Mathematical Programming 44 (1989) 1-26
North-Holland

A POLYNOMIAL-TIME ALGORITHM FOR A CLASS OF LINEAR COMPLEMENTARITY PROBLEMS

Masakazu KOJIMA

Department of Information Sciences, Tokyo Institute of Technology, Oh-Okayama, Meguro-ku, Tokyo 152, Japan

Shinji MIZUNO and Akiko YOSHISE

Department of Industrial Engineering and Management, Tokyo Institute of Technology, Oh-Okayama, Meguro-ku, Tokyo 152, Japan

Received 27 July 1987
Revised manuscript received 8 February 1988

Given an $n \times n$ matrix M and an n-dimensional vector q, the problem of finding n-dimensional vectors x and y satisfying

$$y = Mx + q, \quad x \geqslant 0, \ y \geqslant 0, \ x_i y_i = 0 \ \ (i = 1, 2, \dots, n)$$

is known as a linear complementarity problem. Under the assumption that M is positive semi-definite, this paper presents an algorithm that solves the problem in $O(n^3 L)$ arithmetic operations by tracing the path of centers, $\{(x, y) \in S: x_i y_i = \mu \ (i = 1, 2, \dots, n) \text{ for some } \mu > 0\}$ of the feasible region $S = \{(x, y) \geqslant 0: y = Mx + q\}$, where L denotes the size of the input data of the problem.

Key words: Linear complementarity problem, polynomial-time algorithm, path of centers, Karmarkar's algorithm.

1. Introduction

Let M be an $n \times n$ matrix, and $q \in R^n$. The problem of finding an $(x, y) \in R^{2n}$ satisfying

$$y = Mx + q, \quad (x, y) \geqslant 0, \ x_i y_i = 0 \ \ (i = 1, 2, \dots, n) \tag{1.1}$$

is known as a linear complementarity problem (abbreviated by LCP), which has various important applications in linear and convex quadratic programs, bimatrix games and some other areas of engineering [3, 12, 18, etc.]. Here R^n denotes the n-dimensional Euclidean space.

Several computational methods have been developed for solving LCPs [3, 12, 24, etc.]. These methods apply a sequence of pivoting operations to the system of linear equations $y = Mx + q$ or a certain artificial system of equations associated with the LCP. In some worst cases, they require an exponential number of pivoting operations [1, 4 and 16]. It is well-known that an LCP with an arbitrary matrix M is NP-complete [2].

In the field of linear programs, there have been developed many algorithms with a polynomially bounded computational complexity [5, 7, 9, 10, 11, 19, 25, 26, etc.]. For convex quadratic programs, Kapoor and Vaidya [8] and Ye and Tse [27] have recently proposed polynomially bounded algorithms. These algorithms may be roughly classified into three groups:

1. Ellipsoid algorithms which originated from the first polynomially bounded linear programming algorithm by Khachiyan [10] in 1979.

2. Projective rescaling algorithm by Karmarkar [9], its variations and extensions [5, 8, 26, 27, etc.].

3. Algorithms using the idea of tracing the path of centers of a polytope [7, 11, 19, 25, etc.].

The algorithms in the last group are relatively new and are closely related with the second group (See [6]). Among these algorithms, the ones given by Gonzaga [7] and Vaidya [25] in the last group for solving linear programs have attained the $O(n^3L)$ computational complexity in terms of the number of arithmetic operations. The idea on which the algorithms of the last group are based have been studied in more general framework including convex minimization problems and linear complementarity problems by Megiddo [15], Sonnevend [20] and Tanabe [23].

In the previous paper [11], the authors have proposed an algorithm that solves linear programs in $O(n^4L)$ arithmetic operations by using Newton's method as a numerical tool to trace the path of centers simultaneously in the primal and dual feasible regions. In the present paper we modify and extend their algorithm to a class of LCPs with positive semi-definite matrices. The main emphasis will be laid on the theoretical computational complexity of the algorithm. We do not refer to a practically efficient implementation of the algorithm, which should be studied in the future though.

Let R^n_+ and R^n_{++} denote the nonnegative orthant $\{x \in R^n : x \geq 0\}$ of R^n and the positive orthant $\{x \in R^n : x > 0\}$ of R^n, respectively. We employ the symbol S for the set of all the feasible solutions of the LCP, S_{int} its interior and S_{cp} the set of all the solutions of the LCP;

$$S = \{(x, y) \in R^{2n}_+ : y = Mx + q\},$$

$$S_{\text{int}} = S \cap R^{2n}_{++} = \{(x, y) \in R^{2n}_{++} : y = Mx + q\},$$

$$S_{\text{cp}} = \{(x, y) \in S : x_i y_i = 0 \ (i = 1, 2, \ldots, n)\}.$$

Throughout the paper, we impose the following assumptions on the LCP:

Assumptions. (i) $n \geq 2$. (If $n = 1$ then the LCP could be solved trivially.)

(ii) All the elements of the $n \times n$ matrix M and the vector q are integers.

(iii) The matrix M is positive semi-definite, i.e., $x^{\mathrm{T}} M x \geq 0$ for every $x \in R^n$.

We may further assume without loss of generality that

(iv) each row of the matrix M has at least one nonzero element.

To see this, assume on the contrary that all the elements in the ith row of the matrix M are zero. If q_i is negative then the LCP has no solution. Otherwise we can reduce the size of the LCP to be solved by eliminating the variables x_i and y_i.

We define the size L of the LCP (1.1) by

$$L = \lfloor \sum_{i=1}^{n} \sum_{j=1}^{n+1} \log(|a_{ij}|+1) + \log(n^2) \rfloor + 1, \tag{1.2}$$

where a_{ij} denotes the (i, j)th element of the $n \times (n+1)$ matrix $A = [M \;\; q]$ consisting of the coefficient matrix M and the constant vector q of the system of equations of the LCP (1.1) and $\lfloor \xi \rfloor$ the largest integer not greater than $\xi \in R_+$. The assumption (iv) ensures the inequality $n + \log(n^2) \leqslant L$, which we will need in Section 6.

The size L determines the accuracy to be attained in the following sense: If

$$(\hat{x}, \hat{y}) \in S \quad \text{and} \quad \hat{x}_i \hat{y}_i < 2^{-2L} \quad (i = 1, 2, \ldots, n), \tag{1.3}$$

then there exists a solution (x^*, y^*) of the LCP such that

$$\begin{aligned} x_i^* &= 0 \quad \text{for every } i \in I, \\ y_j^* &= 0 \quad \text{for every } j \in J, \end{aligned} \tag{1.4}$$

where

$$I = \{i : \hat{x}_i < 2^{-L}\} \quad \text{and} \quad J = \{j : \hat{y}_j < 2^{-L}\}. \tag{1.5}$$

Furthermore, using the information (1.3), we can compute the solution (x^*, y^*) in $O(n^3)$ arithmetic operations. This will be shown in Appendix B. It should be noted that (1.3) and (1.5) imply $I \cup J = \{1, 2, \ldots, n\}$. Hence each $(\hat{x}, \hat{y})$ satisfying (1.3) itself can be regarded as an approximate solution of the LCP. The requirement (1.3) may be replaced by a stronger one

$$(\hat{x}, \hat{y}) \in S \quad \text{and} \quad \hat{x}^{\mathrm{T}} \hat{y} < 2^{-2L}. \tag{1.3}'$$

This inequality will be used as a stopping criteria in our algorithm.

Now we describe an outline of the algorithm. We first introduce a family of systems of equations with the nonnegative parameter μ:

$$H(\mu, x, y) = 0 \quad \text{and} \quad (x, y) \in R_+^{2n}, \tag{1.6}$$

where $H : R_+ \times R_+^{2n} \to R^n \times R^n$ is a mapping defined by

$$H(\mu, x, y) = (XYe - \mu e, y - Mx - q) \tag{1.7}$$

for every $(\mu, x, y) \in R_+^{1+2n}$, where X denotes the $n \times n$ diagonal matrix $\operatorname{diag}(x_1, x_2, \ldots, x_n)$, Y the diagonal matrix $\operatorname{diag}(y_1, y_2, \ldots, y_n)$ and e the n-dimensional vector of ones. Obviously (x, y) is a solution of the LCP if and only if it is a solution of the system (1.6) for $\mu = 0$. In other words, the LCP (1.1) is equivalent to the system of equations

$$H(0, x, y) = 0 \quad \text{and} \quad (x, y) \in R_+^{2n}. \tag{1.1}'$$

We call each $(x, y) \in R^{2n}$ satisfying the system (1.6) for some $\mu > 0$ a center (of the feasible region S), and define the path of centers S_{cen} to be the set of all the centers:

$$\begin{aligned} S_{\text{cen}} &= \{(x, y) \in R^{2n}_{+}: H(\mu, x, y) = 0 \text{ for some } \mu > 0\} \\ &= \{(x, y) \in S_{\text{int}}: XYe = \mu e \text{ for some } \mu > 0\}. \end{aligned}$$

When $S_{\text{int}} \neq \emptyset$ and Assumption (iii) is satisfied, the system (1.6) has a unique solution for each $\mu > 0$ and the path S_{cen} of centers forms a smooth curve in the set S_{int}. We can also characterize solutions of the system (1.6) in terms of the logarithmic barrier function method. These facts have been indicated and studied partially by Megiddo [15]. We will give a complete proof for these facts in Appendix A.

Geometrically the path S_{cen} of centers runs through the interior S_{int} of the feasible region S to a solution of the LCP which lies in the boundary of S. Starting from a known initial point in a neighborhood of the path S_{cen}, we trace the path S_{cen} until we attain a sufficiently small parameter μ. The idea of this approach has been suggested by Megiddo [15].

Generally, we are not able to trace the path S_{cen} accurately because it runs nonlinearly through the interior S_{int} of the feasible region S. We are forced to stray from the path S_{cen} even if a given initial point lies on the path. As a measure for the deviation of each $(x, y) \in S_{\text{int}}$ from the path S_{cen}, we employ the quantity

$$\begin{aligned} \min_{\mu \in R_{+}} \|H(\mu, x, y)\| &= \min_{\mu \in R_{+}} \|XYe - \mu e\| \\ &= \|XYe - (x^{\mathrm{T}} y/n)e\|. \end{aligned}$$

It should be noted that the first equality follows from the definition (1.7) of the mapping H and $(x, y) \in S_{\text{int}}$, and the second because the point $(x^{\mathrm{T}} y/n)e$ is an orthogonal projection of the point XYe onto the line $\{\mu e: \mu \in R\}$. Obviously an $(x, y) \in S_{int}$ lies on the path S_{cen} if and only if $\|XYe - (x^{\mathrm{T}} y/n)e\| = 0$. We want to control our approximation of the path so that the deviation $\|XYe - (x^{\mathrm{T}} y/n)e\|$ converges zero at least linearly as the error $x^{\mathrm{T}} y$ for the complementarity slackness tends to zero. This leads to the definition of the α-center neighborhood $S_{\text{cen}}(\alpha)$ of the path S_{cen}:

$$S_{\text{cen}}(\alpha) = \{(x, y) \in S_{\text{int}}: \|XYe - (x^{\mathrm{T}} y/n)e\| \leq (x^{\mathrm{T}} y/n)\alpha\}. \tag{1.8}$$

Here α is a positive number whose value will be specified in the succeeding discussion.

For the time being, we assume that an initial point (x^1, y^1) satisfying

$$(x^1, y^1) \in S_{\text{cen}}(\alpha) \quad \text{and} \quad (x^1)^{\mathrm{T}} y^1 \leq 2^{O(L)} \tag{1.9}$$

is known in advance. In Section 2, we present an algorithm which starts from this point and generates a sequence $\{(x^k, y^k)\} \subset S_{\text{cen}}(\alpha)$. For some positive constant η, each iteration decreases the error $(x^k)^{\mathrm{T}} y^k$ at least linearly with the ratio $(1 - \eta/n^{0.5})$, so that the algorithm stops within $O(n^{0.5}L)$ iterations when an approximate solution $(\hat{x}, \hat{y})$ satisfying (1.3)′ is obtained. Each iteration solves a linear system induced from a local linearization of the system (1.6) to compute a new point in $S_{\text{cen}}(\alpha)$. This requires $O(n^3)$ arithmetic operations. Thus, the total number of the arithmetic operations over all the $O(n^{0.5}L)$ iterations amounts to $O(n^{3.5}L)$.

Sections 3, 4 and 5 are devoted to reducing the total computational complexity to $O(n^3L)$ by incorporating the rank one update technique [7, 9 and 25] into the algorithm given in Section 2. The modified algorithm with the $O(n^3L)$ computational complexity described there could be stated directly without requiring Section 2 but it is rather complicated. So the simpler $O(n^{3.5}L)$ algorithm in Section 2 will be helpful to understand the structure of the $O(n^3L)$ algorithm.

In Section 6, we show that any given LCP satisfying the Assumptions (i), (ii) and (iii) can be converted into an artificial linear complementarity problem, LCP′ which satisfies not only the same assumptions but also has a known initial point from which the algorithms given in Sections 2 and 3 can start. It will be shown there that an application of the algorithms to LCP′ either yields a solution of the original LCP or decides that the LCP has no solution.

2. The $O(n^{3.5}L)$ algorithm

We first present a procedure which is used repeatedly in the $O(n^{3.5}L)$ algorithm, Algorithm 1 described below for tracing the path S_{cen} of centers to solve the LCP (1.1). Given a point $(x, y) \in S_{\text{int}}$ and a parameter $\mu > 0$ as inputs, the procedure generates a new point $(\bar{x}, \bar{y}) \in R^{2n}$ as an output in $O(n^3)$ arithmetic operations.

The procedure can be interpreted as a Newton iteration to the system (1.6) of equations for a fixed parameter $\mu > 0$. If we write the Jacobian matrices of the mapping on the left hand side of (1.6) with respect to x and y by $D_xH(\mu, x, y)$ and $D_yH(\mu, x, y)$ respectively then the Newton direction at (x, y) is defined as a solution $(\Delta x, \Delta y)$ of the system of linear equations:

$$D_xH(\mu, x, y)\Delta x + D_yH(\mu, x, y)\Delta y = H(\mu, x, y),$$

or

$$Y\Delta x + X\Delta y = XYe - \mu e \quad \text{and} \quad \Delta y = M\Delta x, \tag{2.1}$$

where

$$X = \operatorname{diag}(x_1, \ldots, x_n) \quad \text{and} \quad Y = \operatorname{diag}(y_1, \ldots, y_n). \tag{2.2}$$

By a simple calculation, we obtain

$$\Delta x = (M + X^{-1}Y)^{-1}(Ye - \mu X^{-1}e) \quad \text{and} \quad \Delta y = M\Delta x. \tag{2.3}$$

Then the new point $(\bar{x}, \bar{y})$ will be given by

$$(\bar{x}, \bar{y}) = (x, y) - (\Delta x, \Delta y). \tag{2.4}$$

It is easily verified that

$$\bar{y} = M\bar{x} + q \quad \text{for any } (x, y) \in S \text{ and any } \mu > 0. \tag{2.5}$$

Furthermore, if the input point (x, y) lies in a sufficiently small α-center neighborhood $S_{\text{cen}}(\alpha)$ of the path S_{cen} of centers and if we choose a suitable value for the input parameter $\mu > 0$ then the new point $(\bar{x}, \bar{y})$ remains in the α-center neighborhood $S_{\text{cen}}(\alpha)$. More precisely we have the following theorem:

Theorem 1. *Let α be a positive number such that $\alpha \leqslant 0.1$, and $\delta = \alpha/(1-\alpha)$. Suppose that*

$$(x, y) \in S_{\text{cen}}(\alpha) \quad \text{and} \quad \mu = (1 - \delta/n^{1/2})x^{\mathrm{T}}y/n.$$

Then the point $(\bar{x}, \bar{y})$ defined by (2.3) and (2.4) satisfies

$$(\bar{x}, \bar{y}) \in S_{\text{cen}}(\alpha), \tag{2.6}$$

$$\bar{x}^{\mathrm{T}}\bar{y} \leqslant (1 - \delta/(6n^{1/2}))x^{\mathrm{T}}y. \tag{2.7}$$

The theorem above will be derived as a corollary of a more general theorem, Theorem 2 in the succeeding section, whose proof will be given in Section 4.

Remark. The results in Theorem 1 could be strengthened slightly. In fact, if we gave a proof directly to Theorem 1, we could replace the upper bound for α by $\alpha \leqslant 0.2$ and the inequality (2.7) by

$$\bar{x}^{\mathrm{T}}\bar{y} \leqslant (1 - \delta/(2n^{1/2}))x^{\mathrm{T}}y. \tag{2.7$'$}$$

The direct proof of Theorem 1 would be similar to but simpler than the proof of Theorem 2 which will be given in Section 4.

Now we are ready to describe an algorithm:

Algorithm 1. We assume that an initial point $(x^1, y^1) \in S_{\text{int}}$ satisfying (1.9) is known in advance. (We will show how to prepare such an initial point in Section 6.)

Step 0: Let α be a positive constant such that $\alpha \leqslant 0.1$, and $\delta = \alpha/(1-\alpha)$. Let $k = 1$.

Step 1: If $(x^k)^{\mathrm{T}}y^k < 2^{-2L}$ then stop. Otherwise go to Step 2.

Step 2: Let $\mu = (1 - \delta/n^{1/2})(x^k)^{\mathrm{T}}y^k/n$ and $(x, y) = (x^k, y^k)$. Define the diagonal matrices X and Y by (2.2).

Step 3: Compute the Newton direction $(\Delta x, \Delta y)$ by (2.3) and the new point $(x^{k+1}, y^{k+1}) = (\bar{x}, \bar{y})$ by (2.4).

Step 4: Let $k = k+1$. Go to Step 1.

In view of Theorem 1, the sequence $\{(x^k, y^k)\}$ generated by the algorithm lies in the α-center neighborhood and the value $(x^k)^{\mathrm{T}}y^k$ decreases at least linearly with the global convergence ratio $(1 - \delta/(6n^{1/2}))$ along the sequence. Hence Algorithm 1 stops in $\mathrm{O}(n^{0.5}L)$ iterations. On the other hand, each iteration requires $\mathrm{O}(n^3)$ arithmetic operations to compute a new point. Therefore $\mathrm{O}(n^{3.5}L)$ arithmetic operations are required until the algorithm finds at Step 1 an approximate solution $(\hat{x}, \hat{y}) = (x^k, y^k)$ of the LCP satisfying (1.3)$'$. As we have stated, an exact solution of the LCP can be computed in $\mathrm{O}(n^3)$ additional arithmetic operations (see Appendix B).

3. The $O(n^3 L)$ algorithm

In order to improve the computational complexity, we will modify Algorithm 1 so that it requires $O(n^{2.5})$ arithmetic operations on the average per iteration and that it still maintains the linear convergence with the global convergence rate $(1-\eta/n^{0.5})$ for some $\eta>0$; hence the modified algorithm, Algorithm 2 below will attain the $O(n^3 L)$ computational complexity. For this purpose, we introduce approximations $\tilde{X}$ and $\tilde{Y}$ of the diagonal matrices X and Y into the Newton equations (2.1) as follows:

$$\tilde{Y}\Delta x+\tilde{X}\Delta y = XYe-\mu e \quad \text{and} \quad \Delta y = M\Delta x, \tag{3.1}$$

where

$$\begin{aligned} X &= \operatorname{diag}(x_1, x_2, \ldots, x_n), & Y &= \operatorname{diag}(y_1, y_2, \ldots, y_n), \\ \tilde{X} &= \operatorname{diag}(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n), & \tilde{Y} &= \operatorname{diag}(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n). \end{aligned} \tag{3.2}$$

We call $(\Delta x, \Delta y)$ the modified Newton direction. It follows immediately from (3.1) that

$$\Delta x = (M+\tilde{X}^{-1}\tilde{Y})^{-1}\tilde{X}^{-1}(XYe-\mu e) \quad \text{and} \quad \Delta y = M\Delta x. \tag{3.3}$$

The new point $(\bar{x}, \bar{y})$ is given by

$$(\bar{x}, \bar{y}) = (x, y)-(\Delta x, \Delta y). \tag{3.4}$$

Thus, the modified Newton procedure accepts $\mu>0$, $(x, y)\in S_{\text{int}}$ and $(\tilde{x}, \tilde{y})\in R^{2n}_{++}$ as inputs and generates the new point $(\bar{x}, \bar{y})$ as an output, which always satisfies the system of equations

$$\bar{y} = M\bar{x}+q \quad \text{for any } (x, y)\in S_{\text{int}} \text{ and any } \mu>0. \tag{3.5}$$

Concerning the modified Newton procedure we have the following result:

Theorem 2. *Let α, β and δ be constants such that $0<\alpha\leqslant 0.1$, $\delta=\alpha/(1-\alpha)$ and $0\leqslant\beta\leqslant\delta$. Suppose that the inputs $\mu\in R_+$, $(x, y)\in S_{\text{int}}$, $(\tilde{x}, \tilde{y})\in R^{2n}_{++}$ to the modified Newton procedure satisfy*

$$(x, y)\in S_{\text{cen}}(\alpha), \tag{3.6}$$

$$\mu = (1-\delta/n^{1/2})x^{\mathrm{T}}y/n, \tag{3.7}$$

$$\tilde{x}_i/x_i \in [(1+\beta)^{-1}, (1+\beta)] \quad (i=1,2,\ldots,n), \tag{3.8}$$

$$\tilde{y}_i/y_i \in [(1+\beta)^{-1}, (1+\beta)] \quad (i=1,2,\ldots,n). \tag{3.9}$$

Then the new point $(\bar{x}, \bar{y})$ given by (3.3) *and* (3.4) *satisfies*

$$(\bar{x}, \bar{y})\in S_{\text{cen}}(\alpha), \tag{3.10}$$

$$\bar{x}^{\mathrm{T}}\bar{y}\leqslant(1-\delta/(6n^{1/2}))x^{\mathrm{T}}y. \tag{3.11}$$

The proof of this theorem will be given in the next section. If we take $\beta = 0$ then the assertion of Theorem 2 is exactly the same as that of Theorem 1. Hence we can derive Theorem 1 from Theorem 2 as a corollary. Theorem 2 shows the possibility of using the modified Newton procedure instead of using the Newton procedure to get the same order $O(n^{0.5}L)$ of the total iterations as Algorithm 1. The modified procedure will be used repeatedly, as a core of Algorithm 2, together with a device which saves a certain amount of the arithmetic operations to compute the modified Newton direction.

The algorithm starts with an initial point (x^1, y^1) satisfying (1.9). Let

$$\tilde{X} = \mathrm{diag}(x_1^1, x_2^1, \ldots, x_n^1), \qquad \tilde{Y} = \mathrm{diag}(y_1^1, y_2^1, \ldots, y_n^1). \tag{3.12}$$

We also set $\tilde{B} = M + \tilde{X}^{-1}\tilde{Y}$ and compute its inverse $\tilde{B}^{-1}$, which will be used later for computing the modified Newton direction by (3.3). This work requires $O(n^3)$ arithmetic operations. The matrices $\tilde{X}$, $\tilde{Y}$, $\tilde{B}^{-1}$ will be updated and stored throughout the iterations.

Let $k \geqslant 1$. Suppose that we have obtained the $(x^k, y^k) \in S_{\mathrm{cen}}(\alpha)$ and the matrices $\tilde{X}$, $\tilde{Y}$ and $\tilde{B}^{-1} = (M + \tilde{X}^{-1}\tilde{Y})^{-1}$ at the end of the $(k-1)$th iteration. At the beginning of the kth iteration, we set

$$\begin{aligned} &\mu = (1 - \delta/n^{1/2})(x^k)^{\mathrm{T}} y^k/n, \qquad (x, y) = (x^k, y^k), \\ &X = \mathrm{diag}(x_1, x_2, \ldots, x_n), \qquad Y = \mathrm{diag}(y_1, y_2, \ldots, y_n). \end{aligned} \tag{3.13}$$

We then update the diagonal matrices $\tilde{X}$ and $\tilde{Y}$ so that they satisfy the assumption (3.8) and (3.9) of Theorem 2: for every $i = 1, 2, \ldots, n$, if

$$\tilde{x}_i / x_i \notin [(1+\beta)^{-1}, (1+\beta)] \tag{3.14}$$

or

$$\tilde{y}_i / y_i \notin [(1+\beta)^{-1}, (1+\beta)] \tag{3.15}$$

then update $\tilde{x}_i$ and $\tilde{y}_i$ by

$$\tilde{x}_i = x_i \quad \text{and} \quad \tilde{y}_i = y_i. \tag{3.16}$$

Here β denotes a positive number which we have specified in Theorem 2. It should be noted that the number of arithmetic operations required to compute μ and to update the diagonal matrices $\tilde{X}$ and $\tilde{Y}$ is bounded by $O(n)$.

When some of the diagonal elements of the matrices $\tilde{X}$ and $\tilde{Y}$ have been changed by (3.16), we no more have the identity $\tilde{B}^{-1} = (M + \tilde{X}^{-1}\tilde{Y})^{-1}$; hence we need update the matrix $\tilde{B}^{-1}$ before computing the modified Newton direction $(\Delta x, \Delta y)$ by

$$\Delta x = \tilde{B}^{-1}\tilde{X}^{-1}(XYe - \mu e) \quad \text{and} \quad \Delta y = M\Delta x. \tag{3.17}$$

Let Λ^k denote the set of the indices i for which (3.14) occurs, and Γ^k the set of the indices i for which (3.15) occurs. Then we see that the matrix $(M + \tilde{X}^{-1}\tilde{Y})$ whose inverse we want to compute differs from the matrix $\tilde{B}$ which corresponds to the old $(M + \tilde{X}^{-1}\tilde{Y})$ only in the columns with the indices $i \in \Lambda^k \cup \Gamma^k$. Hence we apply a sequence of rank-one updates to the matrix $\tilde{B}^{-1}$ to transform it to the inverse of the matrix $(M+\tilde{X}^{-1}\tilde{Y})$. The number of rank-one updates required amounts to the cardinality $|\Lambda^k \cup \Gamma^k|$ of the index set $\Lambda^k \cup \Gamma^k$. Thus, $|\Lambda^k \cup \Gamma^k| \times \mathrm{O}(n^2)$ arithmetic operations are required to update the matrix $\tilde{B}^{-1}$.

Now we are ready to compute the modified Newton direction $(\Delta x, \Delta y)$ by using (3.17) and the new point (x^{k+1}, y^{k+1}) by

$$(x^{k+1}, y^{k+1}) = (x^k, y^k) - (\Delta x, \Delta y). \tag{3.18}$$

For these computation, $\mathrm{O}(n^2)$ arithmetic operations are required. Since all the assumptions of Theorem 2 are satisfied at $(x, y) = (x^k, y^k)$, its conclusions (3.10) and (3.11) hold at $(\bar{x}, \bar{y}) = (x^{k+1}, y^{k+1})$, i.e.,

$$(x^{k+1}, y^{k+1}) \in S_{\mathrm{cen}}(\alpha),$$

$$(x^{k+1})^{\mathrm{T}} y^{k+1} \leqslant (1-\delta/(6n^{1/2}))(x^k)^{\mathrm{T}} y^k.$$

Summarizing the above discussions, we obtain the algorithm and the theorem below.

Algorithm 2. We assume that an initial point (x^1, y^1) satisfying (1.9) is known in advance.

Step 0: Let α, β and δ be constants such that $0 \leqslant \alpha \leqslant 0.1$, $\delta = \alpha/(1-\alpha)$ and $0 < \beta \leqslant \delta$. Define the diagonal matrices $\tilde{X}$ and $\tilde{Y}$ by (3.12). Compute $\tilde{B}^{-1} = (M+\tilde{X}^{-1}\tilde{Y})^{-1}$. Let $k=1$.

Step 1: If $(x^k)^{\mathrm{T}} y^k \leqslant 2^{-2L}$ then stop. Otherwise go to Step 2.

Step 2: Define μ, (x, y), X and Y by (3.13), i.e.,

$$\mu = (1-\delta/n^{1/2})(x^k)^{\mathrm{T}} y^k/n, \qquad (x, y) = (x^k, y^k),$$

$$X = \mathrm{diag}(x_1, x_2, \ldots, x_n), \qquad Y = \mathrm{diag}(y_1, y_2, \ldots, y_n).$$

Step 3: For every i, if

$$\tilde{x}_i/x_i \notin [(1+\beta)^{-1}, (1+\beta)] \quad ((3.14))$$

or

$$\tilde{y}_i/y_i \notin [(1+\beta)^{-1}, (1+\beta)] \quad ((3.15))$$

occurs then update the diagonal elements $\tilde{x}_i$ of $\tilde{X}$ and $\tilde{y}_i$ of $\tilde{Y}$ by (3.16), i.e.,

$$\tilde{x}_i = x_i \quad \text{and} \quad \tilde{y}_i = y_i.$$

Update the matrix $\tilde{B}^{-1}$ so that it represents the inverse of the matrix $(M+\tilde{X}^{-1}\tilde{Y})$.

Step 4: Compute the modified Newton direction $(\Delta x, \Delta y)$ by (3.17), i.e.,

$$\Delta x = \tilde{B}^{-1}\tilde{X}^{-1}(XYe - \mu e) \quad \text{and} \quad \Delta y = M\Delta x,$$

and the new point (x^{k+1}, y^{k+1}) by (3.18), i.e.,

$$(x^{k+1}, y^{k+1}) = (x^k, y^k) - (\Delta x, \Delta y).$$

Step 5: Let $k = k+1$. Go to Step 1.

Theorem 3. *Algorithm 2 generates a sequence* $\{(x^k, y^k)\}$ *satisfying*

$$(x^k, y^k) \in S_{\text{cen}}(\alpha) \quad \textit{and} \quad (x^k)^{\mathrm{T}} y^k \leqslant (1-\delta/(6n^{1/2}))(x^{k-1})^{\mathrm{T}} y^{k-1}$$

for $k = 2, 3, \ldots,$ *and terminates in* $\mathrm{O}(n^{0.5} L)$ *iterations.*

In the discussions above we have also observed that the kth iteration requires $\mathrm{O}(n)$ arithmetic operations for computing μ by (3.13), $\mathrm{O}(n)$ for updating $\tilde{X}$ and $\tilde{Y}$ (see (3.14), (3.15) and (3.16)), $|\Lambda^k \cup \Gamma^k| \times \mathrm{O}(n^2)$ for updating $\tilde{B}^{-1}$, and $\mathrm{O}(n^2)$ for computing the modified Newton direction $(\Delta x, \Delta y)$ by (3.17) and the new point (x^{k+1}, y^{k+1}) by (3.18). Hence the total number of operations in the kth iteration is bounded by

$$\mathrm{O}(n^2) + |\Lambda^k \cup \Gamma^k| \times \mathrm{O}(n^2).$$

In addition, we need $\mathrm{O}(n^3)$ arithmetic operations to compute the initial $\tilde{B}^{-1}$. Therefore, if we denote the total number of iterations by k^* then the total number of arithmetic operations throughout the iterations is bounded by

$$\mathrm{O}(n^3) + k^* \times \mathrm{O}(n^2) + \left[\sum_{k=1}^{k^*} |\Lambda^k \cup \Gamma^k| \right] \times \mathrm{O}(n^2).$$

It should be noted that the second term $k^* \times \mathrm{O}(n^2)$ is bounded by $\mathrm{O}(n^{2.5} L)$. In Section 5, we prove that the term in brackets $[\cdot]$ is bounded by $k^* \times \mathrm{O}(n^{0.5}) = \mathrm{O}(nL)$. This will establish that the total number of arithmetic operations required in Algorithm 2 is bounded by $\mathrm{O}(n^3 L)$.

4. Proof of Theorem 2

Throughout this section, we use the same symbols α, β, δ, μ, (x, y), X, Y, $(\bar{x}, \bar{y})$, $(\tilde{x}, \tilde{y})$, $(\tilde{X}, \tilde{Y})$ and $(\Delta x, \Delta y)$ as in Theorem 2 and the previous section. In addition, we use the following symbols throughout this section.

$$\begin{aligned}
&\zeta = x^{\mathrm{T}} y / n; \qquad \bar{\zeta} = \bar{x}^{\mathrm{T}} \bar{y} / n;\\
&\Delta X = \operatorname{diag}(\Delta x_1, \Delta x_2, \ldots, \Delta x_n);\\
&\Delta Y = \operatorname{diag}(\Delta y_1, \Delta y_2, \ldots, \Delta y_n);\\
&\bar{X} = \operatorname{diag}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) = X - \Delta X;\\
&\bar{Y} = \operatorname{diag}(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n) = Y - \Delta Y;\\
&\tilde{D} = (\tilde{X}\tilde{Y}^{-1})^{1/2} = \operatorname{diag}(\tilde{x}_1^{1/2}/\tilde{y}_1^{1/2}, \ldots, \tilde{x}_n^{1/2}/\tilde{y}_n^{1/2});\\
&\tilde{u} = (\tilde{X}\tilde{Y})^{-1/2}(XYe - \mu e);\\
&\|A\| = \max\{\|Az\| \colon z \in R^n, \|z\| = 1\} \quad \text{for every } n \times n \text{ matrix } A.
\end{aligned} \tag{4.1}$$

We need five lemmas to prove Theorem 2.

Lemma 1. *If p, r and u in R^n satisfy*

$$p + r = u, \qquad p^{\mathrm{T}} r \geqslant 0$$

then

$$\|p\| \leqslant \|u\|, \qquad \|r\| \leqslant \|u\|,$$

$$\|p\|\,\|r\| \leqslant \|u\|^2/2,$$

$$\|p\| + \|r\| \leqslant 2^{1/2}\|u\|.$$

Proof. It follows from the assumption that

$$\begin{aligned}\|u\|^2 &= \|p+r\|^2 \\ &= \|p\|^2 + 2p^{\mathrm{T}} r + \|r\|^2 \\ &\geqslant \|p\|^2 + \|r\|^2.\end{aligned}$$

This implies that $\|p\| \leqslant \|u\|$ and $\|r\| \leqslant \|u\|$. We also have

$$\|p\|\,\|r\| \leqslant (\|p\|^2 + \|r\|^2)/2 \leqslant \|u\|^2/2,$$

and

$$(\|p\| + \|r\|)^2 \leqslant \|p\|^2 + \|r\|^2 + 2\|p\|\,\|r\| \leqslant 2\|u\|^2. \qquad \square$$

Lemma 2. (a) $\|(XY)^\rho\| \leqslant (1+\delta)^{|\rho|}\zeta^\rho$ *for any* $\rho \in R$.
(b) $\|(\tilde{X}\tilde{Y})^\rho\| \leqslant (1+\delta)^{3|\rho|}\zeta^\rho$ *for any* $\rho \in R$.
Here $\zeta = x^{\mathrm{T}} y/n$.

Proof. Let $\rho \in R$ be fixed. Since $(x, y) \in S_{\mathrm{cen}}(\alpha)$, the definition (1.8) of $S_{\mathrm{cen}}(\alpha)$ implies that, for every $i = 1, 2, \ldots, n$,

$$(1-\alpha)\zeta \leqslant x_i y_i \leqslant (1+\alpha)\zeta.$$

If $\rho \geqslant 0$ then, for every $i = 1, 2, \ldots, n$,

$$(x_i y_i)^\rho \leqslant (1+\alpha)^\rho \zeta^\rho \leqslant (1+\delta)^{|\rho|}\zeta^\rho.$$

The last inequality holds because $0 < \alpha \leqslant \delta$. If $\rho < 0$, for every $i = 1, 2, \ldots, n$,

$$(x_i y_i)^\rho \leqslant (1-\alpha)^\rho \zeta^\rho = (1+\delta)^{|\rho|}\zeta^\rho$$

(the last equality holds because $\delta = \alpha/(1-\alpha)$). Since X and Y are diagonal matrices, we have

$$\begin{aligned}\|(XY)^\rho\| &= \max\{(x_i y_i)^\rho : i = 1, 2, \ldots, n\} \\ &\leqslant (1+\delta)^{|\rho|}\zeta^\rho.\end{aligned}$$

Hence we have shown (a). From the assumptions (3.8) and (3.9) of Theorem 2 and $0 \leq \beta \leq \delta$, we see

$$\|(\tilde{X}X^{-1})^{\rho}\| \leq (1+\beta)^{|\rho|} \leq (1+\delta)^{|\rho|}.$$

Similarly

$$\|(\tilde{Y}Y^{-1})^{\rho}\| \leq (1+\delta)^{|\rho|}.$$

By the inequalities above and (a), we obtain

$$\|(\tilde{X}\tilde{Y})^{\rho}\| \leq \|(XY)^{\rho}\| \, \|(\tilde{X}X^{-1})^{\rho}\| \, \|(\tilde{Y}Y^{-1})^{\rho}\| \leq (1+\delta)^{3|\rho|}\zeta^{\rho}. \qquad \square$$

Lemma 3. $\|\tilde{u}\| \leq 2\delta(1+\delta)^{3/2}\zeta^{1/2}$. *Here* $\tilde{u} = (\tilde{X}\tilde{Y})^{-1/2}(XYe - \mu e)$ *and* $\zeta = (x^{\mathrm{T}}y)/n$.

Proof. By the definition, we have

$$\begin{aligned}
\|\tilde{u}\| &= \|(\tilde{X}\tilde{Y})^{-1/2}(XYe-\mu e)\| \\
&\leq \|(\tilde{X}\tilde{Y})^{-1/2}\| \, \|XYe - \zeta e + (\zeta - \mu)e\| \\
&\leq \|(\tilde{X}\tilde{Y})^{-1/2}\| [\|XYe - \zeta e\| + |\zeta - \mu| \, \|e\|] \\
&\leq \|(\tilde{X}\tilde{Y})^{-1/2}\| [\alpha\zeta + (\delta/n^{1/2})\zeta\|e\|] \\
&\qquad \text{(by the assumptions (3.6) and (3.7) of Theorem 2)} \\
&\leq (1+\delta)^{3/2}\zeta^{-1/2}[\alpha\zeta + \delta\zeta] \\
&\qquad \text{(by Lemma 2 and } \|e\| = n^{1/2}) \\
&\leq 2\delta(1+\delta)^{3/2}\zeta^{1/2} \quad \text{(since } 0 < \alpha \leq \delta). \qquad \square
\end{aligned}$$

Lemma 4. (a) $\|\tilde{D}^{-1}\Delta Xe\| + \|\tilde{D}\Delta Ye\| \leq 2^{1/2}\|\tilde{u}\|$.
(b) $\|\Delta X \Delta Ye\| \leq \|\tilde{u}\|^2/2$.
(c) $\|X^{-1}\Delta x\| \leq 2\delta(1+\delta)^4 < 0.4$.
(d) $\|Y^{-1}\Delta y\| \leq 2\delta(1+\delta)^4 < 0.4$.
(*See* (4.1) *for the definitions of* $\tilde{D}$, ΔX, ΔY.)

Proof. Multiplying the first equality of the modified Newton equations (3.1) by the matrix $(\tilde{X}\tilde{Y})^{-1/2}$ from the left, we have

$$\tilde{D}^{-1}\Delta x + \tilde{D}\Delta y = (\tilde{X}\tilde{Y})^{-1/2}(XYe - \mu e) = \tilde{u}.$$

We also see from the second equality of (3.1) and Assumption (iii) that

$$(\tilde{D}^{-1}\Delta x)^{\mathrm{T}}(\tilde{D}\Delta y) = \Delta x^{\mathrm{T}}\Delta y = \Delta x^{\mathrm{T}} M \Delta x \geq 0.$$

Applying Lemma 1 with $p = \tilde{D}^{-1}\Delta x$ and $r = \tilde{D}\Delta y$, we obtain the inequalities (a), and

$$\begin{aligned}
&\|\tilde{D}^{-1}\Delta x\| \leq \|\tilde{u}\|, \qquad \|\tilde{D}\Delta y\| \leq \|\tilde{u}\|, \\
&\|\tilde{D}^{-1}\Delta x\| \, \|\tilde{D}\Delta y\| \leq \|\tilde{u}\|^2/2.
\end{aligned} \tag{4.2}$$

It follows from the last inequality that

$$\|\Delta X \Delta Y e\| = \|\tilde{D}^{-1}\Delta X \tilde{D} \Delta Y e\| \leq \|\tilde{D}^{-1}\Delta x\| \, \|\tilde{D}\Delta y\| \leq \|\tilde{u}\|^2/2.$$

Thus, we have shown (b). We also see

$$\begin{aligned}
\|X^{-1}\Delta x\| &\leq \|X^{-1}\tilde{X}\| \, \|\tilde{X}\tilde{Y}\|^{-1/2} \|\tilde{D}^{-1}\Delta x\| \\
&\leq (1+\delta)[(1+\delta)^{3/2}/\zeta^{1/2}]\|\tilde{u}\| \\
&\qquad \text{(by (3.8), Lemma 2 and (4.2))} \\
&\leq 2\delta(1+\delta)^4 \quad \text{(by Lemma 3)} \\
&< 0.4 \quad (\text{because } 0 \leq \delta = \alpha/(1-\alpha) < 0.12).
\end{aligned}$$

Thus, we have shown (c). The inequality (d) can be proved similarly. □

Lemma 5. (a) $\|\bar{X}\bar{Y}e - \mu e\| \leq 0.82\alpha\zeta$.
(b) $|\bar{\zeta} - \mu| \leq 0.82\alpha\zeta/n^{1/2}$.
(*See* (4.1) *for the definitions of* $\bar{X}$, $\bar{Y}$, ζ *and* $\bar{\zeta}$.)

Proof. First we observe

$$\begin{aligned}
\bar{X}\bar{Y}e &= XYe - (Y\Delta Xe + X\Delta Ye) + \Delta X\Delta Ye \\
&= XYe - (\tilde{Y}\Delta Xe + \tilde{X}\Delta Ye) + ((\tilde{Y}-Y)\Delta Xe + (\tilde{X}-X)\Delta Ye) + \Delta X\Delta Ye \\
&= \mu e + ((\tilde{Y}-Y)\Delta Xe + (\tilde{X}-X)\Delta Ye) + \Delta X\Delta Ye.
\end{aligned}$$

The last equality follows from the fact that $(\Delta x, \Delta y) = (\Delta Xe, \Delta Ye)$ is a solution of the modified Newton equations (3.1). Hence

$$\begin{aligned}
\|\bar{X}\bar{Y}e - \mu e\| &\leq \|((\tilde{Y}-Y)\Delta Xe + (\tilde{X}-X)\Delta Ye)\| + \|\Delta X\Delta Ye\| \\
&\leq \|(\tilde{X}\tilde{Y})^{1/2}[\tilde{Y}^{-1}(\tilde{Y}-Y)\tilde{D}^{-1}\Delta Xe + \tilde{X}^{-1}(\tilde{X}-X)\tilde{D}\Delta Ye]\| \\
&\qquad + \|\Delta X\Delta Ye\| \quad (\text{since } \tilde{D} = (\tilde{X}\tilde{Y}^{-1})^{1/2}) \\
&\leq \|(\tilde{X}\tilde{Y})^{1/2}\|[\|\tilde{Y}^{-1}(\tilde{Y}-Y)\| \, \|\tilde{D}^{-1}\Delta Xe\| \\
&\qquad\qquad + \|\tilde{X}^{-1}(\tilde{X}-X)\| \, \|\tilde{D}\Delta Ye\|] + \|\Delta X\Delta Ye\| \\
&\leq (1+\delta)^{3/2}\zeta^{1/2}(\delta\|\tilde{D}^{-1}\Delta Xe\| + \delta\|\tilde{D}\Delta Ye\|) + \|\tilde{u}\|^2/2 \\
&\qquad \text{(by Lemma 2 and 4, and the Assumptions (3.8) and (3.9))} \\
&\leq (1+\delta)^{3/2}\zeta^{1/2}\delta 2^{1/2}\|\tilde{u}\| + \|\tilde{u}\|^2/2 \quad \text{(by Lemma 4)} \\
&\leq (2+2^{3/2})\delta^2(1+\delta)^3\zeta \quad \text{(by Lemma 3)} \\
&= (2+2^{3/2})\alpha^2(1-\alpha)^{-5}\zeta \quad (\text{since } \delta = \alpha/(1-\alpha)) \\
&\leq 0.82\alpha\zeta \quad (\text{since } 0 < \alpha \leq 0.1).
\end{aligned}$$

Thus, we have shown the inequality (a). The inequality (b) follows immediately from (a) and

$$\begin{aligned}|\bar{\zeta}-\mu| &= |e^{\mathrm{T}}\bar{X}\bar{Y}e/n-\mu e^{\mathrm{T}}e/n| \\ &= |e^{\mathrm{T}}[\bar{X}\bar{Y}e-\mu e]/n| \\ &\leqslant \|e\|\,\|\bar{X}\bar{Y}e-\mu e\|/n \\ &= \|\bar{X}\bar{Y}e-\mu e\|/n^{1/2}. \qquad \square\end{aligned}$$

Now we are ready to prove Theorem 2. In view of (c) and (d) of Lemma 4, we see that

$$X^{-1}\bar{x}=X^{-1}(x-\Delta x)=e-X^{-1}\Delta x>0$$

and

$$Y^{-1}\bar{y}=Y^{-1}(y-\Delta y)=e-Y^{-1}\Delta y>0,$$

respectively. This implies $(\bar{x},\bar{y})\in R^{2n}_{++}$. Taking account of (3.5), we obtain $(\bar{x},\bar{y})\in S_{\mathrm{int}}$. Now we shall show $(\bar{x},\bar{y})\in S_{\mathrm{cen}}(\alpha)$. By (b) of Lemma 5, we have

$$\begin{aligned}\bar{\zeta} &\geqslant \mu-0.82\alpha\zeta/n^{1/2} \\ &\geqslant [1-\delta/n^{1/2}-0.82\alpha/n^{1/2}]\zeta \quad \text{(by (3.7))} \\ &\geqslant 0.85\zeta \quad (\text{since } 0<\alpha\leqslant 0.1,\ \delta=\alpha/(1-\alpha)<0.12 \text{ and } n\geqslant 2).\end{aligned} \tag{4.3}$$

On the other hand, we see,

$$\begin{aligned}\|\bar{X}\bar{Y}e-\bar{\zeta}e\| &\leqslant \|\bar{X}\bar{Y}e-\mu e\| \\ &\leqslant 0.82\alpha\zeta \quad \text{(by (a) of Lemma 5)} \\ &\leqslant \alpha\bar{\zeta} \quad \text{(by (4.3))}.\end{aligned}$$

Thus, we have shown $(\bar{x},\bar{y})\in S_{\mathrm{cen}}(\alpha)$.

Finally, by (b) of Lemma 5, we obtain

$$\begin{aligned}\bar{\zeta} &\leqslant \mu+0.82\alpha\zeta/n^{1/2} \\ &\leqslant (1-\delta/n^{1/2}+0.82\delta/n^{1/2})\zeta \\ &\qquad (\text{since } \mu=(1-\delta/n^{1/2})\zeta \text{ and } \delta=\alpha/(1-\alpha)\geqslant\alpha) \\ &\leqslant (1-\delta/(6n^{1/2}))\zeta,\end{aligned}$$

which shows (3.11). This completes the proof of Theorem 2.

5. Evaluation of the total arithmetic operations

As we have stated at the end of Section 3, in order to prove that the total number of arithmetic operations required by Algorithm 2 is bounded by $O(n^3 L)$, we need only show

$$\sum_{k=1}^{k^*} |\Lambda^k \cup \Gamma^k| \leq k^* \times O(n^{0.5}). \tag{5.1}$$

Recall that k^* denote the total number of iterations, Λ^k the set of indices i for which (3.14) occurs at the kth iteration, and Γ^k the set of indices i for which (3.15) occurs at the kth iteration. The method of the proof here is based on the one given in section 5 of [7]. In fact, we shall utilize the following lemma whose assertion and proof are closely related with Lemma 5.1 and Theorem 5.2 of [7] and their proof.

Lemma 6. *Let $0<\varepsilon<1$ and $0<\beta$. Let $\{x^k\,(k \in K)\} \subset R^n_{++}$ be a finite sequence such that*

$$\|(X^k)^{-1}(x^{k+1}-x^k)\| \leq \varepsilon, \tag{5.2}$$

where

$$K=\{1,2,\ldots,k^*\} \quad \textit{and} \quad X^k = \operatorname{diag}(x_1^k, x_2^k, \ldots, x_n^k).$$

For every $j=1,2,\ldots,n$, let K_j be a subsequence of K such that $1 \in K_j$ and that

$$x_j^{\bar{k}}/x_j^{\tilde{k}} \notin [(1+\beta)^{-1}, (1+\beta)] \tag{5.3}$$

whenever $\tilde{k}$ and $\bar{k}$ are consecutive indices in K_j. Then

$$\sum_{j=1}^{n} |K_j| \leq n + \varepsilon[(1-\varepsilon)\log(1+\beta)]^{-1} n^{1/2} k^*.$$

Here $|K_j|$ denotes the cardinality of the subsequence K_j.

Proof. We first establish the inequality

$$|K_j| < 1 + [(1-\varepsilon)\log(1+\beta)]^{-1} \sum_{k=1}^{k^*} (|x_j^{k+1}-x_j^k|/x_j^k) \tag{5.4}$$

for every $j=1,2,\ldots,n$. Let $j \in \{1,2,\ldots,n\}$ be fixed. By the assumption (5.2) on the sequence $\{x^k\}$, we have

$$|x_j^{k+1}-x_j^k| \leq \varepsilon x_j^k \quad \text{for every } k=1,2,\ldots,k^*.$$

Hence

$$|x_j^{k+1}-x_j^k|/x_j^{k+1} \leq (1-\varepsilon)^{-1}[|x_j^{k+1}-x_j^k|/x_j^k] \quad \text{for every } k=1,2,\ldots,k^*. \tag{5.5}$$

Let $\tilde{k}$ and $\bar{k}$ be consecutive indices in the subsequence K_j. According to the property imposed on the subsequence K_j, the inequality (5.3) holds. Hence we see

$$\begin{aligned}
\log(1+\beta) &\leq |\log x_j^{\bar{k}} - \log x_j^{\tilde{k}}| \\
&= \left| \int_{x_j^{\tilde{k}}}^{x_j^{\bar{k}}} (1/\xi)\, \mathrm{d}\xi \right| \\
&\leq \sum_{k=\tilde{k}}^{\bar{k}-1} \max\{|x_j^{k+1} - x_j^k|/x_j^k, |x_j^{k+1} - x_j^k|/x_j^{k+1}\} \\
&\leq \sum_{k=\tilde{k}}^{\bar{k}-1} (1-\varepsilon)^{-1} |x_j^{k+1} - x_j^k|/x_j^k.
\end{aligned} \tag{5.6}$$

The last inequality follows from (5.5). Since the inequality (5.6) holds for every pair of consecutive indices $\tilde{k}$ and $\bar{k}$ in K_j, taking the summation of those inequalities, we obtain the inequality (5.4).

Finally, from the inequality (5.4), we have

$$\begin{aligned}
\sum_{j=1}^{n} |K_j| &\leq n + [(1-\varepsilon)\log(1+\beta)]^{-1} \sum_{j=1}^{n} \sum_{k=1}^{k^*} (|x_j^{k+1} - x_j^k|/x_j^k) \\
&\leq n + [(1-\varepsilon)\log(1+\beta)]^{-1} \sum_{k=1}^{k^*} \sum_{j=1}^{n} (|x_j^{k+1} - x_j^k|/x_j^k) \\
&\leq n + [(1-\varepsilon)\log(1+\beta)]^{-1} \sum_{k=1}^{k^*} n^{1/2} \|(X^k)^{-1}(x^{k+1} - x^k)\| \\
&\leq n + \varepsilon[(1-\varepsilon)\log(1+\beta)]^{-1} n^{1/2} k^* \quad \text{(by the assumption (5.2)).} \quad \square
\end{aligned}$$

Now we are ready to prove (5.1). Since

$$\sum_{k=1}^{k^*} |\Lambda^k \cup \Gamma^k| \leq \sum_{k=1}^{k^*} |\Lambda^k| + \sum_{k=1}^{k^*} |\Gamma^k|,$$

it suffices to show that

$$\sum_{k=1}^{k^*} |\Lambda^k| \leq k^* \times \mathrm{O}(n^{0.5}) \tag{5.7}$$

and

$$\sum_{k=1}^{k^*} |\Gamma^k| \leq k^* \times \mathrm{O}(n^{0.5}). \tag{5.8}$$

To show the inequality (5.7), we construct a subsequence K_j of $\{1, 2, \ldots, k^*\}$ for each $j = 1, 2, \ldots, n$ as follows: Let $K_j = \{1\}$, and add $k \in \{2, 3, \ldots, k^*\}$ to K_j if and only if $j \in \Lambda^k$. Then, in view of (c) of Lemma 4 and the construction of the index set K_j $(j=1,2,\dots,n)$, we see that the sequence $\{x^k: k=1,2,\dots,k^*\}$ satisfies all the hypothesis of Lemma 6 with $\varepsilon=0.4$ and that $0<\beta\leq\delta=\alpha/(1-\alpha)\leq 0.12$. Thus, we obtain

$$\sum_{k=1}^{k^*}|A^k|=\sum_{j=1}^{n}|K_j|-n\leq k^*\times \mathrm{O}(n^{0.5}).$$

The inequality (5.8) can be shown similarly by using (d) of Lemma 4. This completes the proof.

Remark. Although we have not evaluated the computational complexity of Algorithms 1 and 2 in bit operations, we conjecture that it is enough to keep all the elements of the kth iterate (x^k, y^k) in rational numbers with denominator $2^{\mathrm{O}(L)}$. Then a similar bit analysis as in Section 8 of the paper [19] by Renegar would suggest that Algorithms 1 and 2 have the total bit computational complexity $\mathrm{O}(n^{4.5}L^2(\log L)(\log\log L))$ and $\mathrm{O}(n^4L^2(\log L)(\log\log L))$, respectively.

6. An artificial problem having a trivial initial point

In this section we show how to prepare an initial point (x^1, y^1) from which Algorithms 1 and 2 start. For this purpose, we shall construct an artificial linear complementarity problem, LCP′ which not only satisfies Assumptions (i), (ii) and (iii) in Section 1 but also has a trivial initial point.

Let

$$\bar{L}=\sum_{i=1}^{n}\sum_{j=1}^{n+1}\log(|a_{ij}|+1)+\log(n^2),$$

where a_{ij} denotes the (i,j)th element of the $n\times(n+1)$ matrix $A=[M\ \ q]$. By the definition (1.2) of L,

$$L=\lfloor\bar{L}\rfloor+1 \quad\text{and}\quad L-1\leq\bar{L}\leq L. \tag{6.1}$$

Define

$$q_0=2^{\bar{L}}(n+1)/n^2, \qquad q'=(q_0,q)\in R^{1+n},$$

$$M'=\begin{bmatrix}0 & -e^{\mathrm{T}}\\ e & M\end{bmatrix}.$$

It is easily verified that all the elements of the vector q' and the matrix M' are integers. We consider the artificial linear complementarity problem, LCP′: Find an $(x',y')=(x_0,x,y_0,y)$ such that

$$y'=M'x'+q', \quad (x',y')\in R_+^{2(1+n)},$$

$$x_iy_i=0 \quad (i=0,1,\dots,n).$$

The LCP′ obviously satisfies Assumptions (i), (ii) and (iii).

We use the symbols S', $S'_{\rm int}$, $S'_{\rm cen}$ and $S'_{\rm cen}(\alpha)$ for the feasible region of the LCP′, its interior, the path of centers of S' and its α-center neighborhood, respectively. If we denote the (i, j)th element of the $(1+n)\times(n+2)$ matrix $A' = [M' \;\; q']$ by a'_{ij}, the size L' of the LCP′ is given by

$$L' = \left\lfloor \sum_{i=1}^{n+1} \sum_{j=1}^{n+2} \log(|a'_{ij}|+1) + \log(1+n)^2 \right\rfloor + 1.$$

Taking account of the inequality $n + \log n^2 \leqslant \bar{L} \leqslant L$, which follows from Assumption (iv) in Section 1, we obtain

$$L \leqslant L' \leqslant 4L. \tag{6.2}$$

Define

$$x_0^1 = 2^{2\bar{L}}, \qquad x^1 = (2^{\bar{L}}/n^2)e,$$
$$y_0^1 = q_0 - e^{\rm T}x^1 = 2^{\bar{L}}(n+1)/n^2 - (2^{\bar{L}}/n^2)e^{\rm T}e = 2^{\bar{L}}/n^2,$$
$$y^1 = (x_0^1)e + Mx^1 + q = (2^{2\bar{L}})e + (2^{\bar{L}}/n^2)Me + q,$$
$$(x'^1, y'^1) = (x_0^1, x^1, y_0^1, y^1).$$

Lemma 7.
(a) $0 < (15/16)\times(2^{2\bar{L}})e \leqslant 2^{2\bar{L}}(1 - 1/n^4)e \leqslant y^1$
$\leqslant 2^{2\bar{L}}(1 + 1/n^4)e \leqslant (17/16)\times(2^{2\bar{L}})e.$

(b) $(x'^1, y'^1) \in S'_{\rm int}$.

Proof. We see

$$(2^{\bar{L}}/n^2)Me + q = (2^{\bar{L}}/n^2)[Me + (n^2 2^{-\bar{L}})q]$$

and, by the definition (6.1) of $\bar{L}$,

$$-(2^{\bar{L}}/n^2)e \leqslant Me + (n^2 2^{-\bar{L}})q \leqslant (2^{\bar{L}}/n^2)e.$$

Hence

$$-(2^{2\bar{L}}/n^4)e \leqslant (2^{\bar{L}}/n^2)Me + q \leqslant (2^{2\bar{L}}/n^4)e.$$

Thus, (a) follows from the definition of y^1 and $n \geqslant 2$. Since $x'^1 > 0$ and $y_0^1 > 0$ are obvious by the definition, we have $(x', y') > 0$. By the definition we also see $y'^1 = M'x'^1 + q'$. Hence we obtain (b). □

The theorem below shows that (x'^1, y'^1) can serve as an initial point for applications of Algorithms 1 and 2 to the LCP′.

Theorem 4.
(a) $(x'^1)^{\rm T} y'^1 \leqslant 2^{3L} \leqslant 2^{3L'}$.

(b) $(x'^1, y'^1) \in S_{\rm cen}(0.1)$.

Proof. By the definition, we have

$$x_0^1 y_0^1 = 2^{3\bar{L}}/n^2, \tag{6.3}$$

and by Lemma 7, for $i = 1, 2, \ldots, n$,

$$\begin{aligned}(15/16) \times 2^{3\bar{L}}/n^2 \leqslant 2^{3\bar{L}}(1-1/n^4)/n^2 \leqslant x_i^1 y_i^1 \\ \leqslant 2^{3\bar{L}}(1+1/n^4)/n^2 \leqslant (17/16) \times 2^{3\bar{L}}/n^2.\end{aligned} \tag{6.4}$$

Letting $\zeta = (x'^1)^{\mathrm{T}} y'^1/(1+n)$, we then see

$$(15/16) \times 2^{3\bar{L}}/n^2 \leqslant \zeta \leqslant (17/16) \times 2^{3\bar{L}}/n^2. \tag{6.5}$$

Hence, by (6.1) and (6.2), we obtain (a). From (6.3), (6.4) and (6.5), we also see

$$\begin{aligned}
\left[\sum_{i=0}^{n} |x_i^1 y_i^1 - \zeta|^2\right]^{1/2} &= \min_{\mu \in R} \left[\sum_{i=0}^{n} |x_i^1 y_i^1 - \mu|^2\right]^{1/2} \\
&\leqslant \left[\sum_{i=0}^{n} |x_i^1 y_i^1 - 2^{3\bar{L}}/n^2|^2\right]^{1/2} \\
&\leqslant \left[\sum_{i=1}^{n} |2^{3\bar{L}}/n^6|^2\right]^{1/2} \quad \text{(by (6.3) and (6.4))} \\
&\leqslant 2^{3\bar{L}}/n^{5.5} \\
&\leqslant (2^{3\bar{L}}/n^{5.5})\zeta/[(15/16) \times 2^{3\bar{L}}/n^2] \quad \text{(by (6.5))} \\
&\leqslant (16/15)[1/(n^{3.5})]\zeta \\
&< 0.1\zeta \quad (\text{by } n \geqslant 2).
\end{aligned}$$

Thus, we have shown (b). □

In view of the theorem above, we can apply Algorithm 2 to the LCP′ from the initial point (x'^1, y'^1) to compute an approximate solution $(\hat{x}_0, \hat{x}, \hat{y}_0, \hat{y}) \in S'_{\text{int}}$ such that

$$\hat{x}_i \hat{y}_i \leqslant 2^{-2L'} \quad (i = 0, 1, 2, \ldots, n) \tag{6.6}$$

in $O((1+n)^{0.5} L') = O(n^{0.5} L)$ iterations, requiring $O((1+n)^3 L') = O(n^3 L)$ arithmetic operations. Furthermore, using the information (6.6) on the approximate solution, we can compute an exact solution $(\bar{x}_0, \bar{x}, \bar{y}_0, \bar{y})$ of the LCP′ in additional $O((1+n)^3) = O(n^3)$ arithmetic operations. (See Appendix B.) If $\bar{x}_0 = 0$ then $(\bar{x}, \bar{y})$ turns out to be a solution of the LCP (1.1). Otherwise, the theorem below ensures that the LCP has no solution. Therefore we can conclude that the application of Algorithm 2 to the LCP′ solves the LCP in $O(n^3 L)$ arithmetic operations.

If we apply Algorithm 1 to the LCP′ instead of Algorithm 2, a total of $O((1+n)^{3.5}L')=O(n^{3.5}L)$ arithmetic operations will be required to solve the LCP.

Theorem 5. *Suppose that the LCP* (1.1) *has a solution. Then* $\bar{x}_0=0$ *for any solution* $(\bar{x}_0, \bar{x}, \bar{y}_0, \bar{y})$ *of the LCP′.*

Proof. Since the LCP has a solution, we can find a solution $(\tilde{x}, \tilde{y})$ in the set of basic feasible solutions of the system of equations $y-Mx=q$, $(x, y)\geqslant 0$. We then see that each coordinate of $\tilde{x}$ is bounded by $2^{\bar{L}}/n^2$, i.e., $\tilde{x}\leqslant(2^{\bar{L}}/n^2)e$. For every coordinate of the basic feasible solution $(\tilde{x}, \tilde{y})$ can be represented as the ratio Δ_1/Δ_2 of the determinants Δ_1 and Δ_2 of some $n\times n$ submatrices of $[E \;\; -M \;\; q]$ such that $1\leqslant|\Delta_2|$ and $0\leqslant|\Delta_1|\leqslant 2^{\bar{L}}/n^2$. See, for example, [21]. Here E denotes the $n\times n$ identity matrix. Let $\tilde{x}_0=0$ and $\tilde{y}_0=q_0-e^{\mathrm{T}}\tilde{x}$. It follows from $\tilde{x}\leqslant(2^{\bar{L}}/n^2)e$ and the definition of q_0 that $\tilde{y}_0>0$. Obviously the point $(\tilde{x}', \tilde{y}')=(\tilde{x}_0, \tilde{x}, \tilde{y}_0, \tilde{y})$ satisfies the other requirements of the LCP′, so that it is a solution of the LCP′. Let $(\bar{x}', \bar{y}')=(\bar{x}_0, \bar{x}, \bar{y}_0, \bar{y})$ be an arbitrary solution of the LCP′. To show $\bar{x}_0=0$, we utilize the identity

$$(\tilde{x}')^{\mathrm{T}}\bar{y}'=(\bar{x}')^{\mathrm{T}}\bar{y}'+(\bar{x}')^{\mathrm{T}}\tilde{y}'+(\tilde{x}'-\bar{x}')^{\mathrm{T}}M'(\tilde{x}'-\bar{x}')$$

given by Mangasarian [14]. This identity can be verified directly by using $(\bar{x}')^{\mathrm{T}}\bar{y}'=0$. Since $(\tilde{x}')^{\mathrm{T}}\tilde{y}'=0$ and M' is positive semi-definite, we have $(\tilde{x}')^{\mathrm{T}}\bar{y}'+(\bar{x}')^{\mathrm{T}}\tilde{y}'\leqslant 0$. Hence, by $(\tilde{x}', \tilde{y}')\geqslant 0$, $(\bar{x}', \bar{y}')\geqslant 0$ and $\tilde{y}_0>0$, we obtain $\bar{x}_0=0$. □

7. Conclusions

We have presented two algorithms, Algorithms 1 and 2 that solve an LCP satisfying Assumptions (i), (ii) and (iii) in $O(n^{0.5}L)$ iterations, where L denotes the size of the input data of the LCP (see (1.2)). An essential idea behind the algorithms is "tracing the path of centers of the feasible region by using Newton's method", which has been successfully utilized by several authors to develop polynomial-time algorithms for linear programs. Algorithm 1 requires $O(n^3)$ arithmetic operations per iteration; hence it has the $O(n^{3.5}L)$ computational complexity in terms of arithmetic operations. Algorithm 2 is rather complicated mainly because it incorporates the rank-one update procedure to save $O(n^{0.5})$ arithmetic operations on the average per iteration, but it attains $O(n^3L)$ computational complexity.

Acknowledgement

The authors are grateful to the referees for helpful comments which have improved the paper. The remarks on bit computational complexity in Section 5 and in Appendix B have been added according to a suggestion by one of the referees.

Appendix A. A characterization of centers in terms of the logarithmic barrier function method

Consider the following quadratic programming problem (QP):

$$\begin{array}{ll}\text{Minimize} & x^{\mathrm{T}} y \\ \text{subject to} & (x, y) \in S,\end{array}$$

where $S = \{(x, y) : y - Mx = q, x \geq 0, y \geq 0\}$. The QP is equivalent to the LCP in the sense that (x, y) is a solution of the LCP if and only if it is a minimal solution of the QP with the objective value 0. We now apply the logarithmic barrier function method to the QP to replace the nonnegativity condition $(x, y) \geq 0$ by additional logarithmic barrier function terms to the objective function.

$$\begin{array}{lll} L(\mu): & \text{Minimize} & x^{\mathrm{T}} y - \mu \sum_{j=1}^{n} \log x_j - \mu \sum_{j=1}^{n} \log y_j \\ & \text{subject to} & (x, y) \in S_{\mathrm{int}}. \end{array}$$

The following two theorems characterize a point in the path S_{cen} of centers, i.e., a solution of the system (1.6) with $\mu > 0$, in terms of the problem $L(\mu)$.

Theorem A.1. *Let $\mu > 0$. If (x, y) satisfies the system* (1.6) *of equations then it is a minimal solution of $L(\mu)$.*

Proof. The objective function of the problem $L(\mu)$ can be rewritten as

$$\sum_{j=1}^{n} [x_j y_j - \mu \log(x_j y_j)].$$

Since each term $x_j y_j - \mu \log(x_j y_j)$ in brackets $[\cdot]$ attains the minimum under the condition $(x_j, y_j) > 0$ if and only if $x_j y_j = \mu$, the desired result follows. □

Theorem A.2. *Let $\mu > 0$. Let the LCP satisfy Assumption* (iii) *in Section* 1. *Suppose that (x, y) is a minimal solution of $L(\mu)$. Then it is a solution of the system* (1.6).

Proof. We first observe that (x, y) satisfies the Karush–Kuhn–Tucker optimality condition (see, for example, [13]) with a Lagrangian multiplier vector $u \in R^n$:

$$y - \mu X^{-1} e + M^{\mathrm{T}} u = 0, \quad x - \mu Y^{-1} e - u = 0 \quad \text{and} \quad y - Mx = q.$$

We shall show that $u = 0$. Then the condition above is equivalent to the system (1.6). Multiplying the diagonal matrices

$$X = \operatorname{diag}(x_1, x_2, \ldots, x_n) \quad \text{and} \quad Y = \operatorname{diag}(y_1, y_2, \ldots, y_n)$$

to the first and the second equalities above, respectively, we obtain

$$Xy - \mu e + XM^{\mathrm{T}} u = 0 \quad \text{and} \quad Xy - \mu e - Yu = 0,$$

which imply

$$X(M^{\mathrm{T}}+X^{-1}Y)u=0.$$

By Assumption (iii) and $(x, y)>0$, we see that the $n\times n$ matrix $X(M^{\mathrm{T}}+X^{-1}Y)$ is nonsingular, so that we can conclude that the Lagrange multiplier vector u is zero. □

Theorem A.3. *Let the LCP satisfy Assumption* (iii) *and* $S_{\mathrm{int}}\neq\emptyset$. *Then the problem* $L(\mu)$ *has a unique minimal solution for every* $\mu>0$.

Proof. Let $\mu>0$ be fixed. The problem $L(\mu)$ can be rewritten as

$$\begin{array}{ll}\text{Minimize} & x^{\mathrm{T}}(Mx+q)-\mu\sum_{j=1}^{n}\log x_j-\mu\sum_{j=1}^{n}\log y_j\\ \text{subject to} & (x, y)\in S_{\mathrm{int}}.\end{array}$$

Since the objective function to be minimized is a strictly convex function on a nonempty convex constraint set S_{int}, a minimal solution is unique if it exists. In order to see the existence of a solution to this problem, we need only prove that the set $S_{\mathrm{int}}(\omega)$ of all the feasible solutions of the problem $L(\mu)$ with the objective value not greater than ω is nonempty, closed and bounded for some sufficiently large ω. The closedness of the set $S_{\mathrm{int}}(\omega)$ can be verified easily. By the assumption $S_{\mathrm{int}}\neq\emptyset$, the set of all the solutions to the LCP is bounded. (See [14].) Hence so is the set

$$\{(x, y)\in S\colon x^{\mathrm{T}}(Mx+q)\leqslant 0\}.$$

Since the function $x\to x^{\mathrm{T}}(Mx+q)$ is convex by Assumption (iii), the set

$$\{(x, y)\in S\colon x^{\mathrm{T}}(Mx+q)\leqslant \omega\}$$

is bounded for every $\omega\geqslant 0$. This implies that the minimum of the quadratic function $x^{\mathrm{T}}(Mx+q)$ over the subset

$$S(r)=\{(x, y)\in S\colon \|(x, y)\|=r\}$$

of S grows at least linearly in r for sufficiently large r. Hence the minimum of the objective function of the problem $L(\mu)$, which differs only the logarithmic barrier terms

$$-\mu\sum_{j=1}^{n}\log x_j-\mu\sum_{j=1}^{n}\log y_j$$

from the quadratic function $x^{\mathrm{T}}(Mx+q)$, over the set $S(r)$ diverges as r tends to the infinity. This ensures that the set $S_{\mathrm{int}}(\omega)$ is bounded for any ω. Finally, we see by the assumption $S_{\mathrm{int}}\neq\emptyset$ that the set $S_{\mathrm{int}}(\omega)$ is nonempty for sufficiently large ω. □

Theorem A.4. *Let the LCP satisfy Assumptions* (iii) *in Section* 1, *and* $S_{\rm int} \neq \emptyset$. *Let* $(x(\mu), y(\mu))$ *denote the solution of the system* (1.6). *Then* $(x(\mu), y(\mu))$ *is* C^1 (*continuously differentiable*) *at every* $\mu > 0$.

Proof. It is easily verified under Assumptions (iii) that the Jacobian matrix of the mapping H with respect to (x, y) is nonsingular and C^1 at every solution (x, y) of the system (1.6) with $\mu > 0$. Thus, the desired result follows from the well-known implicit function theorem (see, for example, [17]). □

Appendix B. Computing an exact solution of the LCP

We shall assume that an approximate solution $(\hat{x}, \hat{y}) \in S$ of the LCP (1.1) satisfying the relation (1.3) and show how we compute an exact solution by using this information. We employ the notation z for the $2n$-dimensional variable vector (x, y) and A for the $n \times 2n$ matrix $[E \ \ -M]$, where E denotes the $n \times n$ identity matrix. Then the feasible region S of the LCP (1.1) can be represented as the set of solutions of the system of equations

$$Az = q, \ z = (x, y) \in R_+^{2n}. \tag{B.1}$$

Lemma B. *Suppose* $\bar{z} \in S$. *Let* $\bar{K} = \{k \colon \bar{z}_k < 2^{-L}\}$. *Then there is a vertex* z^* *of* S *such that*

$$z_k^* = 0 \quad \text{for all } k \in \bar{K}. \tag{B.2}$$

Proof. By Cramer's rule, we know that each element of a basic feasible solution of the system (B.1) can be represented as the ratio Δ_1/Δ_2 of the determinants of some $n \times n$ submatrix of $[A \ \ q]$. From the definition (1.2) of L, we see the absolute value of the determinant Δ of any $n \times n$ submatrix of $[A \ \ q]$ is bounded by $2^L/n^2$. See, for example, [21]. This implies that, for any basic feasible solution v,

$$v_i = 0 \quad \text{if } v_i < n^2 2^{-L}$$

or equivalently

$$v_i \geq n^2 2^{-L} \quad \text{if } v_i > 0 \tag{B.3}$$

holds. Since v is a vertex of S if and only if it is a basic feasible solution of (B.1), (B.3) is true for any vertex of S. On the other hand, the point $\bar{z}$ can be represented as the sum of the convex combination of some vertices $v^1, v^2, \ldots, v^p$ of S and some unbounded direction u of S such that

$$\bar{z} = \sum_{j=1}^{p} c_j v^j + u, \qquad \sum_{j=1}^{p} c_j = 1, \quad c_j \geq 0 \ (j = 1, 2, \ldots, p).$$

See, for example, [22, Theorem 2.12.6]. In view of Caratheodory's theorem [20, Theorem 2.2.12], we may assume that $p \leqslant 1+n$. Hence we can find an index r such that $c_r \geqslant 1/(1+n)$. We shall show that $z^* = v^r$ satisfies (B.2). Assume on the contrary that

$$v_k^r > 0 \quad \text{for some } k \in \bar{K}.$$

Then

$$v_k^r \geqslant n^2 2^{-L}$$

because v^r is a vertex of S so that (B.3) holds with $v = v^r$. Since all the components of the vectors $v^1, v^2, \ldots, v^p$ and u are nonnegative, we obtain

$$\bar{z}_k = \sum_{j=1}^{p} c_j v_k^j + u_k \geqslant c_r v_k^r \geqslant (1/(1+n)) n^2 2^{-L} > 2^{-L} \quad (\text{since } 2 \leqslant n).$$

This contradicts $k \in \bar{K}$. □

Suppose that $\hat{z} = (\hat{x}, \hat{y}) \in S$ satisfies the condition (1.3). For every $\bar{z} \in S$, define

$$K(\bar{z}) = \{k\colon \bar{z}_k < 2^{-L}\}$$

and

$$K(\bar{z})^{\mathrm{c}} = \{k\colon \bar{z}_k \geqslant 2^{-L}\}.$$

As we will see below, we can move from $\hat{z}$ to a point $\bar{z} \in S$ in $\mathrm{O}(n^3)$ arithmetic operations such that $K(\hat{z}) \subset K(\bar{z})$ and that the set of columns of the matrix A with indices in $K(\bar{z})^{\mathrm{c}}$ is linearly independent. We now consider the system of equations

$$\begin{aligned} &Az = q, \quad z \in R_+^{2n}, \\ &z_k = 0 \quad \text{for every } k \in K(\bar{z}). \end{aligned} \tag{B.4}$$

Applying Lemma B, we see that this system of equations has a solution $z^* = (x^*, y^*)$, and by the assumption (1.3) and $K(\hat{z}) \subset K(\bar{z})$ that (x^*, y^*) is an exact solution of the LCP satisfying (1.4). On the other hand, since the columns of the matrix A associated with indices in $K(\bar{z})^{\mathrm{c}}$ is linearly independent, the solution $z^* = (x^*, y^*)$ is unique and can be computed in $\mathrm{O}(n^3)$ arithmetic operations.

Now we shall show how to move from the point $\hat{z} \in S$ to a point $\bar{z} \in S$ such that $K(\hat{z}) \subset K(\bar{z})$ and that the set of columns of the matrix A with indices in $K(\bar{z})^{\mathrm{c}}$ is linearly independent. Let $\bar{z} = \hat{z}$. We consider the polyhedral set $P(\bar{z})$ consisting of the solutions $z = (x, y)$ of the system

$$\begin{aligned} &Az = q, \quad z \in R_+^{2n}, \\ &z_k = \bar{z}_k \quad \text{for every } k \in K(\bar{z}). \end{aligned} \tag{B.5}$$

By applying the Gaussian elimination to the homogeneous system

$$Au = 0, \tag{B.6}$$

we compute a solution u satisfying $u_j = 0$ ($j \in K(\bar{z})$) and $u_k > 0$ for some $k \in K(\bar{z})^c$ if it exists, and a maximal index subset K of $K(\bar{z})^c$ such that the set of the columns of the matrix A with the indices in K is linearly independent. This requires $O(n^3)$ arithmetic operations. If $K = K(\bar{z})^c$ then $\bar{z}$ itself is a desired point in S. Otherwise we have a solution u of the system (B.6) satisfying $u_j = 0$ ($j \in K(\bar{z})$) and $u_k > 0$ for some $k \in K(\bar{z})^c$. In this case we can move from $\bar{z}$ toward the direction $-u$ to obtain a point $z \in P(\bar{z})$ such that $|K(z)| < |K(\bar{z})|$ by applying a ratio test to the solution $\bar{z}$ of the nonhomogeneous system (B.5) with the solution u of the homogeneous system (B.6). Here $|K|$ denotes the number of elements in an index set K. Then, replacing $\bar{z}$ by z, we perform pivot operations to the homogeneous system (B.6) to generate a new solution u such that $u_j = 0$ ($j \in K(\bar{z})$) and $u_k > 0$ for some $k \in K(\bar{z})^c$ if it exists, and repeat the same procedure until we find a point $\bar{z} \in S$ satisfying the desired property. This iteration terminates in at most n steps since $0 \leq |K(z)| < |K(\bar{z})| \leq n$. The total number of pivot operations, each of which requires $O(n^2)$ arithmetic operations, is bounded by n. Each ratio test requires $O(n)$ arithmetic operations. Therefore, the total number of arithmetic operations amounts to $O(n^3)$.

Remark. To estimate the bit computational complexity for computing such a $\bar{z} \in S$, we assume that each element of $\hat{z}$ has been represented in a rational number with a denominator $2^{O(L)}$ and a numerator $2^{O(L)}$. Then we can execute each iteration above such that each element of z is a rational number with a denominator $2^{O(L)}$. If we assume, in addition, that all the elements of z generated at each iteration have numerators $2^{O(L)}$, we can conclude that the total number of bit operations amounts to $O(n^3 L(\log L)(\log\log L))$ because each pivoting operation requires $O(n^2 L(\log L)(\log\log L))$ bit operations and each ratio test $O(nL(\log L)(\log\log L))$ bit operations. See also the Remark at the end of Section 5.

References

[1] J.R. Birge and A. Gana, "Computational complexity of van der Heyden's variable dimensional algorithm and Dantzig-Cottle's principal pivoting method for solving LCP's," *Mathematical Programming* 26 (1983) 316–325.

[2] S.J. Chung, "A note on the complexity of LCP: the LCP is strongly NP-complete," Technical Report 792, Dept. of Industrial Engineering and Operations Engineering, The University of Michigan (Ann Arbor, Michigan, 1979).

[3] G.B. Dantzig and R.W. Cottle, "Positive (semi-definite) matrices and mathematical programming," Report ORC 63-18, (RR) 13, University of Berkeley (California, 1963).

[4] Y. Fathi, "Computational complexity of LCPs associated with positive definite symmetric matrices," *Mathematical Programming* 17 (1979) 335–344.

[5] D.M. Gay, "A variant of Karmarkar's linear programming algorithm for problems in standard form," *Mathematical Programming* 37 (1987) 81–90.

[6] P.E. Gill, W. Murray, M.A. Saunders, J.A. Tomlin and M.H. Wright, "On projected Newton barrier methods for linear programming and an equivalence to Karmarkar's projective method," *Mathematical Programming* 36 (1986) 183–209.

[7] C.C. Gonzaga, "An algorithm for solving linear programming problems in $O(n^3 L)$ operations," in: N. Megiddo, ed., *Progress in Mathematical Programming* (Springer-Verlag, New York, 1988), to appear.

[8] S. Kapoor and P.M. Vaidya, "Fast algorithms for convex quadratic programming and multicommodity flows," *Proceedings of the 18th Annual ACM Symposium on Theory of Computing* (Berkeley, California, 1986) 147–159.

[9] N. Karmarkar, "A new polynomial-time algorithm for linear programming," *Combinatorica* 4 (1984) 373–395.

[10] L.G. Khachiyan, "A polynomial algorithm in linear programming," *Soviet Mathematics Doklady* 20 (1979) 191–194.

[11] M. Kojima, S. Mizuno and A. Yoshise, "A primal–dual interior point algorithm for linear programming," in: N. Megiddo, ed., *Progress in Mathematical Programming* (Springer-Verlag, New York, 1988), to appear.

[12] C.E. Lemke, "Bimatrix equilibrium points and mathematical programming," *Management Science* 11 (1965) 681–689.

[13] O.L. Mangasarian, *Nonlinear Programming* (McGraw-Hill, New York, 1969).

[14] O.L. Mangasarian, "Simple computable bounds for solutions of linear complementarity problems and linear programs," *Mathematical Programming Study* 25 (1985) 1–12.

[15] N. Megiddo, "Pathways to the optimal set in linear programming," *Proceedings of the 6th Mathematical Programming symposium of Japan* (Nagoya, Japan, 1986) 1–35.

[16] K.G. Murty, "Computational complexity of complementary pivot methods," *Mathematical Programming Study* 7 (1978) 61–73.

[17] J.M. Ortega and W.C. Rheinboldt, *Iterative Solutions of Nonlinear Equations of Several Variables* (Academic Press, New York, 1970).

[18] J.S. Pang, I. Kaneko and W.P. Hallman, "On the solution of some (parametric) linear complementarity problems with application to portfolio selection, structural engineering and actuarial graduation," *Mathematical Programming* 16 (1979) 325–347.

[19] J. Renegar, "A polynomial-time algorithm, based on Newton's method, for linear programming," *Mathematical Programming* 40 (1988) 59–94.

[20] G. Sonnevend, "An 'analytic center' for polyhedrons and new classes of global algorithms for linear (smooth, convex) programming," *Proceedings of the 12th IFIP Conference on System Modeling and Optimization* (Budapest, 1985), to appear in *Lecture Notes in Control and Information Sciences* (Springer-Verlag).

[21] A. Schrijver, *Theory of Linear and Integer Programming* (John Wiley & Sons, 1986).

[22] J. Stoer and C. Witzgall, *Convexity and Optimization in Finite Dimensions* (Springer, New York, 1970).

[23] K. Tanabe, "Complementarity-enforcing centered Newton method for linear programming: Global method," Symposium, "New Method for Linear Programming" at The Institute of Statistical Mathematics (Tokyo, 1987).

[24] L. Van der Heyden, "A variable dimension algorithm for the linear complementarity problem," *Mathematical Programming* 19 (1980) 328–346.

[25] P.M. Vaidya, "An algorithm for linear programming which requires $O(((m+n)n^2+(m+n)^{1.5}n)L)$ arithmetic operations," AT&T Bell Laboratories, Murray Hill (New Jersey, 1987).

[26] Y. Ye and M. Kojima, "Recovering optimal dual solutions in Karmarkar's polynomial time algorithm for linear programming," *Mathematical Programming* 39 (1987) 305–317.

[27] Y. Ye and E. Tse, "A polynomial-time algorithm for convex quadratic programming," Working Paper, Dept. of Engineering-Economic Systems, Stanford University (California, 1986).