all but a denumerable number of starting points.

Key words: Location theory, The Fermat-Weber location problem, Weiszfeld's iterative algorithm.

Let $a^1, \ldots, a^m$ be m distinct points in $\mathbb{R}^n$. Suppose that each point a^i, $1 \leq i \leq m$, is associated with a positive weight w_i. The *Fermat-Weber location problem* [6, 9] is to find a point in $\mathbb{R}^n$ that will minimize the sum of the (weighted) Euclidean distances from the m given points:

$$\min f(x) = \sum_{i=1}^{m} w_i \|x - a^i\|. \tag{1}$$

It is well known that if the data points are not collinear the objective is strictly convex, and therefore has a unique optimum. (In the collinear case at least one of the points $a^1, \ldots, a^m$ is optimal and it can be found in linear time by the algorithm in [1].)

There are several infinite schemes to solve the Fermat-Weber location problem [2-4, 6, 8, 10]. One of the most popular algorithms was discovered by Weiszfeld [10]. It has since been analysed extensively in [5] and [7]. The algorithm is based on the following mapping of $\mathbb{R}^n$ into the convex hull of $a^1, \ldots, a^m$;

$$T(x) = \begin{cases} \dfrac{\sum_{i=1}^m w_i \|x - a^i\|^{-1} a^i}{\sum_{i=1}^m w_i \|x - a^i\|^{-1}} & \text{if } x \neq a^1, \ldots, a^m, \\ a^i & \text{if } x = a^i \text{ for some } i = 1, \ldots, m. \end{cases} \tag{2}$$

Weiszfeld's algorithm is defined by the following iterative scheme:

$$x^{r+1} = T(x^r). \tag{3}$$

Notice that the sequence of iterates $\{x^r\}$, $r=0, 1, 2, \ldots$, may in general contain irrational elements. Theoretically, it is not even clear to us what should be the correct precision for rational approximations to the iterates, such that the scheme (3) will generate an ε-approximation in efficient time. (Assume rational data.)

Convergence results of the above algorithm are discussed in [5, 7]. In [7] Kuhn claimed that the sequence $\{x^r\}$, $r=0, 1, 2, \ldots$, converges to the unique optimum for all but a denumerable number of starting points x^0. Katz [5] derived results on the (local) rapidity of the convergence. Specifically, if the optimum is not one of the m original points then the convergence is always linear. However, when the optimum coincides with an original point convergence can be either linear, superlinear, or sublinear. (It is not at all clear whether testing the optimality of an original point can be done in polynomial time.)

In this note we point out a flaw in the main convergence result of [7]. It is stated in [7] that if the points $a^1, \ldots, a^m$ are non-collinear, then for all but a denumerable number of initial points x^0, the sequence $\{x^r\}$, defined iteratively by (3), converges to the unique optimal solution. The argument of the proof is based on the claim that for each a^i, $i=1, \ldots, m$, the algebraic system $T(x)=a^i$ has a finite number of solutions. We show that the non-collinearity is not sufficient to ensure the validity of Kuhn's convergence theorem. In particular, we demonstrate that the system $T(x)=a^i$ can have a continuum set of solutions even when the points $a^1, \ldots, a^m$ are not collinear. Using (2) and (3), it will then follow that Weiszfeld's algorithm fails to converge when it starts at any point of the above continuum set.

Example 1. Consider the unweighted problem in $\mathbb{R}^3$ defined by the four points $a^1=(1, 0, 0)$, $a^2=(0, 1, 0)$, $a^3=(-1, -1, 0)$ and $a^4=(0, 0, 0)$. It is easily verified that $T(x)=a^4$ for any $x=(-\frac{1}{6}, -\frac{1}{6}, x_3)$ in $\mathbb{R}^3$. However, the point a^4 is the optimal point.

Example 1 contradicts a statement in [5, Section 5], which claims that if a^i is the optimal point the system $T(x)=a^i$ has a finite number of solutions.

The next example demonstrates that the system $T(x)=a^i$ can have a continuum set of solutions even if a^i is not optimal.

Example 2. Consider the problem in $\mathbb{R}^3$ defined by $a^1=(1, 0, 0)$, $a^2=(-1, 0, 0)$, $a^3=(0, 0, 0)$, $a^4=(0, 2, 0)$ and $a^5=(0, -2, 0)$. Let $w_1=w_2=w_3=w_5=1$ and $w_4=3$. Consider the point a^3. (a^3 is not optimal since $f(a^3)=10>f((0, 1, 0))=2\sqrt{2}+1+3+3=7+2\sqrt{2}$.) We show that the algebraic system $T(x)=a^3$ has an infinite number of solutions. Consider the points $x=(0, x_2, x_3)$. $T(x)=a^3$ is equivalent to the system

$$\frac{3a^4}{\|x-a^4\|}+\frac{a^5}{\|x-a^5\|}=0.$$

Therefore, $6\sqrt{(x_2+2)^2+x_3^2}=2\sqrt{(x_2-2)^2+x_3^2}$. Thus, all points $(0, x_2, x_3)$ on the circle $(x_2+\frac{5}{2})^2+x_3^2=\frac{9}{4}$ satisfy $T(x)=a^3$.

In view of the above examples we conjecture that if the non-collinearity is replaced by the stronger assumption that the convex hull of the points $a^1, \ldots, a^m$ is of full dimension, then the algebraic system $T(x) = a^i$ has a finite number of solutions for $i = 1, \ldots, m$. Phrased differently, the conjecture is that for each $i = 1, \ldots, m$, there is a finite number of solutions to $T(x) = a^i$ in the minimal affine set containing the points $a^1, \ldots, a^m$. (Note that if a^i is an extreme point of the convex hull of $a^1, \ldots, a^m$, then $x = a^i$ is the unique solution to the system $T(x) = a^i$.) If the conjecture is true then Kuhn's arguments will imply that whenever Weiszfeld's scheme is initiated at the affine set containing the points $a^1, \ldots, a^m$, convergence is guaranteed for all but a denumerable number of starting points.

Finally, we pose an interesting question which follows from the above discussion. Using the results in [5, 7] one can easily find, in finite time, a rational initial point x^0 for which the respective sequence of iterates generated by Weiszfeld's scheme converges to the optimal solution. Assuming that all data are integer, can such a point x^0 be found and computed in polynomial time?

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