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A PROBABILISTIC ANALYSIS OF THE SWITCHING ALGORITHM FOR THE EUCLIDEAN TSP

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The well-known switching algorithm proposed by Lin and Kernighan for the Euclidean Travelling Salesman Problem has proved to be a simple efficient algorithm for medium size problems (though it often gets trapped in local optima). Although its complexity status is still open, it has been observed to be polynomially bounded in practice, when applied to uniformly distributed points in the unit square. In this paper this polynomial behaviour is derived theoretically. (However, we will come up with a bound of $O(n^{18})$ with probability $1-c/n$, whereas in practice the algorithm works slightly better.)

1. Introduction

Suppose we are given n points, labelled $1,\ldots,n$ in the plane. A *tour* is then any sequence (permutation) $\sigma=(\sigma_1,\ldots,\sigma_n)$ of the n points. The *length* of a tour σ is the sum of all distances of consecutive points:

$$L(\sigma)=\sum_{i=1}^{n} d(\sigma_i,\sigma_{i+1}) \quad (\text{with } \sigma_{n+1}=\sigma_1).$$

With these notations, the *Euclidean Travelling Salesman Problem* can be stated as follows: Given a set of n points, labelled by $1,\ldots,n$, find a tour of minimal length. This problem is (well-) known to be NP-complete and several heuristics have been developed for solving it. Apart from the cutting plane approach taken by several authors (e.g. Grötschel, Fleischmann, Padberg, to mention just three of them), there are two well-known "probabilistic" algorithms. The first of them is based on partitioning the set of points, constructing subtours in each part and to connect them to a tour (cf. [2] for further information). The second one, due to Lin and Kernighan (cf. [3]) is based on a switching procedure (described below) which improves a given current solution step by step until a "local" optimum is reached. It is this method we will be concerned with in the following.

Suppose we are given a tour $\sigma=(\sigma_1,\ldots,\sigma_n)$. If $i<j$, then we will say that the tour $\tilde{\sigma}$, defined by

$$\tilde{\sigma}=(\sigma_1,\ldots,\sigma_{i-1},\sigma_j,\sigma_{j-1},\ldots,\sigma_i,\sigma_{j+1},\ldots,\sigma_n)$$

is obtained from σ by switching i and j (cf. Fig. 1). This switching operation naturally defines a neighbourhood $N(\sigma)$ of σ, consisting all $\tilde{\sigma}$ obtainable form σ by switching some i and j.

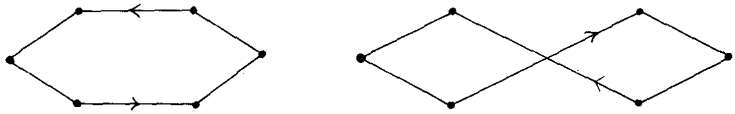

Fig. 1

Now the algorithm of Lin and Kernigham—denoted by ALK for short—simply works as follows:

INIT: Choose an initial solution σ

LOOP: If there is any $\tilde{\sigma} \in N(\sigma)$ such that $L(\tilde{\sigma}) < L(\sigma)$, then replace σ by $\tilde{\sigma}$ and return to LOOP. Else STOP.

There are two major problems with this algorithm. First, one can show that, in general, it does not yield an optimal solution, but can get "trapped in a local optimum". Usually, one tries to overcome this difficulty by running ALK several times with randomly chosen initial solutions. The second problem is, that nothing is known about its complexity so far. However, if the points are uniformly distributed, say, in the unit square, then experimental results show that the running time is polynomially bounded in n. It is this result, we are going to derive theoretically in Section 2. More precisely, we will show that, if ALK is applied to a random problem as above, then its running time is $O(n^{18})$ with probability $1 - c/n$ for some constant $c > 0$.

We would like to note that, during the last years, another approach has been taken to provide the algorithm from getting trapped in a local solution. This approach is well known as "Simulated annealing" or "Metropolis algorithm". Essentially, the idea is to allow the algorithm to switch from σ to $\tilde{\sigma}$ even if $L(\tilde{\sigma}) > L(\sigma)$, however, with a decreasing probability for such "bad" switches. It would be an interesting question, whether our result can be used to derive polynomial bounds for the Simulated annealing version. Anyway, experimental results of this method are very encouraging (cf. [5, 1]) and have revived the interest in the original algorithm of Lin and Kernighan. Therefore, we found it worthwhile to write this paper.

2. A probabilistic analysis of the algorithm

Suppose we are given an instance of the Euclidean TSP, defined by n points $1, \ldots, n$ in the unit square. Let

$$d^* := \min_{\sigma} \{ |L(\sigma) - L(\tilde{\sigma})|,\ \tilde{\sigma} \in N(\sigma) \text{ and } L(\sigma) \neq L(\tilde{\sigma}) \}.$$

Since the algorithm, given any (current) solution σ, will either stop or find a better solution $\tilde{\sigma}$ by switching after at most n^2 steps, we see that its running time T will be bounded by $n^2 L_0 / d^*$, where L_0 is the length of the initial solution. In particular, $L_0 \leq n\sqrt{2}$, thus

$$T \leq \sqrt{2} n^3 / d^*.$$

Unfortunately, d^* may be arbitrarily small, and therefore, no polynomial bound can be derived from this. However, as we will see, in case the n points are uniformly distributed in $[0, 1]^2$, there exists $c > 0$ such that $d^* \geqslant n^{-15}$ with probability $\geqslant 1 - c/n$. This, of course, implies that with probability $\geqslant 1 - c/n$, the algorithm stops after executing at most $O(n^{18})$ elementary steps.

To derive this, let us first note, that d^* can be defined as follows: Let d denote the euclidean distance function. Furthermore, given a quadruple of points (i, j, k, l), let

$$F(i, j, k, l) = d(i, l) - d(j, l) + d(j, k) - d(i, k).$$

Then, obviously,

$$d^* = \min |F(i, j, k, l)|,$$

the minimum being taken over all quadruples (i, j, k, l) of distinct points of $1, \ldots, n$ such that $F(i, j, k, l) \neq 0$.

Now, let i, j and k be fixed. For $\varepsilon > 0$, let $A_\varepsilon(i, j, k)$ denote the set of all points $x \in [0, 1]^2$ satisfying $|F(i, j, k, x)| \leqslant \varepsilon$.

Lemma 2.1. *There exists a constant $K > 0$ such that the area of $A_\varepsilon(i, j, k)$ is bounded by $K\sqrt{\varepsilon}/d(i, j)$.*

Proof. It is easy to see (by applying a distance preserving transformation moving i and j to $(0, -d)$ and $(0, d)$, resp., where $d = d(i, j)/2$, that it suffices to prove the following:

Let $i = (0, -d)$ and $j = (0, d)$ with $0 \leqslant d \leqslant 1/\sqrt{2}$ and consider the following curve (cf. Fig. 2):

$$\Gamma_c\colon \quad d(x, i) - d(x, j) = c \quad \text{with } -2d < c < 2d.$$

Finally let $B_\varepsilon(c)$ denote the set of all points $x \in [0, \sqrt{2}]^2$ lying between the two curves Γ_c and $\Gamma_{c+\varepsilon}$. Then there exists a constant $K > 0$ such that the area of $B_\varepsilon(c)$ is bounded by $K\sqrt{\varepsilon}/d$ for all $c \in (0, 2d)$.

In order to prove this, we proceed with the following.

Claim 1. *There exists a constant $L > 0$ such that for every $c \in [0, 2d]$ and every $x \in \Gamma_c \cap [0, \sqrt{2}]^2$ the following holds:*

Let $\delta := L\sqrt{\varepsilon}/d$, and let $p(x)$ denote the path consisting of the two line segments $[x, x + \delta e_2]$ and $[x + \delta e_2, x + \delta e_2 - \delta e_1]$, where e_i denotes the ith unit vector in $\mathbb{R}^2$, i.e. $p(x)$ denotes the path starting at x and then moving first up and then to the left each time by an amount of δ. Then $p(x)$ meets $\Gamma_{c+\varepsilon}$.

Proof. Let us start by rewriting the equation for Γ_c explicitly (recall that $i = (0, -d)$ and $j = (0, d)$):

$$\Gamma_c\colon \quad \sqrt{x_1^2 + (x_2 + d)^2} - \sqrt{x_1^2 + (x_2 - d)^2} = c.$$

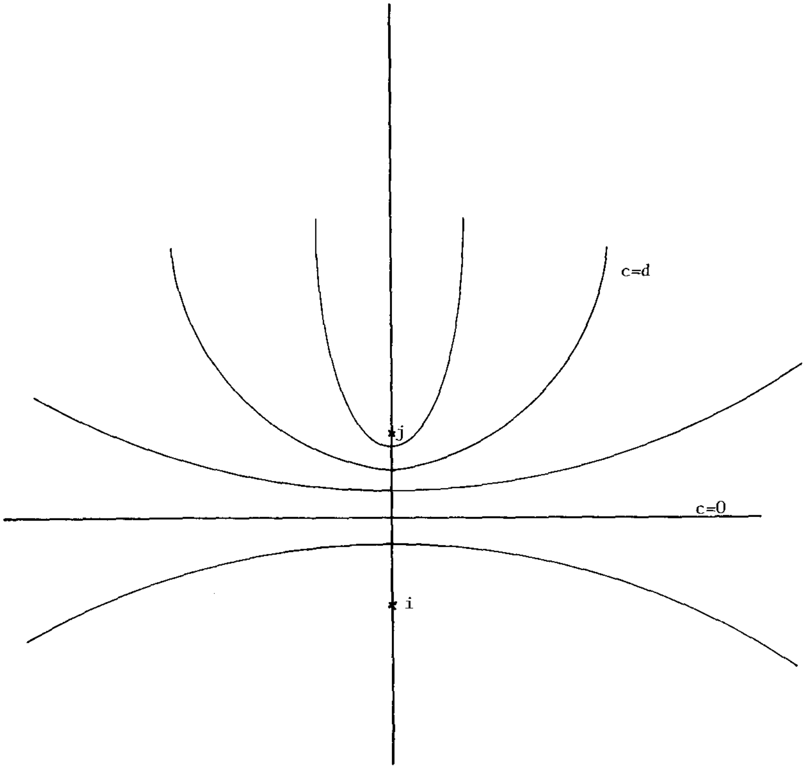

Fig. 2

Now let us investigate how c varies with x. We have

$$\frac{\partial c}{\partial x_1}=\frac{x_1}{\sqrt{x_1^2+(x_2+d)^2}}-\frac{x_1}{\sqrt{x_1^2+(x_2-d)^2}}=-x_1c\frac{1}{\sqrt{x_1^2+(x_2+d)^2}\sqrt{x_1^2+(x_2-d)^2}},$$

$$\frac{\partial c}{\partial x_2}=\frac{x_2+d}{\sqrt{x_1^2+(x_2+d)^2}}-\frac{x_2-d}{\sqrt{x_1^2+(x_2-d)^2}}=\frac{(x_2+d)(-c)+2d\sqrt{x_1^2+(x_2+d)^2}}{\sqrt{x_1^2+(x_2+d)^2}\sqrt{x_1^2+(x_2-d)^2}}.$$

It is easy to see, that $\partial c/\partial x_1 \leqslant 0$ and $\partial c/\partial x_2 \geqslant 0$ (cf. also Fig. 2). Since $0 \leqslant x_1, x_2 \leqslant \sqrt{2}$ and $d \leqslant 1/\sqrt{2}$, all the square roots above are bounded by $\sqrt{7}$. Thus

$$\left|\frac{\partial x}{\partial x_1}\right| \geqslant \tfrac{1}{7}x_1c,$$

$$\left|\frac{\partial c}{\partial x_2}\right| \geqslant \tfrac{1}{7}[2d\sqrt{x_1^2+(x_2+d)^2}-(x_2+d)c].$$

Since $x_2+d \leqslant \sqrt{x_1^2+(x_2+d)^2}$, we get

$$\left|\frac{\partial c}{\partial x_2}\right| \geqslant \tfrac{1}{7}(2d-c)\sqrt{x_1^2+(x_2+d)^2} \geqslant \tfrac{1}{7}x_1(2d-c).$$

Hence we get

$$\left|\frac{\partial c}{\partial x_1}\right| + \left|\frac{\partial c}{\partial x_2}\right| \geqslant \tfrac{2}{7} x_1 d. \tag{1}$$

Now let us fix a point x on Γ_c and let $L = \sqrt{7\sqrt{2}}$.

Case 1: $x_1 \leqslant L \cdot \sqrt{\varepsilon}/d$. In this case we move from x to the left until we meet $x_1 = 0$ (thereby increasing c by a negligibly small amount). Then we move up in x_2-direction by an amount of $L\sqrt{\varepsilon}/d$. This yields an increase of c by $2L\sqrt{\varepsilon}/d \geqslant \varepsilon$, since $\partial c/\partial x_2 = 2$ at $x_1 = 0$. Thus we have crossed $\Gamma_{c+\varepsilon}$ on our way. And hence we will also cross $\Gamma_{c+\varepsilon}$ by first moving up in x_2-direction and then to the left, each time by an amount of $L\sqrt{\varepsilon}/d$.

Case 2: $x_1 \geqslant L\sqrt{\varepsilon}/d$. In this case we go from x up in x_2-direction by an amount of $\Delta x_2 = (L/2)\sqrt{\varepsilon}/d$ and then move to the left by an amount of $\Delta x_1 = (L/2)\sqrt{\varepsilon}/d$. All along this way $x_1 \geqslant (L/2)\sqrt{\varepsilon}/d$, and hence (1) shows that the increase in c is

$$\Delta c \geqslant \left|\frac{\partial c}{\partial x_1}\right| \Delta x_1 + \left|\frac{\partial c}{\partial x_2}\right| \Delta x_2 \geqslant \left\{\left|\frac{\partial c}{\partial x_1}\right| + \left|\frac{\partial c}{\partial x_2}\right|\right\}(L/2)\sqrt{\varepsilon}/d$$
$$\geqslant \{\tfrac{2}{7}(L/2\sqrt{\varepsilon}/d)d\}(L/2)\sqrt{\varepsilon}/d \geqslant \varepsilon/(\sqrt{2d}) \geqslant \varepsilon.$$

Thus we must cross over $\Gamma_{c+\varepsilon}$ on our way, and the more we will cross over $\Gamma_{c+\varepsilon}$ if we move up and then to the left by an amount of $L\sqrt{\varepsilon}/d$ instead of $(L/2)\sqrt{\varepsilon}/d$. This finishes the proof of Claim 1.

The rest is easy: Recall that $B_c(\varepsilon)$ has been defined to be the set of points in the square $[0, \sqrt{2}]^2$ lying between Γ_c and $\Gamma_{c+\varepsilon}$. From Claim 1 we conclude that all of $B_c(\varepsilon)$, except possibly a part of area $\leqslant \sqrt{2}L\sqrt{\varepsilon}/d$, is covered by the paths $p(x)$ (mentioned in Claim 1), where x runs over $\Gamma_c \cap [0, \sqrt{2}]^2$ (cf. Fig. 3).

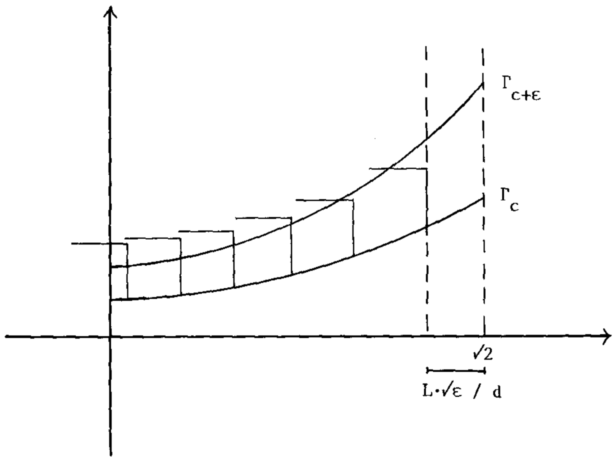

Fig. 3

This shows that $B_c(\varepsilon)$ has area less than $(\gamma_c+\sqrt{2})L\sqrt{\varepsilon}/d$, where γ_c denotes the length of $\Gamma_c\cap[0,\sqrt{2}]^2$. Since Γ_c is monotone, γ_c is bounded by $2\sqrt{2}$. This finishes the proof of Lemma 2.1. □

Now let $1,\dots,n$ denote independent random variables, which are uniformly distributed in the unit square. Given any four of them, say i, j, k, l, then

$$F(i,j,k,l)\leqslant\varepsilon \quad\Leftrightarrow\quad l\in A_\varepsilon(i,j,k).$$

By Lemma 2.1, we get

$$\text{Prob}(F(i,j,k,l)\leqslant\varepsilon|d(i,j))\leqslant K\sqrt{\varepsilon}/d(i,j).$$

This yields, for every $d>0$,

$$\begin{aligned}\text{Prob}(F(i,j,k,l)\leqslant\varepsilon)&\leqslant\text{Prob}(F(i,j,k,l)\leqslant\varepsilon\,|\,d(i,j)\geqslant d)+\text{Prob}(d(i,j)<d)\\&\leqslant K\sqrt{\varepsilon}/d+\pi d^2.\end{aligned}$$

Choosing $d=\varepsilon^{1/6}$, we get

$$\text{Prob}(F(i,j,k,l)\leqslant\varepsilon)\leqslant c\varepsilon^{1/3}\quad\text{for some constant } c>0.$$

Summing up over all quadruples (i,j,k,l), we get

$$\text{Prob}(d^*\leqslant\varepsilon)\leqslant n^4c\varepsilon^{1/3}$$

and hence, for $\varepsilon=n^{-15}$,

$$\text{Prob}(d^*\geqslant n^{-15})\geqslant 1-c/n,$$

which, as we noted already, implies that the algorithm stops after at most $O(n^{18})$ steps with probability $1-c/n$.

Remarks

1. Obviously, a similar analysis can be carried out for the so-called "k-switching algorithm" (an obvious extension of the "2-switching algorithm" considered here). Furthermore, it can be applied for switching algorithms designed for similar combinatorial optimization problem (e.g. minimum Euclidean perfect matching or matching- and cutproblems in appropriate classes of random graphs).

2. It is known that, in contrast to the speed of local search algorithms, the quality of the solutions they came up with is usually rather poor (cf. [7]). So far, however, this has been proven only for TSP in general graphs (cf. [6]), allowing edge-weightings which do not satisfy the triangle inequality. It would be interesting to compute the average quality of solutions in our case.

3. See [4, Chapter 6] for more about probabilistic analysis of the Traveling Salesman Problem. For example, it has been shown that Karp's "dissection algorithm", mentioned in the introduction has an expected running time of $O(n^2\log n)$ with variance $O(n^4(\log n)^3)$. Our approach does not seem to yield interesting results about average running times.

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