ON THE ANALYTIC LUNAR AND SOLAR PERTURBATIONS OF A NEAR EARTH SATELLITE

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Abstract. The disturbing function of the Moon (Sun) is expanded as a sum of products of two harmonic functions, one depending on the position of the satellite and the other on the position of the Moon (Sun). The harmonic functions depending on the position of the perturbing body are developed into trigonometric series with the ecliptic elements l, l', F, D and Γ of the lunar theory which are nearly linear with respect to time. Perturbation of elements are in the form of trigonometric series with the ecliptic lunar elements and the equatorial elements ω and Ω of the satellite so that analytic integration is simple and the results accurate over a long period of time.

Notation

G	– the gravitational constant
M	– the mass of the Earth
m'	– the mass of the Moon
m''	– the mass of the Sun
$\mathbf{r}$	– the geocentric position vector of the satellite
r	$= \lvert\mathbf{r}\rvert$,
$\hat{r}$	$= \mathbf{r}/r$
$\mathbf{r}'$	– the geocentric position vector of the Moon
r'	$= \lvert\mathbf{r}'\rvert$
$\hat{r}'$	$= \mathbf{r}'/r'$
$\mathbf{r}''$	– the geocentric position vector of the Sun
r''	$= \lvert\mathbf{r}''\rvert$
$\hat{r}''$	$= \mathbf{r}''/r''$
a'	– the mean distance of the Moon from the Earth, defined in such a manner that the constant part in the expansion of the lunar parallax equals unity
a''	– the mean geocentric distance of the Sun
γ'	– the angle between $\mathbf{r}$ and $\mathbf{r}'$
γ''	– the angle between $\mathbf{r}$ and $\mathbf{r}''$
$\lambda, \mu, \nu, \lambda', \mu', \nu', \lambda'', \mu''. \nu''$	– the rectangular components of $\hat{r}$, $\hat{r}'$ and $\hat{r}''$ respectively in the geocentric equatorial coordinate system
R_e	– the equatorial radius of the Earth
g	– the mean anomaly of the satellite
n	– the mean motion of the satellite
f	– the true anomaly of the satellite
e	– the eccentricity of the satellite orbit
i	– the inclination of the satellite orbit
Ω	– the longitude of the ascending node of the satellite
ω	– the argument of perigee of the satellite
$\tilde{\omega}$	$= \omega + \Omega$

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a	– the semimajor axis of the satellite orbit
δa	– the perturbations in a caused by the Moon (primed)/Sun (double primed)
δe	– the perturbations in e caused by the Moon (primed)/Sun (double primed)
δi	– the perturbations in i caused by the Moon (primed)/Sun (double primed)
δg	– the perturbations in g caused by the Moon (primed)/Sun (double primed)
$\delta \Omega$	– the perturbations in Ω caused by the Moon (primed)/Sun (double primed)
$\delta \omega$	– the perturbations in ω caused by the Moon (primed)/Sun (double primed)
ε	– the obliquity of the ecliptic
J_2, J_4	– zonal harmonic coefficients in the Earth's gravitational potential ($J_2 = 1.082\,19 \times 10^{-3}$ and $J_4 = -2.123 \times 10^{-6}$)
t_0	– Julian date of January 0.5, 1900 (2 415 020.0 days)
$t - t_0$	– number of days from January 0.5, 1900
T	– number of Julian Centuries (36 525 days) from January 0.5, 1900
$l_☾$	– geocentric mean longitude of the Moon
$\Omega_☾$	– geocentric mean longitude of the lunar node
$\tilde{\omega}_☾$	– geocentric mean longitude of the lunar perigee
$l_\odot$	– geocentric mean longitude of the Sun
$\Gamma \equiv \tilde{\omega}_\odot$	– geocentric mean longitude of the solar perigee
$l \equiv l_☾ - \tilde{\omega}_☾$	– argument of the principal elliptic term
$l' \equiv l_\odot - \tilde{\omega}_\odot$	– argument of the annual equation
$F \equiv l_☾ - \Omega_☾$	– argument of the principal term in latitude
$D \equiv l_☾ - l_\odot$	– half argument of the variation

1. Introduction

The usual analytical treatment of the secular and long period effects of the Sun and Moon upon a close Earth satellite is based on a trigonometric expansion of the disturbing function with the angular equatorial elements of the Sun and Moon as arguments (Kozai, 1959; Musen *et al.*, 1961; Kaula, 1962; Murphy and Felsentreger, 1966). The disturbing function is then integrated by approximating the equatorial elements as linear functions of time.

The expansion presented in this paper, following Musen and Estes (1971), represents the disturbing function as a sum of products of two harmonic functions, one depending on the position of the satellite and the other on the position of the Moon (Sun) as given by the Hill-Brown lunar theory (Newcomb's solar theory) instead of osculating elements which are not known precisely (Musen and Felsentreger, 1972). The harmonic functions depending on the lunar and solar positions are represented as trigonometric series with the ecliptic elements l, l', F, D, and Γ of the lunar theory which are very nearly linear with respect to time, in contrast to the equatorial node of the lunar orbit which oscillates between two limits with a period of nearly eighteen years. The expansion of perturbations is then in the form of trigonometric series with the ecliptic lunar elements and the equatorial elements ω and Ω of the satellite so that it is easy to include perturbations in the lunar orbit into the disturbing function and the analytic integration is simple and valid for a long time span.

As with other analytic theories of lunar and solar perturbations based on the development of the disturbing function into a trigonometric series, there will appear small divisors for some terms. In particular the time derivatives of the satellite elements ω and Ω which appear in denominators after integration are calculated from expressions given by Brouwer (1959) so that oblateness produces resonances at some critical inclination angles such as 46.4°, 56.1°, 63.4°, 69.0°, and 73.1°. Such resonances must be treated as special cases.

2. The Lunar and Solar Disturbing Function

The expansion of the lunar disturbing function is

$$\begin{aligned} R' &= Gm'\left\{\frac{1}{r'}\left[\frac{r}{r'}P_1(\cos\gamma') + \frac{r^2}{r'^2}P_2(\cos\gamma') + \right.\right. \\ &\quad \left.\left. + \frac{r^3}{r'^3}P_3(\cos\gamma') + \cdots\right] - \frac{r}{r'^2}\cos\gamma'\right\} \\ &= Gm'\left\{\frac{r^2}{r'^3}P_2(\cos\gamma') + \frac{r^3}{r'^4}P_3(\cos\gamma') + \cdots\right\}, \end{aligned} \tag{1}$$

where $P_j(\cos\gamma')$ are Legendre polynomials and

$$\mathbf{r}\cdot\mathbf{r}' = rr'\cos\gamma'.$$

Then

$$\begin{aligned} P_2(\cos\gamma') &= \tfrac{1}{2}(3\cos^2\gamma' - 1) \\ &= \tfrac{3}{2}[\lambda^2\lambda'^2 + 2\lambda\mu\lambda'\mu' + \mu^2\mu'^2 + \\ &\quad + 2\lambda\nu\lambda'\nu' + 2\mu\nu\mu'\nu' + \nu^2\nu'^2] - \tfrac{1}{2} \\ &= \tfrac{1}{4}(1 - 3\nu^2)[1 - 3\nu'^2] + \tfrac{3}{4}(\lambda^2 - \mu^2)[\lambda'^2 - \mu'^2] + \\ &\quad + 3\lambda\mu[\lambda'\mu'] + 3\lambda\nu[\lambda'\nu'] + 3\mu\nu[\mu'\nu'], \end{aligned}$$

$$\begin{aligned} P_3(\cos\gamma') &= \tfrac{1}{2}\cos\gamma'(5\cos^2\gamma' - 3) \\ &= \tfrac{3}{8}\lambda(1 - 5\nu^2)[\lambda'(1 - 5\nu'^2)] + \\ &\quad + \tfrac{3}{8}\mu(1 - 5\nu^2)[\mu'(1 - 5\nu'^2)] + \\ &\quad + \tfrac{1}{4}\nu(3 - 5\nu^2)[\nu'(3 - 5\nu'^2)] + \\ &\quad + \tfrac{5}{8}\lambda(\lambda^2 - 3\mu^2)[\lambda'(\lambda'^2 - 3\mu'^2)] + \\ &\quad + \tfrac{5}{8}\mu(3\lambda^2 - \mu^2)[\mu'(3\lambda'^2 - \mu'^2)] + \\ &\quad + \tfrac{15}{2}\lambda\mu\nu[2\lambda'\mu'\nu'] + \\ &\quad + \tfrac{15}{4}\nu(\lambda^2 - \mu^2)[\nu'(\lambda'^2 - \mu'^2)], \end{aligned}$$

and we have, in notation similar to that of Musen and Estes (1971)

$$R' = \frac{Gm'a^2}{a'^3}\{a_{20}C'_{20} + a_{21}C'_{21} + b_{21}S'_{21} + a_{22}C'_{22} + b_{22}S'_{22}\} +$$
$$+ \frac{Gm'a^3}{a'^4}\{a_{31}C'_{31} + b_{31}S'_{31} + b_{32}S'_{32} + a_{33}C'_{33} + b_{33}S'_{33} +$$
$$+ a_{34}C'_{34} + b_{34}S'_{34}\} +$$
$$+ \cdots, \tag{2}$$

where

$$a_{20} = \frac{1}{4}\left(\frac{r}{a}\right)^2 (1 - 3\nu^2),$$

$$a_{21} = \frac{3}{4}\left(\frac{r}{a}\right)^2 (\lambda^2 - \mu^2), \qquad b_{21} = \frac{3}{2}\left(\frac{r}{a}\right)^2 \lambda\mu,$$

$$a_{22} = 3\left(\frac{r}{a}\right)^2 \mu\nu, \qquad b_{22} = 3\left(\frac{r}{a}\right)^2 \lambda\nu,$$

$$a_{31} = \frac{3}{8}\left(\frac{r}{a}\right)^3 \lambda(1 - 5\nu^2), \qquad b_{31} = \frac{3}{8}\left(\frac{r}{a}\right)^3 \mu(1 - 5\nu^2),$$

$$b_{32} = \frac{1}{4}\left(\frac{r}{a}\right)^3 \nu(3 - 5\nu^2),$$

$$a_{33} = \frac{5}{8}\left(\frac{r}{a}\right)^3 \lambda(\lambda^2 - 3\mu^2), \qquad b_{33} = \frac{5}{8}\left(\frac{r}{a}\right)^3 \mu(3\lambda^2 - \mu^2),$$

$$a_{34} = \frac{15}{2}\left(\frac{r}{a}\right)^3 \lambda\mu\nu, \qquad b_{34} = \frac{15}{4}\left(\frac{r}{a}\right)^3 \nu(\lambda^2 - \mu^2),$$

$$C'_{20} = \left(\frac{a'}{r'}\right)^3 (1 - 3\nu'^2),$$

$$C'_{21} = \left(\frac{a'}{r'}\right)^3 (\lambda'^2 - \mu'^2), \qquad S'_{21} = 2\left(\frac{a'}{r'}\right)^3 \lambda'\mu',$$

$$C'_{22} = \left(\frac{a'}{r'}\right)^3 \mu'\nu', \qquad S'_{22} = \left(\frac{a'}{r'}\right)^3 \lambda'\nu',$$

$$C'_{31} = \left(\frac{a'}{r'}\right)^4 \lambda'(1 - 5\nu'^2), \qquad S'_{31} = \left(\frac{a'}{r'}\right)^4 \mu'(1 - 5\nu'^2),$$

$$S'_{32} = \left(\frac{a'}{r'}\right)^4 \nu'(3 - 5\nu'^2),$$

$$C'_{33} = \left(\frac{a'}{r'}\right)^4 \lambda'(\lambda'^2 - 3\mu'^2), \qquad S'_{33} = \left(\frac{a'}{r'}\right)^4 \mu'(3\lambda'^2 - \mu'^2),$$

$$C'_{34} = 2\left(\frac{a'}{r'}\right)^4 \lambda'\mu'\nu', \qquad S'_{34} = \left(\frac{a'}{r'}\right)^4 \nu'(\lambda'^2 - \mu'^2).$$

The solar disturbing function is of the same form with all primes replaced by double primes.

In the equatorial elements of the satellite,

$$\lambda = \cos(f + \omega)\cos\Omega - \sin(f + \omega)\sin\Omega\cos i$$
$$\mu = \cos(f + \omega)\sin\Omega + \sin(f + \omega)\cos\Omega\cos i$$
$$\nu = \sin(f + \omega)\sin i.$$

Then

$$a_{20} = \frac{1}{4}\left(\frac{r}{a}\right)^2\left\{1 - \tfrac{3}{2}\sin^2 i + \tfrac{3}{2}\sin^2 i\cos(2f + 2\omega)\right\}$$

$$a_{21} = \frac{1}{4}\left(\frac{r}{a}\right)^2\{\tfrac{1}{2}\sin^2 i\cos 2\Omega + \tfrac{1}{2}\cos 2\Omega\cos(2f + 2\omega) -$$
$$- \cos i\sin 2\Omega\sin(2f + 2\omega) + \tfrac{1}{2}\cos^2 i\cos 2\Omega\cos(2f + 2\omega)\}$$

$$b_{21} = \frac{3}{2}\left(\frac{r}{a}\right)^2\{\tfrac{1}{2}\sin^2 i\sin\Omega\cos\Omega +$$
$$+ \tfrac{1}{2}(1 + \cos^2 i)\sin\Omega\cos\Omega\cos(2f + 2\omega) +$$
$$+ \tfrac{1}{2}\cos i(\cos^2\Omega - \sin^2\Omega)\sin(2f + 2\omega)\},$$

$$a_{22} = 3\left(\frac{r}{a}\right)^2\{\tfrac{1}{2}\sin i\sin\Omega\sin(2f + 2\omega) + \tfrac{1}{2}\sin i\cos i\cos\Omega -$$
$$- \tfrac{1}{2}\sin i\cos i\cos\Omega\cos(2f + 2\omega)\},$$

$$b_{22} = 3\left(\frac{r}{a}\right)^2\{- \tfrac{1}{2}\sin i\cos i\sin\Omega + \tfrac{1}{2}\sin i\cos\Omega\sin(2f + 2\omega) +$$
$$+ \tfrac{1}{2}\sin i\cos i\sin\Omega\cos(2f + 2\omega)\},$$

$$a_{31} = \frac{3}{8}\left(\frac{r}{a}\right)^3\{\cos\Omega(1 - \tfrac{3}{4}\sin^2 i)\cos(f + \omega) +$$
$$+ \tfrac{3}{4}\sin^2 i\cos\Omega\cos(3f + 3\omega) +$$
$$+ \cos i\sin\Omega(\tfrac{9}{4}\sin^2 i - 1)\sin(f + \omega) -$$
$$- \tfrac{3}{4}\sin^2 i\cos i\sin\Omega\sin(3f + 3\omega)\},$$

$$b_{31} = \frac{3}{8}\left(\frac{r}{a}\right)^3\{\sin\Omega(1 - \tfrac{3}{4}\sin^2 i)\cos(f + \omega) +$$
$$+ \tfrac{3}{4}\sin^2 i\cos\Omega\cos(3f + 3\omega) +$$
$$+ \cos i\cos\Omega(1 - \tfrac{9}{4}\sin^2 i)\sin(f + \omega)$$
$$+ \tfrac{3}{4}\sin^2\cos i\cos\Omega\sin(3f + 3\omega)\},$$

$$b_{32} = \frac{1}{4}\left(\frac{r}{a}\right)^3 \{\sin i(3 - \tfrac{15}{4}\sin^2 i)\sin(f+\omega) + \tfrac{5}{4}\sin^3 i \sin(3f+3\omega)\},$$

$$\begin{aligned} a_{33} = \frac{5}{8}\left(\frac{r}{a}\right)^3 \{&\tfrac{3}{4}\sin^2 i\cos\Omega(\cos^2\Omega - 3\sin^2\Omega)\cos(f+\omega) + \\ &+ \tfrac{3}{4}\cos i\sin^2 i\sin\Omega(\sin^2\Omega - 3\cos^2\Omega)\sin(f+\omega) + \\ &+ \tfrac{1}{4}(1 + 3\cos^2 i)\cos\Omega(\cos^2\Omega - 3\sin^2\Omega)\cos(3f+3\omega) + \\ &+ \tfrac{1}{4}\cos i(3 + \cos^2 i)\sin\Omega(\sin^2\Omega - 3\cos^2\Omega)\sin(3f+3\omega)\}, \end{aligned}$$

$$\begin{aligned} b_{33} = \frac{5}{8}\left(\frac{r}{a}\right)^3 \{&\tfrac{3}{4}\sin^2 i\sin\Omega(3\cos^2\Omega - \sin^2\Omega)\cos(f+\omega) + \\ &+ \tfrac{3}{4}\sin^2 i\cos i\cos\Omega(\cos^2\Omega - 3\sin^2\Omega)\sin(f+\omega) + \\ &+ \tfrac{1}{4}\sin\Omega(1 + 3\cos^2 i)(3\cos^2\Omega - \sin^2\Omega)\cos(3f+3\omega) + \\ &+ \tfrac{1}{4}\cos\Omega\cos i(3 + \cos^2 i)(\cos^2\Omega - 3\sin^2\Omega)\sin(3f+3\omega)\}, \end{aligned}$$

$$\begin{aligned} a_{34} = \frac{15}{2}\left(\frac{r}{a}\right)^3 \{&\tfrac{1}{4}\cos i\sin i\cos 2\Omega\cos(f+\omega) + \\ &+ \tfrac{1}{4}\sin i(1 - 3\cos^2 i)\cos\Omega\sin\Omega\sin(f+\omega) - \\ &- \tfrac{1}{4}\cos i\sin i\cos 2\Omega\cos(3f+3\omega) + \\ &+ \tfrac{1}{4}\sin i(1 + \cos^2 i)\sin\Omega\cos\Omega\sin(3f+3\omega)\}, \end{aligned}$$

$$\begin{aligned} b_{34} = \frac{15}{4}\left(\frac{r}{a}\right)^3 \{&-\tfrac{1}{4}\sin 2i\sin 2\Omega\cos(f+\omega) + \\ &+ \tfrac{1}{4}\sin i(1 - 3\cos^2 i)\cos 2\Omega\sin(f+\omega) + \\ &+ \tfrac{1}{4}\sin 2i\sin 2\Omega\cos(3f+3\omega) + \\ &+ \tfrac{1}{4}\sin i(1 + \cos^2 i)\cos 2\Omega\sin(3f+3\omega)\}. \end{aligned}$$

Using the well-known formulas of elliptic motion

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2 dg = 1 + \tfrac{3}{2}e^2,$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2 \cos 2f\, dg = \tfrac{5}{2}e^2,$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^3 \cos f\, dg = -\tfrac{5}{2}e - \tfrac{15}{8}e^3,$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^3 \cos 3f\, dg = -\tfrac{35}{8}e^3,$$

we find for the averaged potential due to the P_2 term

$$[R_2'] = \frac{Gm'a^2}{a'^3}\{\tfrac{1}{4}(1 + \tfrac{3}{2}e^2)(1 - \tfrac{3}{2}\sin^2 i)C_{20}' +$$
$$+ \tfrac{15}{16}e^2 \sin^2 i(\cos 2\omega C_{20}') +$$
$$+ \tfrac{3}{8}\sin^2 i(1 + \tfrac{3}{2}e^2)C_{210}' +$$
$$+ \tfrac{15}{16}e^2(1 + \cos^2 i)(\cos 2\omega C_{210}') +$$
$$+ \tfrac{15}{8}e^2 \cos i(\sin 2\omega S_{210}') + \tfrac{15}{4}e^2 \sin i(\sin 2\omega S_{220}') -$$
$$- \tfrac{15}{4}e^2 \sin i \cos i(\cos 2\omega C_{220}') +$$
$$+ \tfrac{3}{2}\sin i \cos i(1 + \tfrac{3}{2}e^2)C_{220}'\}, \qquad (3)$$

where

$$C_{210}' = \cos 2\Omega C_{21}' + \sin 2\Omega S_{21}',$$
$$C_{220}' = \cos \Omega C_{22}' - \sin \Omega S_{22}',$$
$$S_{210}' = \cos 2\Omega S_{21}' - \sin 2\Omega C_{21}',$$
$$S_{220}' = \sin \Omega C_{22}' + \cos \Omega S_{22}',$$

and that due to the P_3 term

$$[R_3'] = \frac{Gm'a^3}{a'^4}\{\tfrac{15}{16}e(1 + \tfrac{3}{4}e^2)(\tfrac{3}{4}\sin^2 i - 1)[\cos \omega C_{310}'] +$$
$$+ \tfrac{15}{16}e(1 + \tfrac{3}{4}e^2)\cos i(1 - \tfrac{9}{4}\sin^2 i)[\sin \omega S_{310}'] -$$
$$- \tfrac{315}{256}e^3 \sin^2 i[\cos 3\omega C_{310}'] +$$
$$+ \tfrac{315}{256}e^3 \sin^2 \cos i[\sin 3\omega S_{310}'] +$$
$$+ \tfrac{15}{8}e(1 + \tfrac{3}{4}e^2)\sin i(\tfrac{5}{4}\sin^2 i - 1)[\sin \omega S_{32}'] -$$
$$- \tfrac{175}{128}e^3 \sin^3 i[\sin 3\omega S_{32}'] -$$
$$- \tfrac{75}{64}e(1 + \tfrac{3}{4}e^2)\sin^2 i[\cos \omega C_{330}'] +$$
$$+ \tfrac{75}{64}e(1 + \tfrac{3}{4}e^2)\cos i \sin^2 i[\sin \omega S_{330}'] -$$
$$- \tfrac{175}{256}e^3(1 + 3\cos^2 i)[\cos 3\Omega C_{330}'] +$$
$$+ \tfrac{175}{256}e^3(3 + \cos^2 i)\cos i[\sin 3\omega S_{330}'] +$$
$$+ \tfrac{75}{32}e(1 + \tfrac{3}{4}e^2)\sin i(3\cos^2 i - 1)[\sin \omega S_{340}'] -$$
$$- \tfrac{75}{16}e(1 + \tfrac{3}{4}e^2)\sin i \cos i[\cos \omega C_{340}'] -$$
$$- \tfrac{525}{128}e^3 \sin i(1 + \cos^2 i)[\sin 3\omega S_{340}'] +$$
$$+ \tfrac{525}{64}e^3 \cos i \sin i[\cos 3\omega C_{340}']\}, \qquad (4)$$

where

$$\begin{aligned}
C'_{310} &= \cos \Omega C'_{31} + \sin \Omega S'_{31}, \\
S'_{310} &= \sin \Omega C'_{31} - \cos \Omega S'_{31}, \\
C'_{330} &= \cos 3\Omega C'_{33} + \sin 3\Omega S'_{33}, \\
S'_{330} &= \sin 3\Omega C'_{33} - \cos 3\Omega S'_{33}, \\
C'_{340} &= \cos 2\Omega C'_{34} - \sin 2\Omega S'_{34}, \\
S'_{340} &= \sin 2\Omega C'_{34} + \cos 2\Omega S'_{34}.
\end{aligned}$$

The satellite elements appearing in these equations and those which follow now designate mean instead of osculating elements.

Primed (double primed) quantities appearing in the disturbing function depend only upon the position of the Moon (Sun). The coordinates of the Moon are obtained analytically from E. W. Brown's theory where if $\lambda_☾$ is the true longitude of the Moon measured in the plane of the ecliptic and β is the latitude above the plane of the ecliptic, Brown's tables express a'/r', β and $\delta\lambda \equiv \lambda_☾ - l_☾$ by sums of periodic terms whose arguments are algebraic sums of multiples of l, l', F, D and Γ. Then

$$\begin{aligned}
\lambda' &= \cos \lambda_☾ \cos \beta, \\
\mu' &= \sin \lambda_☾ \cos \beta \cos \varepsilon - \sin \beta \sin \varepsilon, \\
\nu' &= \sin \beta \cos \varepsilon + \sin \lambda_☾ \cos \beta \sin \varepsilon,
\end{aligned}$$

where

$$\begin{aligned}
\cos \lambda_☾ &= \cos (l' + D + \Gamma + \delta\lambda) \\
&= (l' + D + \Gamma) \cos \delta\lambda - \sin (l' + D + \Gamma) \sin \delta\lambda, \\
\sin \lambda_☾ &= \sin (l' + D + \Gamma) \cos \delta\lambda + \cos (l' + D + \Gamma) \sin \delta\lambda, \\
\sin \delta\lambda &= \delta\lambda - \tfrac{1}{6}(\delta\lambda)^3 + \cdots, \\
\cos \delta\lambda &= 1 - \tfrac{1}{2}(\delta\lambda)^2 + \tfrac{1}{24}(\delta\lambda)^4 - \cdots, \\
\sin \beta &= \beta - \tfrac{1}{6}\beta^3 + \cdots, \\
\cos \beta &= 1 - \tfrac{1}{2}\beta^2 + \cdots.
\end{aligned}$$

The coordinates of the sun are obtained in similar fashion from Newcomb's theory which expresses the ecliptic longitude and latitude and the common logarithm of r''/a'' by sums of periodic terms with the same ecliptic elements as the lunar theory. From the relation $r^n = e^{n \ln r}$ we have

$$\begin{aligned}
\left(\frac{a''}{r''}\right)^n &= \exp\left\{-\frac{n}{M_0} \log \left(\frac{r''}{a''}\right)\right\} \\
&= 1 - \frac{n}{M_0}\left(\log \frac{r''}{a''}\right) + \frac{n^2}{2M_0^2}\left(\log \frac{r''}{a''}\right)^2 - \cdots,
\end{aligned}$$

where M_0 is the logarithmic modulus,

$$\frac{1}{M_0} = 2.302\,585\,1\,.$$

The series manipulations involved in the above calculations were performed by electronic computer with programs developed by Musen and Estes (1971) for expanding the Earth's tidal potential.

3. The Main Problem

The secular and long period perturbations in the orbital elements are given by the variation equations

$$\frac{d\delta a}{dt} = \frac{2}{na}\frac{\partial[R]}{\partial g},$$

$$\frac{d\delta e}{dt} = \frac{1-e^2}{na^2e}\frac{\partial[R]}{\partial g} - \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial[R]}{\partial\omega},$$

$$\frac{d\delta i}{dt} = \frac{\cos i}{na^2\sqrt{(1-e^2)}\sin i}\frac{\partial[R]}{\partial\omega} - \frac{1}{na^2\sqrt{(1-e^2)}\sin i}\frac{\partial[R]}{\partial\Omega},$$

$$\frac{d\delta\Omega}{dt} = \frac{1}{na^2\sqrt{(1-e^2)}\sin i}\frac{\partial[R]}{\partial i} + \frac{d\dot{\Omega}}{de}\delta e + \frac{d\dot{\Omega}}{di}\delta i,$$

$$\frac{d\delta\omega}{dt} = \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial[R]}{\partial e} - \frac{\cos i}{na^2\sqrt{(1-e^2)}\sin i}\frac{\partial[R]}{\partial i} + \frac{d\dot{\omega}}{de}\delta e + \frac{d\dot{\omega}}{di}\delta i,$$

$$\frac{d\delta g}{dt} = -\frac{1-e^2}{na^2e}\frac{\partial[R]}{\partial e} - \frac{2}{na}\frac{\partial[R]}{\partial a} + \frac{d\dot{g}}{de}\delta e + \frac{d\dot{g}}{di}\delta i, \tag{5}$$

where (Brouwer, 1959)

$$\dot{g} = n\left\{1 - \frac{3J_2R_e^2}{4a^2(1-e^2)^{3/2}}(1-3\cos^2 i) + \frac{3J_2^2R_e^2}{128a^4(1-e^2)^{7/2}} \times \right.$$

$$\times\,[10 - 25e^2 + 16\sqrt{1-e^2} - 6(10 - 15e^2 + 16\sqrt{1-e^2})\cos^2 i +$$

$$+\,(130 - 25e^2 + 144\sqrt{1-e^2})\cos^4 i] -$$

$$\left. - \frac{45J_4R_e^4e^2}{128a^4(1-e^2)^{7/2}}(3 - 30\cos^2 i + 35\cos^4 i)\right\},$$

$$\dot{\omega} = n\left\{-\frac{3J_2R_e^2}{4a^2(1-e^2)^2}(1-5\cos^2 i) + \frac{3J_2^2R_e^4}{128a^4(1-e^2)^4}[-10-25e^2 + \right.$$
$$+ 24\sqrt{1-e^2} - 6(6-21e^2+32\sqrt{1-e^2})\cos^2 i +$$
$$+ 5(86-9e^2+72\sqrt{1-e^2})\cos^4 i] -$$
$$- \frac{15J_4R_e^4}{128a^4(1-e^2)^4}[3(4+3e^2) - 18(8+7e^2)\cos^2 i +$$
$$\left. + 7(28+27e^2)\cos^4 i]\right\},$$

$$\dot{\Omega} = n\left\{-\frac{3J_2R_e^2\cos i}{2a^2(1-e^2)^2} + \frac{3J_2^2R_e^4\cos i}{32a^4(1-e^2)^4}[4-9e^2+12\sqrt{1-e^2} - \right.$$
$$- (40-5e^2+36\sqrt{1-e^2})\cos^2 i] -$$
$$\left. - \frac{15J_4R_e^4(2+3e^2)\cos i}{32a^4(1-e^2)^4}(3-7\cos^2 i)\right\}.$$

The main problem denotes the secular and long period effects resulting from setting $[R]$ to be $[R_2'] + [R_2'']$. Then

$$\frac{\mathrm{d}\delta a}{\mathrm{d}t} = 0$$

$$\left(\frac{\mathrm{d}\delta e}{\mathrm{d}t}\right)' = -ne\sqrt{1-e^2}\frac{M'}{M}\frac{a^3}{a'^3}\{-\tfrac{15}{8}(1+\cos^2 i)[\sin 2\omega C_{210}'] +$$
$$+ \tfrac{15}{4}\cos i[\cos 2\omega S_{210}'] +$$
$$+ \tfrac{15}{2}\sin i[\cos 2\omega S_{220}'] +$$
$$+ \tfrac{15}{2}\sin i\cos i[\sin 2\omega C_{220}'] -$$
$$- \tfrac{15}{8}\sin^2 i[\sin 2\omega C_{20}']\},$$

$$\left(\frac{\mathrm{d}\delta i}{\mathrm{d}t}\right)' = \frac{n\frac{M'}{M}}{\sqrt{(1-e^2)}}\frac{a^3}{a'^3}\{\tfrac{15}{8}e^2\sin i\cos i[\sin 2\omega C_{210}'] -$$
$$- \tfrac{15}{8}e^2\sin i[\cos 2\omega S_{210}'] -$$
$$- \tfrac{15}{4}e^2\sin^2 i[\sin 2\omega C_{220}'] +$$
$$+ \tfrac{15}{4}e^2\cos i[\cos 2\omega S_{220}'] -$$
$$- \tfrac{3}{4}\sin i(1+\tfrac{3}{2}e^2)S_{210}' +$$
$$+ \tfrac{3}{2}\cos i(1+\tfrac{3}{2}e^2)S_{220}' -$$
$$- \tfrac{15}{8}e^2\sin i\cos i[\sin 2\omega C_{20}']\},$$

$$\left(\frac{\mathrm{d}\delta\Omega}{\mathrm{d}t}\right)' = \frac{n\dfrac{M'}{M}}{\sqrt{(1-e^2)}}\frac{a^3}{a'^3}\{-\tfrac{3}{4}(1+\tfrac{3}{2}e^2)\cos iC'_{20} +$$
$$+ \tfrac{3}{4}\cos i(1+\tfrac{3}{2}e^2)C'_{210} +$$
$$+ \frac{3}{2}\frac{\cos 2i}{\sin i}(1+\tfrac{3}{2}e^2)C'_{220} +$$
$$+ \tfrac{30}{16}e^2\cos i[\cos 2\omega C'_{20}] - \tfrac{30}{16}e^2\cos i[\cos 2\omega C'_{210}] -$$
$$- \tfrac{15}{8}e^2[\sin 2\omega S'_{210}] + \tfrac{15}{4}e^2\operatorname{ctn} i[\sin 2\omega S'_{220}] -$$
$$- \tfrac{15}{4}e^2\frac{\cos 2i}{\sin i}[\cos 2\omega C'_{220}]\} +$$
$$+ \frac{\mathrm{d}\dot{\Omega}}{\mathrm{d}e}(\delta e)' + \frac{\mathrm{d}\dot{\Omega}}{\mathrm{d}i}(\delta i)',$$

$$\left(\frac{\mathrm{d}\delta\omega}{\mathrm{d}t}\right)' = \frac{n\dfrac{M'}{M}}{\sqrt{(1-e^2)}}\frac{a^3}{a'^3}\{\tfrac{3}{8}(4-5\sin^2 i+e^2)C'_{20} +$$
$$+ \tfrac{3}{8}(5\sin^2 i-2-3e^2)C'_{210} +$$
$$+ \tfrac{3}{2}\operatorname{ctn} i(5\sin^2 i-1-\tfrac{3}{2}e^2)C'_{220} +$$
$$+ \tfrac{30}{16}(\sin^2 i-e^2)[\cos 2\omega C'_{20}] +$$
$$+ \tfrac{30}{16}(1+\cos^2 i-e^2)[\cos 2\omega C'_{210}] +$$
$$+ \tfrac{15}{8}(2-e^2)\cos i[\sin 2\omega S'_{210}] +$$
$$+ \tfrac{15}{4}\operatorname{ctn} i(e^2-2\sin^2 i)[\cos 2\omega C'_{220}] +$$
$$+ \frac{15}{4}\left(2\sin i - e^2\frac{(1+\sin^2 i)}{\sin i}\right)[\sin 2\omega S'_{220}]\} +$$
$$+ \frac{\mathrm{d}\dot{\omega}}{\mathrm{d}e}(\delta e)' + \frac{\mathrm{d}\dot{\omega}}{\mathrm{d}i}(\delta i)',$$

$$\left(\frac{\mathrm{d}\delta g}{\mathrm{d}t}\right)' = -\frac{n\left(\dfrac{M'}{M}\right)a^3}{a'^3}\{\tfrac{1}{4}(7+3e^2)(1-\tfrac{3}{2}\sin^2 i)C'_{20} +$$
$$+ \tfrac{3}{8}(7+3e^2)\sin^2 iC'_{210} +$$
$$+ \tfrac{3}{2}(7+3e^2)\sin i\cos iC'_{220} +$$
$$+ \tfrac{30}{16}(1+e^2)\sin^2 i[\cos 2\omega C'_{20}] +$$
$$+ \tfrac{30}{16}(1+e^2)(1+\cos^2 i)[\cos 2\omega C'_{210}] +$$
$$+ \tfrac{30}{8}(1+e^2)\cos i[\sin 2\omega S'_{210}] -$$
$$- \tfrac{15}{2}(1+e^2)\sin i\cos i[\cos 2\omega C'_{220}] +$$
$$+ \tfrac{30}{4}(1+e^2)\sin i[\sin 2\omega S'_{220}]\} + \frac{\mathrm{d}\dot{g}}{\mathrm{d}e}(\delta e)' + \frac{\mathrm{d}\dot{g}}{\mathrm{d}i}(\delta i)'. \tag{6}$$

Similar expressions are obtained for the solar perturbations where all primes are replaced by double primes.

Tables I–III display the principal terms of the lunar longitude, latitude and (a'/r') as given by Brown's theory. For the purpose of this paper, the lunar and solar terms in longitude, latitude and parallax whose coefficients are less than 5×10^{-6} rad in magnitude are omitted. In addition all planetary terms and cosine terms in $\delta\lambda$ and β are omitted. The resulting accuracy of the functions depending on the position of the Moon and Sun is 1×10^{-4} with 231 terms in C'_{200}, 235 terms in C'_{210}, 255 terms in C'_{220}, 11 terms in C''_{200}, 17 terms in C''_{210} and 16 terms in C''_{220}. Tables V–VIII list the terms of these functions whose coefficients exceed 10^{-5} rad.

TABLE I

Longitude of Moon $\delta\lambda$

Coef. × 10^5 of sine	Multiples of					Coef. × 10^5 of sine	Multiples of				
	l	l'	F	D	Γ		l	l'	F	D	Γ
−61	0	0	0	1	0	1149	0	0	0	2	0
7	0	0	0	4	0	−27	0	0	2	−2	0
−200	0	0	2	0	0	−3	0	0	2	2	0
−80	0	1	0	−2	0	−324	0	1	0	0	0
9	0	1	0	1	0	−12	0	1	0	2	0
−4	0	2	0	−2	0	−4	0	2	0	0	0
1	1	−2	0	−2	0	1	1	−2	0	0	0
14	1	−1	0	−2	0	72	1	−1	0	0	0
7	1	−1	0	2	0	4	1	0	−2	−2	0
19	1	0	−2	0	0	−3	1	0	−2	2	0
−19	1	0	0	−4	0	2	1	0	0	−3	0
−2224	1	0	0	−2	0	9	1	0	0	−1	0
10976	1	0	0	0	0	−4	1	0	0	1	0
93	1	0	0	2	0	−22	1	0	2	0	0
−2	1	1	0	−4	0	−100	1	1	0	−2	0
−53	1	1	0	0	0	−1	1	1	0	2	0
−4	1	2	0	−2	0	−1	2	−1	0	−2	0
5	2	−1	0	0	0	−15	2	0	0	−4	0
−103	2	0	0	−2	0	373	2	0	0	0	0
7	2	0	0	2	0	−2	2	0	2	0	0
−1	2	1	0	−4	0	−4	2	1	0	−2	0
−4	2	1	0	0	0	−6	3	0	0	−2	0
17	3	0	0	0	0						

Figures 1 and 2 compare perturbations of orbital elements obtained using the method of this paper with those obtained from the Geodyn program (numerical integration of the disturbing function, Equation (1)) and an analytic theory of long period and secular effects (Murphy and Felsentreger, 1966) using the equatorial elements of the Moon.

TABLE II

Latitude of Moon

Coef. × 10^5 of sine	Multiples of					Coef. × 10^5 of sine	Multiples of				
	l	l'	F	D	Γ		l	l'	F	D	Γ
−2	0	0	1	−4	0	−302	0	0	1	−2	0
2	0	0	1	−1	0	8950	0	0	1	0	0
−3	0	0	1	1	0	57	0	0	1	2	0
−1	0	0	3	−2	0	−3	0	0	3	0	0
−4	0	1	−1	−2	0	−2	0	1	−1	0	0
−6	0	1	−1	2	0	−14	0	1	1	−2	0
−3	0	1	1	0	0	3	1	−1	−1	0	0
3	1	−1	1	0	0	1	1	0	−3	0	0
−1	1	0	−1	−4	0	−97	1	0	−1	−2	0
485	1	0	−1	0	0	16	1	0	−1	2	0
−3	1	0	1	−4	0	−81	1	0	1	−2	0
490	1	0	1	0	0	7	1	0	1	2	0
−4	1	1	−1	−2	0	−2	1	1	−1	0	0
−4	1	1	1	−2	0	−3	1	1	1	0	0
−1	2	0	−1	−4	0	15	2	0	−1	0	0
−7	2	0	1	−2	0	30	2	0	1	0	0
2	3	0	1	0	0						

TABLE III

$\frac{a'}{r'}$

Coef. × 10^5 of cosine	Multiples of					Coef. × 10^5 of cosine	Multiples of				
	l	l'	F	D	Γ		l	l'	F	D	Γ
100 000	0	0	0	0	0	−29	0	0	0	1	0
825	0	0	0	2	0	8	0	0	0	4	0
−3	0	0	2	−2	0	56	0	1	0	−2	0
−12	0	1	0	0	0	4	0	1	0	1	0
−9	0	1	0	2	0	3	0	2	0	−2	0
−7	1	−1	0	−2	0	34	1	−1	0	0	0
7	1	−1	0	2	0	−21	1	0	−2	0	0
−1	1	0	−2	2	0	18	1	0	0	−4	0
−1	1	0	0	−3	0	1002	1	0	0	−2	0
5450	1	0	0	0	0	−3	1	0	0	1	0
90	1	0	0	2	0	1	1	0	0	4	0
−2	1	0	2	−2	0	2	1	1	0	−4	0
42	1	1	0	−2	0	−28	1	1	0	0	0
−1	1	1	0	2	0	1	1	2	0	−2	0
4	2	−1	0	0	0	11	2	0	0	−4	0
−9	2	0	0	−2	0	297	2	0	0	0	0
8	2	0	0	2	0	−3	2	1	0	0	0
−3	3	0	0	−2	0	18	3	0	0	0	0
1	4	0	0	0	0						

TABLE IV

Development of sin λ'' and cos λ''

Coef. × 10^5 of cosine in cos λ'' of sine in sin λ''	Multiples of				
	l	l'	F	D	Γ
99 972	0	1	0	0	1
1674 − 4.2T	0	2	0	0	1
32	0	3	0	0	1
1	0	4	0	0	1
2	0	1	0	1	1
−1675 + 4.2T	0	0	0	0	1
−4	0	−1	0	0	1
−2	0	1	0	−1	1
4	0	0	1	−1	0
−4	0	2	−1	1	2

Development of $(a''/r'')^3$

Coef. × 10^5 of cosine	Multiples of				
	l	l'	F	D	Γ
100 042 − 0.2T	0	0	0	0	0
−1	0	0	0	1	0
5027 − 12.5T	0	1	0	0	0
126 − 0.63T	0	2	0	0	0
3	0	3	0	0	0

TABLE V

Development of C'_{200}

Coef. × 10^5 of cosine	Multiples of					
	l	l'	F	D	Γ	Ω
75 695	0	0	0	0	0	0
−60	0	0	2	−2	0	0
909	0	0	2	0	0	0
2369	1	0	0	−2	0	0
326	1	0	0	2	0	0
147	0	1	−1	3	1	0
−9826	0	1	−1	1	1	0
6	1	−1	1	−5	−1	0
−86	1	1	1	−1	1	0
−170	1	−1	1	−3	−1	0
5	2	1	−1	3	1	0
−5	0	2	−1	3	1	0
3	0	0	1	−3	−1	0
1	0	1	1	0	1	0

Coef. × 10^5 of cosine	Multiples of					
	l	l'	F	D	Γ	Ω
6	0	1	−1	5	1	0
2055	0	0	0	2	0	0
−73	1	0	−2	0	0	0
−16	1	0	2	−2	0	0
−272	0	1	1	−1	1	0
9720	0	1	1	1	1	0
−813	1	1	−1	1	1	0
−269	1	−1	−1	−1	−1	0
33	1	1	−1	3	1	0
−27	2	1	1	−1	1	0
−12	0	2	1	−1	1	0
5	0	0	1	−1	1	0
−54	2	1	−1	1	1	0
28	0	0	2	2	0	0

Table V (Continued)

Coef. × 10^5 of cosine	Multiples of						Coef. × 10^5 of cosine	Multiples of					
	l	l'	F	D	Γ	Ω		l	l'	F	D	Γ	Ω
−3	0	1	2	−2	0	0	32	1	0	−2	−2	0	0
175	1	0	2	0	0	0	1016	2	0	0	0	0	0
−3	2	0	2	−2	0	0	23	2	0	2	0	0	0
297	0	1	1	3	1	0	14	1	−1	−1	−5	−1	0
−4	0	1	3	−1	1	0	−19	0	1	−3	1	1	0
19	0	0	1	3	1	0	−7	1	−1	−3	−1	−1	0
354	1	−1	−1	−3	−1	0	1861	1	1	1	1	1	0
72	1	1	1	3	1	0	247	2	1	1	1	1	0
−13	2	−1	−1	−1	−1	0	11	2	1	1	3	1	0
−21	0	2	1	1	1	0	15	0	2	−1	1	1	0
−5	2	−1	−1	−3	−1	0	17	0	0	1	1	1	0
−11	0	0	1	−1	−1	0	−10	0	1	1	2	1	0
−4	1	2	1	−1	1	0	16	1	0	−1	−3	−1	0
28	3	1	1	1	1	0	9	2	−1	−1	−5	−1	0
15	1	0	1	1	1	0	−13	1	2	1	1	1	0
−5	3	1	1	−1	1	0	−1	1	−2	−1	−1	−1	0
1	1	0	−1	−1	−1	0	23 457	0	2	0	2	2	0
717	0	2	0	4	2	0	19	0	2	0	0	2	0
18	0	2	0	6	2	0	34	1	−2	0	−6	−2	0
−10	0	2	2	0	2	0	4	0	2	−2	4	2	0
96	0	2	−2	2	2	0	48	0	1	0	4	2	0
−10	0	3	0	4	2	0	10	1	2	−2	2	2	0
−17	1	−2	−2	−2	−2	0	−173	1	2	0	0	2	0
853	1	−2	0	−4	−2	0	4491	1	2	0	2	2	0
−663	1	−2	0	−2	−2	0	174	1	2	0	4	2	0
2	1	−2	0	0	−2	0	594	2	2	0	2	2	0
29	2	2	0	4	2	0	−81	0	3	0	2	2	0
−13	2	−2	0	−4	−2	0	72	0	1	0	2	2	0
−71	2	2	0	0	2	0	−24	0	2	0	3	2	0
4	0	2	0	1	2	0	−9	1	3	0	0	2	0
−8	1	−3	0	−4	−2	0	40	1	−1	0	−4	−2	0
13	1	1	0	4	2	0	3	1	2	0	1	2	0
67	3	2	0	2	2	0	22	2	−2	0	−6	−2	0
42	1	1	0	2	2	0	−38	1	3	0	2	2	0
−14	3	2	0	0	2	0	−3	1	−3	0	−2	−2	0
−6	1	2	0	3	2	0	9	2	1	0	2	2	0
−3	2	3	0	0	2	0	−8	2	3	0	2	2	0
−66	0	0	0	1	0	0	4	0	1	−1	2	1	0
4	0	1	−1	0	1	0	−2	1	1	1	2	1	0
31	0	0	0	4	0	0	−9	1	0	−2	2	0	0
12 386	1	0	0	0	0	0	62	1	0	0	−4	0	0
6	1	0	0	4	0	0	−4	0	1	1	−3	1	0
−130	0	1	−1	−1	1	0	−154	1	1	−1	−1	1	0
−25	1	−1	−1	1	−1	0	−5	1	1	1	−3	1	0
−804	1	−1	1	−1	−1	0	−1	2	1	1	−3	1	0
−7	2	1	−1	−1	1	0	5	1	0	2	2	0	0
5	0	1	1	5	1	0	1	1	1	1	5	1	0
−2	2	−1	−1	1	−1	0	−2	0	2	1	3	1	0

Table V (Continued)

Coef. × 10^5 of cosine	Multiples of						Coef. × 10^5 of cosine	Multiples of					
	l	l'	F	D	Γ	Ω		l	l'	F	D	Γ	Ω
−8	0	2	−1	−1	1	0	1	0	0	1	1	−1	0
1	3	1	1	3	1	0	3	1	0	1	3	1	0
−1	1	−2	−1	1	−1	0	−1	1	−2	−1	−3	−1	0
−2	0	2	2	2	2	0	1	0	2	−2	0	2	0
1	0	1	0	6	2	0	2	1	2	−2	0	2	0
−1	1	−2	−2	0	−2	0	−1	1	2	0	−2	2	0
4	1	2	0	6	2	0	−2	0	1	0	0	2	0
2	1	−1	0	−6	−2	0	3	3	2	0	4	2	0
−1	1	1	0	0	2	0	−2	1	3	0	4	2	0
−2	0	1	−1	−3	1	0	−2	1	2	2	0	2	0
1	1	−2	2	−4	−2	0	8	1	−2	2	−2	−2	0
4	0	1	0	−4	0	0	133	0	1	0	−2	0	0
2	2	1	0	4	2	0	1	0	0	0	4	2	0
−24	0	1	0	0	0	0	10	0	1	0	1	0	0
2	0	3	0	3	2	0	2	0	1	0	1	2	0
−22	0	1	0	2	0	0	6	0	2	0	−2	0	0
−1	1	−2	0	−2	0	0	1	1	−2	0	0	0	0
1	1	−2	0	2	0	0	−16	1	−1	0	−2	0	0
76	1	−1	0	0	0	0	−6	1	0	−1	1	1	0
−4	1	−2	1	−1	−1	0	2	2	0	1	1	1	0
23	1	−1	0	2	0	0	−2	1	−2	1	1	−1	0
1	1	1	−3	1	1	0	−4	1	1	−1	−3	1	0
−3	1	0	0	−3	0	0	39	2	0	0	−4	0	0
−2	2	−1	1	−5	−1	0	5	1	2	−1	1	1	0
−6	1	2	−1	−1	1	0	−6	1	0	1	−3	−1	0
1	2	−1	0	−6	−2	0	−1	4	2	0	0	2	0
−4	1	0	0	−1	0	0	−5	2	0	−2	0	0	0
110	2	0	0	−2	0	0	38	2	0	0	2	0	0
−9	2	−1	1	−3	−1	0	−4	3	1	−1	1	1	0
81	3	0	0	0	0	0	2	3	0	2	0	0	0
−2	1	−1	3	−1	−1	0	−1	3	−1	−1	−1	−1	0
3	1	0	1	−1	−1	0	2	4	1	1	1	1	0
−2	2	2	1	1	1	0	5	4	2	0	2	2	0
−11	1	0	0	1	0	0	−1	2	0	−2	2	0	0
−20	1	−1	1	1	−1	0	−66	2	−1	1	−1	−1	0
7	1	1	0	−4	0	0	102	1	1	0	−2	0	0
−64	1	1	0	0	0	0	2	1	1	0	1	0	0
−5	1	1	0	2	0	0	3	1	2	0	−2	0	0
13	2	−1	0	0	0	0	3	2	−1	0	2	0	0
−3	2	1	−1	−3	1	0	−2	2	−1	1	1	−1	0
1	2	0	0	4	0	0	3	2	1	0	−4	0	0
4	2	1	0	−2	0	0	−10	2	1	0	0	0	0
1	3	−1	0	0	0	0	2	3	0	0	−4	0	0
−5	3	−1	1	−1	−1	0	4	3	0	0	2	0	0
−1	3	1	0	0	0	0	6	4	0	0	0	0	0

TABLE VI

Development of C'_{210}

Coef. × 10^5 of cosine	Multiples of						Coef. × 10^5 of cosine	Multiples of					
	l	l'	F	D	Γ	Ω		l	l'	F	D	Γ	Ω
168	0	2	0	2	2	2	90 813	0	2	0	2	2	−2
−94	0	2	0	3	2	−2	16	0	2	0	1	2	−2
5	0	2	0	4	2	2	2777	0	2	0	4	2	−2
73	0	2	0	0	2	−2	−3	0	2	0	5	2	−2
69	0	2	0	6	2	−2	−39	0	2	2	0	2	−2
17	0	2	−2	4	2	−2	−14	0	2	2	2	2	−2
371	0	2	−2	2	2	−2	9	0	1	0	6	2	−2
3	0	3	0	0	2	−2	188	0	1	0	4	2	−2
−313	0	3	0	2	2	−2	275	0	1	0	2	2	−2
14	0	3	0	3	2	−2	−2	0	1	0	1	2	−2
−39	0	3	0	4	2	−2	−2	0	3	2	0	2	−2
8	0	0	0	4	2	−2	−3	0	4	0	2	2	−2
3	0	0	0	2	2	−2	−2	1	−4	0	−4	−2	2
3	1	0	0	2	2	−2	2	1	0	0	4	2	−2
−32	1	−3	0	−4	−2	2	163	1	1	0	2	2	−2
−10	1	−3	0	−2	−2	2	51	1	1	0	4	2	−2
2	1	1	0	6	2	−2	6	1	2	−2	0	2	−2
−3	1	−2	−2	−4	−2	2	40	1	2	−2	2	2	−2
−67	1	−2	−2	−2	−2	2	2	1	−2	−2	0	−2	2
130	1	−2	0	−6	−2	2	−6	1	−2	0	−5	−2	2
−1	1	2	0	0	2	2	−669	1	2	0	0	2	−2
3303	1	−2	0	−4	−2	2	6	1	−2	0	−4	−2	−2
14	1	2	0	1	2	−2	−3	1	−2	0	−3	−2	2
32	1	2	0	2	2	2	17 388	1	2	0	2	2	−2
−2567	1	−2	0	−2	−2	2	−5	1	−2	0	−2	−2	−2
−25	1	2	0	3	2	−2	1	1	2	0	4	2	2
671	1	2	0	4	2	−2	6	1	−2	0	0	−2	2
18	1	2	0	6	2	−2	−15	1	2	2	0	2	−2
7	1	−2	2	−4	−2	2	15	1	−1	0	−6	−2	2
−32	1	3	0	0	2	−2	153	1	−1	0	−4	−2	2
−147	1	3	0	2	2	−2	3	1	−1	0	−2	−2	2
4	1	3	0	3	2	−2	−13	1	3	0	4	2	−2
5	1	0	0	−4	−2	2	36	2	1	0	2	2	−2
9	2	1	0	4	2	−2	−3	2	−2	−2	−4	−2	2
85	2	−2	0	−6	−2	2	−273	2	2	0	0	2	−2
−52	2	−2	0	−4	−2	2	4	2	2	0	1	2	−2
4	2	2	0	2	2	2	2301	2	2	0	2	2	−2
111	2	2	0	4	2	−2	3	2	2	0	6	2	−2
−3	2	2	2	0	2	−2	8	2	−1	0	−6	−2	2
−11	2	3	0	0	2	−2	−2	2	−1	0	−4	−2	2
−31	2	3	0	2	2	−2	−2	2	3	0	4	2	−2
6	3	1	0	2	2	−2	−56	3	2	0	0	2	−2
260	3	2	0	2	2	−2	15	3	2	0	4	2	−2
−5	3	3	0	2	2	−2	−9	4	2	0	0	2	−2
27	4	2	0	2	2	−2	−4	2	2	0	3	2	−2
5	0	2	−2	0	2	−2	−2	1	−3	0	−6	−2	2

Table VI (Continued)

Coef. × 10^5 of cosine	Multiples of						Coef. × 10^5 of cosine	Multiples of					
	l	l'	F	D	Γ	Ω		l	l'	F	D	Γ	Ω
2	1	−2	0	−8	−2	2	−2	1	2	2	2	2	−2
30	1	−2	2	−2	−2	2	4	2	2	−2	2	2	−2
3	2	−2	−2	−2	−2	2	3	2	−2	0	−8	−2	2
2	5	2	0	2	2	−2	7855	0	0	0	0	0	2
−3	0	0	0	1	0	2	−3	0	0	0	1	0	−2
106	0	0	0	2	0	2	106	0	0	0	2	0	−2
2	0	0	0	4	0	2	2	0	0	0	4	0	−2
−3	0	0	2	−2	0	2	−3	0	0	2	−2	0	−2
47	0	0	2	0	0	2	47	0	0	2	0	0	−2
7	0	1	0	−2	0	2	7	0	1	0	−2	0	−2
−1	0	1	0	0	0	2	−1	0	1	0	0	0	−2
−1	0	1	0	2	0	2	−1	0	1	0	2	0	−2
4	1	−1	0	0	0	2	4	1	−1	0	0	0	−2
1	1	−1	0	2	0	2	1	1	−1	0	2	0	−2
2	1	0	−2	−2	0	2	2	1	0	−2	−2	0	−2
−4	1	0	−2	0	0	2	−4	1	0	−2	0	0	−2
3	1	0	0	−4	0	2	3	1	0	0	−4	0	−2
123	1	0	0	−2	0	2	123	1	0	0	−2	0	−2
643	1	0	0	0	0	2	643	1	0	0	0	0	−2
17	1	0	0	2	0	2	17	1	0	0	2	0	−2
5	1	1	0	−2	0	2	5	1	1	0	−2	0	−2
−3	1	1	0	0	0	2	−3	1	1	0	0	0	−2
2	2	0	0	−4	0	2	2	2	0	0	−4	0	−2
6	2	0	0	−2	0	2	6	2	0	0	−2	0	−2
53	2	0	0	0	0	2	53	2	0	0	0	0	−2
2	2	0	0	2	0	2	2	2	0	0	2	0	−2
4	3	0	0	0	0	2	4	3	0	0	0	0	−2
−2	2	1	0	0	2	−2	1	0	0	2	2	0	2
1	0	0	2	2	0	−2	9	1	0	2	0	0	2
9	1	0	2	0	0	−2	1	2	0	2	0	0	2
1	2	0	2	0	0	−2	2	2	−2	2	−2	−2	2
−2	3	3	0	0	2	−2	−2	0	1	−1	5	1	−2
−2	1	−1	1	−5	−1	2	−4	0	1	1	−1	1	2
95	0	1	1	−1	1	−2	146	0	1	1	1	1	2
−3386	0	1	1	1	1	−2	−6	0	0	1	1	1	−2
94	1	−1	−1	−1	−1	2	−4	1	−1	−1	−1	−1	−2
−1	1	1	1	−1	1	2	30	1	1	1	−1	1	−2
8	1	−1	−1	1	−1	2	28	1	1	1	1	1	2
−648	1	1	1	1	1	−2	5	2	−1	−1	−1	−1	2
9	2	1	1	−1	1	−2	2	3	1	1	−1	1	−2
2	0	1	−1	3	1	2	−51	0	1	−1	3	1	−2
−147	0	1	−1	1	1	2	3423	0	1	−1	1	1	−2
−5	0	2	−1	1	1	−2	−12	1	1	−1	1	1	2
283	1	1	−1	1	1	−2	−12	1	1	−1	3	1	−2
59	1	−1	1	−3	−1	2	−3	1	−1	1	−3	−1	−2
280	1	−1	1	−1	−1	2	−12	1	−1	1	−1	−1	−2
−2	1	2	−1	1	1	−2	19	2	1	−1	1	1	−2
−2	2	1	−1	3	1	−2	3	2	−1	1	−3	−1	2

Table VI (*Continued*)

Coef. × 10^5 of cosine	Multiples of						Coef. × 10^5 of cosine	Multiples of					
	l	l'	F	D	Γ	Ω		l	l'	F	D	Γ	Ω
3	0	1	1	2	1	−2	4	0	1	1	3	1	2
−104	0	1	1	3	1	−2	−2	0	1	1	5	1	−2
4	0	2	1	−1	1	−2	−7	0	0	1	3	1	−2
7	0	2	1	1	1	−2	−2	0	0	1	−1	1	−2
−5	1	0	1	1	1	−2	2	1	−1	−3	−1	−1	2
−5	1	−1	−1	−5	−1	2	−123	1	−1	−1	−3	−1	2
5	1	−1	−1	−3	−1	−2	1	1	1	1	3	1	2
−25	1	1	1	3	1	−2	−6	1	0	−1	−3	−1	2
5	1	2	1	1	1	−2	−3	2	−1	−1	−5	−1	2
2	2	−1	−1	−3	−1	2	4	2	1	1	1	1	2
−86	2	1	1	1	1	−2	−4	2	1	1	3	1	−2
−10	3	1	1	1	1	−2	−2	0	1	−1	−1	1	2
46	0	1	−1	−1	1	−2	7	0	1	−3	1	1	−2
3	0	2	−1	−1	1	−2	4	0	0	1	−1	−1	2
2	1	0	−1	1	1	−2	−2	1	1	−1	−1	1	2
54	1	1	−1	−1	1	−2	7	1	−1	1	1	−1	2
2	1	2	−1	−1	1	−2	3	1	0	1	−3	−1	2
2	2	1	−1	−1	1	−2	23	2	−1	1	−1	−1	2
2	3	−1	1	−1	−1	2							

TABLE VII

Development of C'_{220}

Coef. × 10^5 of cosine	Multiples of						Coef. × 10^5 of cosine	Multiples of					
	l	l'	F	D	Γ	Ω		l	l'	F	D	Γ	Ω
−2	0	1	−1	5	1	−1	−2	1	−1	1	−5	−1	1
−15	0	1	1	−1	1	1	99	0	1	1	−1	1	−1
519	0	1	1	1	1	1	−3553	0	1	1	1	1	−1
−6	0	0	1	1	1	−1	99	1	−1	−1	−1	−1	1
−14	1	−1	−1	−1	−1	−1	−5	1	1	1	−1	1	1
31	1	1	1	−1	1	−1	8	1	−1	−1	1	−1	1
−1	1	−1	−1	1	−1	−1	99	1	1	1	1	1	1
−680	1	1	1	1	1	−1	5	2	−1	−1	−1	−1	1
−1	2	1	1	−1	1	1	10	2	1	1	−1	1	−1
2	3	1	1	−1	1	−1	8	0	1	−1	3	1	1
−54	0	1	−1	3	1	−1	−525	0	1	−1	1	1	1
3592	0	1	−1	1	1	−1	−6	0	2	−1	1	1	−1
−43	1	1	−1	1	1	1	297	1	1	−1	1	1	−1
2	1	1	−1	3	1	1	−12	1	1	−1	3	1	−1
62	1	−1	1	−3	−1	1	−9	1	−1	1	−3	−1	−1
294	1	−1	1	−1	−1	1	−43	1	−1	1	−1	−1	−1
−3	2	1	−1	1	1	1	20	2	1	−1	1	1	−1
4	2	−1	1	−3	−1	1	4	0	1	1	2	1	−1

Table VII (Continued)

Coef. × 10^5 of cosine	Multiples of						Coef. × 10^5 of cosine	Multiples of					
	l	l'	F	D	Γ	Ω		l	l'	F	D	Γ	Ω
16	0	1	1	3	1	1	−109	0	1	1	3	1	−1
−3	0	1	1	5	1	−1	4	0	2	1	−1	1	−1
1	0	0	1	3	1	1	−7	0	0	1	3	1	−1
−1	0	2	1	1	1	1	8	0	2	1	1	1	−1
−5	1	0	1	1	1	−1	−2	1	0	1	3	1	−1
3	1	−1	−3	−1	−1	1	−5	1	−1	−1	−5	−1	1
−129	1	−1	−1	−3	−1	1	19	1	−1	−1	−3	−1	−1
4	1	1	1	3	1	1	−26	1	1	1	3	1	−1
−6	1	0	−1	−3	−1	1	5	1	2	1	1	1	−1
−3	2	−1	−1	−5	−1	1	2	2	−1	−1	−3	−1	1
13	2	1	1	1	1	1	−90	2	1	1	1	1	−1
−4	2	1	1	3	1	−1	1	3	1	1	1	1	1
−10	3	1	1	1	1	−1	−7	0	1	−1	−1	1	1
48	0	1	−1	−1	1	−1	−1	0	1	−3	1	1	1
7	0	1	−3	1	1	−1	3	0	2	−1	−1	1	−1
4	0	0	1	−1	−1	1	2	1	0	−1	1	1	−1
−8	1	1	−1	−1	1	1	56	1	1	−1	−1	1	−1
8	1	−1	1	1	−1	1	−1	1	−1	1	1	−1	−1
2	1	2	−1	−1	1	−1	3	1	0	1	−3	−1	1
3	2	1	−1	−1	1	−1	24	2	−1	1	−1	−1	1
−3	2	−1	1	−1	−1	−1	2	3	−1	1	−1	−1	1
811	0	2	0	2	2	1	−18 842	0	2	0	2	2	−1
20	0	2	0	3	2	−1	−3	0	2	0	1	2	−1
25	0	2	0	4	2	1	−576	0	2	0	4	2	−1
−15	0	2	0	0	2	−1	−14	0	2	0	6	2	−1
8	0	2	2	0	2	−1	−4	0	2	−2	4	2	−1
3	0	2	2	2	2	−1	3	0	2	−2	2	2	1
−77	0	2	−2	2	2	−1	−2	0	1	0	6	2	−1
2	0	1	0	4	2	1	−39	0	1	0	4	2	−1
−3	0	3	0	2	2	1	65	0	3	0	2	2	−1
2	0	1	0	2	2	1	−57	0	1	0	2	2	−1
−3	0	3	0	3	2	−1	8	0	3	0	4	2	−1
−2	0	0	0	4	2	−1	7	1	−3	0	−4	−2	1
1	1	1	0	2	2	1	−34	1	1	0	2	2	−1
2	1	−3	0	−2	−2	1	−10	1	1	0	4	2	−1
−8	1	2	−2	2	2	−1	14	1	−2	−2	−2	−2	1
−27	1	−2	0	−6	−2	1	1	1	−2	0	−6	−2	−1
−6	1	2	0	0	2	1	139	1	2	0	0	2	−1
−685	1	−2	0	−4	−2	1	30	1	−2	0	−4	−2	−1
−3	1	2	0	1	2	−1	155	1	2	0	2	2	1
−3 608	1	2	0	2	2	−1	533	1	−2	0	−2	−2	1
−23	1	−2	0	−2	−2	−1	5	1	2	0	3	2	−1
6	1	2	0	4	2	1	−139	1	2	0	4	2	−1
−4	1	2	0	6	2	−1	3	1	2	2	0	2	−1
−3	1	−1	0	−6	−2	1	7	1	3	0	0	2	−1
−32	1	−1	0	−4	−2	1	1	1	−1	0	−4	−2	−1
−1	1	3	0	2	2	1	30	1	3	0	2	2	−1
2	1	3	0	4	2	−1	−7	2	1	0	2	2	−1

Table VII (Continued)

Coef. × 10^5 of cosine	Multiples of						Coef. × 10^5 of cosine	Multiples of					
	l	l'	F	D	Γ	Ω		l	l'	F	D	Γ	Ω
−18	2	−2	0	−6	−2	1	−2	2	2	0	0	2	1
57	2	2	0	0	2	−1	11	2	−2	0	−4	−2	1
21	2	2	0	2	2	1	−478	2	2	0	2	2	−1
−23	2	2	0	4	2	−1	2	2	3	0	0	2	−1
6	2	3	0	2	2	−1	2	3	2	0	2	2	1
−54	3	2	0	2	2	−1	−3	3	2	0	4	2	−1
2	4	2	0	0	2	−1	−6	4	2	0	2	2	−1
18 115	0	0	0	0	0	1	−8	0	0	0	1	0	1
−8	0	0	0	1	0	−1	246	0	0	0	2	0	1
246	0	0	0	2	0	−1	4	0	0	0	4	0	1
4	0	0	0	4	0	−1	−7	0	0	2	−2	0	1
−7	0	0	2	−2	0	−1	109	0	0	2	0	0	1
109	0	0	2	0	0	−1	16	0	1	0	−2	0	1
16	0	1	0	−2	0	−1	−3	0	1	0	0	0	1
−3	0	1	0	0	0	−1	1	0	1	0	1	0	1
1	0	1	0	1	0	−1	−3	0	1	0	2	0	1
−3	0	1	0	2	0	−1	−2	1	−1	0	−2	0	1
−2	1	−1	0	−2	0	−1	9	1	−1	0	0	0	1
9	1	−1	0	0	0	−1	3	1	−1	0	2	0	1
3	1	−1	0	2	0	−1	4	1	0	−2	−2	0	1
4	1	0	−2	−2	0	−1	−9	1	0	−2	0	0	1
−9	1	0	−2	0	0	−1	8	1	0	0	−4	0	1
8	1	0	0	−4	0	−1	283	1	0	0	−2	0	1
283	1	0	0	−2	0	−1	1482	1	0	0	0	0	1
1482	1	0	0	0	0	−1	−1	1	0	0	1	0	1
−1	1	0	0	1	0	−1	39	1	0	0	2	0	1
39	1	0	0	2	0	−1	−2	1	0	2	−2	0	1
−2	1	0	2	−2	0	−1	12	1	1	0	−2	0	1
12	1	1	0	−2	0	−1	−8	1	1	0	0	0	1
−8	1	1	0	0	0	−1	2	2	−1	0	0	0	1
2	2	−1	0	0	0	−1	5	2	0	0	−4	0	1
5	2	0	0	−4	0	−1	13	2	0	0	−2	0	1
13	2	0	0	−2	0	−1	121	2	0	0	0	0	1
121	2	0	0	0	0	−1	5	2	0	0	2	0	1
5	2	0	0	2	0	−1	−1	2	1	0	0	0	1
−1	2	1	0	0	0	−1	10	3	0	0	0	0	1
10	3	0	0	0	0	−1	12	3	2	0	0	2	−1
3	0	0	2	2	0	1	3	0	0	2	2	0	−1
21	1	0	2	0	0	1	21	1	0	2	0	0	−1
3	2	0	2	0	0	1	3	2	0	2	0	0	−1
−6	1	−2	2	−2	−2	1	1	1	2	−1	1	1	1
−1	1	2	−1	1	1	−1	1	2	1	−1	3	1	1
−1	2	1	−1	3	1	−1	1	0	0	1	−1	1	1
−1	0	0	1	−1	1	−1							

TABLE VIII

Development of C''_{200}

Coef. × 10^5 of cosine	Multiples of l	l'	F	D	Γ	Ω	Coef. × 10^5 of cosine	Multiples of l	l'	F	D	Γ	Ω
76 291	0	0	0	0	0	0	−198	0	1	0	0	2	0
23 724	0	2	0	0	2	0	2	0	1	1	−1	1	0
−2	0	3	−1	1	3	0	1389	0	3	0	0	2	0
56	0	4	0	0	2	0	3827	0	1	0	0	0	0
1	0	5	0	0	2	0	96	0	2	0	0	0	0
2	0	3	0	0	0	0							

Development of C''_{210}

−1	0	1	0	0	2	2	−768	0	1	0	0	2	−2
7917	0	0	0	0	0	2	199	0	1	0	0	0	2
199	0	1	0	0	0	−2	5	0	2	0	0	0	2
5	0	2	0	0	0	−2	170	0	2	0	0	2	2
91 851	0	2	0	0	2	−2	10	0	3	0	0	2	2
5376	0	3	0	0	2	−2	3	0	2	0	1	2	−2
−3	0	2	0	−1	2	−2	219	0	4	0	0	2	−2
8	0	5	0	0	2	−2	8	0	1	1	−1	1	−2
−8	0	3	−1	1	3	−2							

Development of C''_{220}

−7	0	1	0	0	2	1	159	0	1	0	0	2	−1
18 257	0	0	0	0	0	1	458	0	1	0	0	0	1
458	0	1	0	0	0	−1	11	0	2	0	0	0	1
11	0	2	0	0	0	−1	820	0	2	0	0	2	1
−19 057	0	2	0	0	2	−1	48	0	3	0	0	2	1
−1115	0	3	0	0	2	−1	2	0	4	0	0	2	1
−45	0	4	0	0	2	−1	−2	0	5	0	0	2	−1
−2	0	1	1	−1	1	−1	2	0	3	−1	1	3	−1

The lunar elements used in the calculation of the perturbed orbital elements [assumed linear in the formal integration of the variation Equations (6)] are given by

$$l = 296\overset{\circ}{.}104\,608 + 13\overset{\circ}{.}064\,992\,446\,5(t - t_0) + $$
$$+ 0\overset{\circ}{.}000\,688\,9((t - t_0) \times 10^{-4})^2,$$

$$l' = 358\overset{\circ}{.}475\,845 + 0\overset{\circ}{.}985\,600\,267\,0(t - t_0) - $$
$$- 0\overset{\circ}{.}000\,011\,2((t - t_0) \times 10^{-4})^2,$$

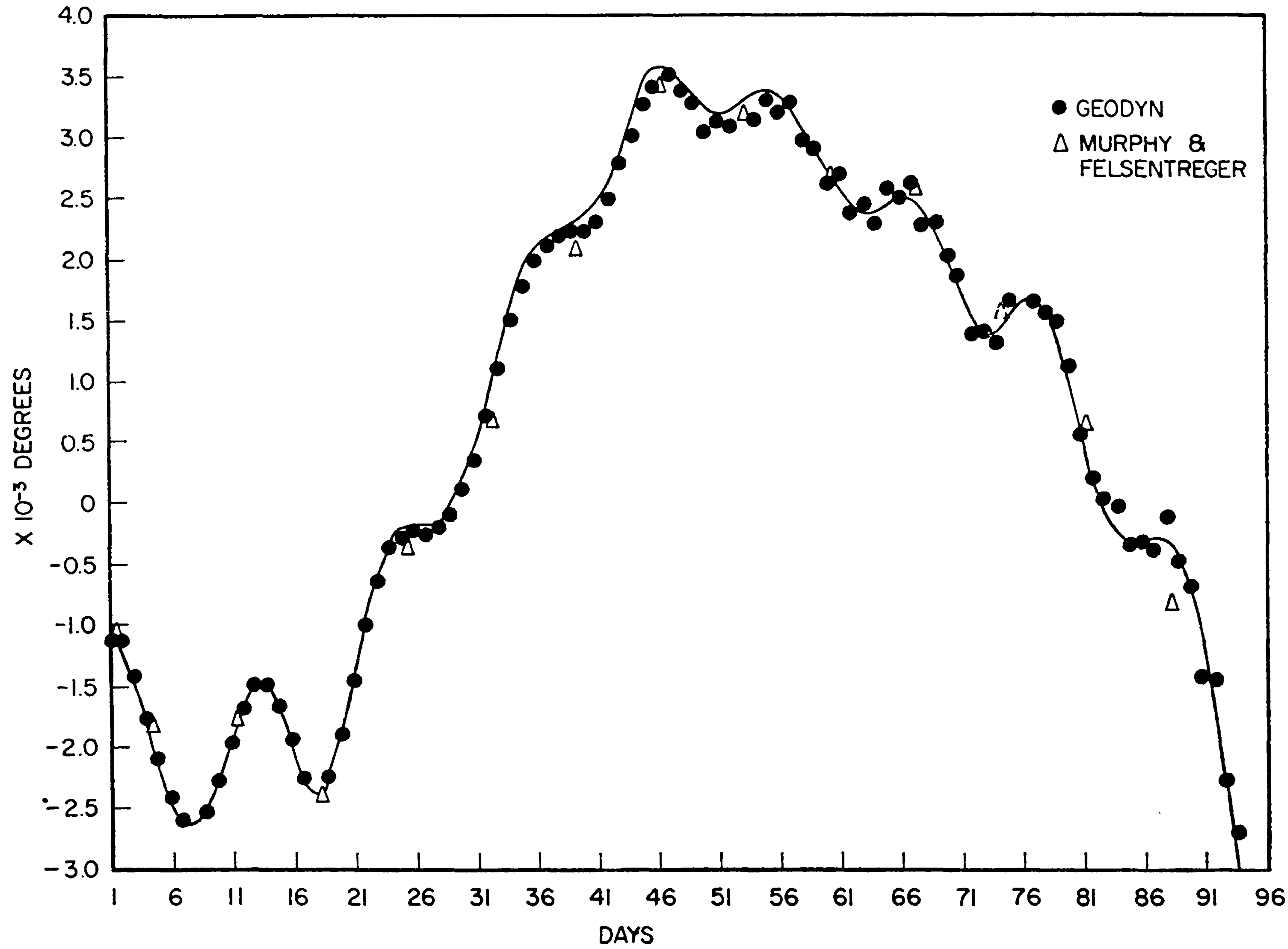

Fig. 1. δi at the epoch March 17.5075, 1958 for the Vanguard I satellite.

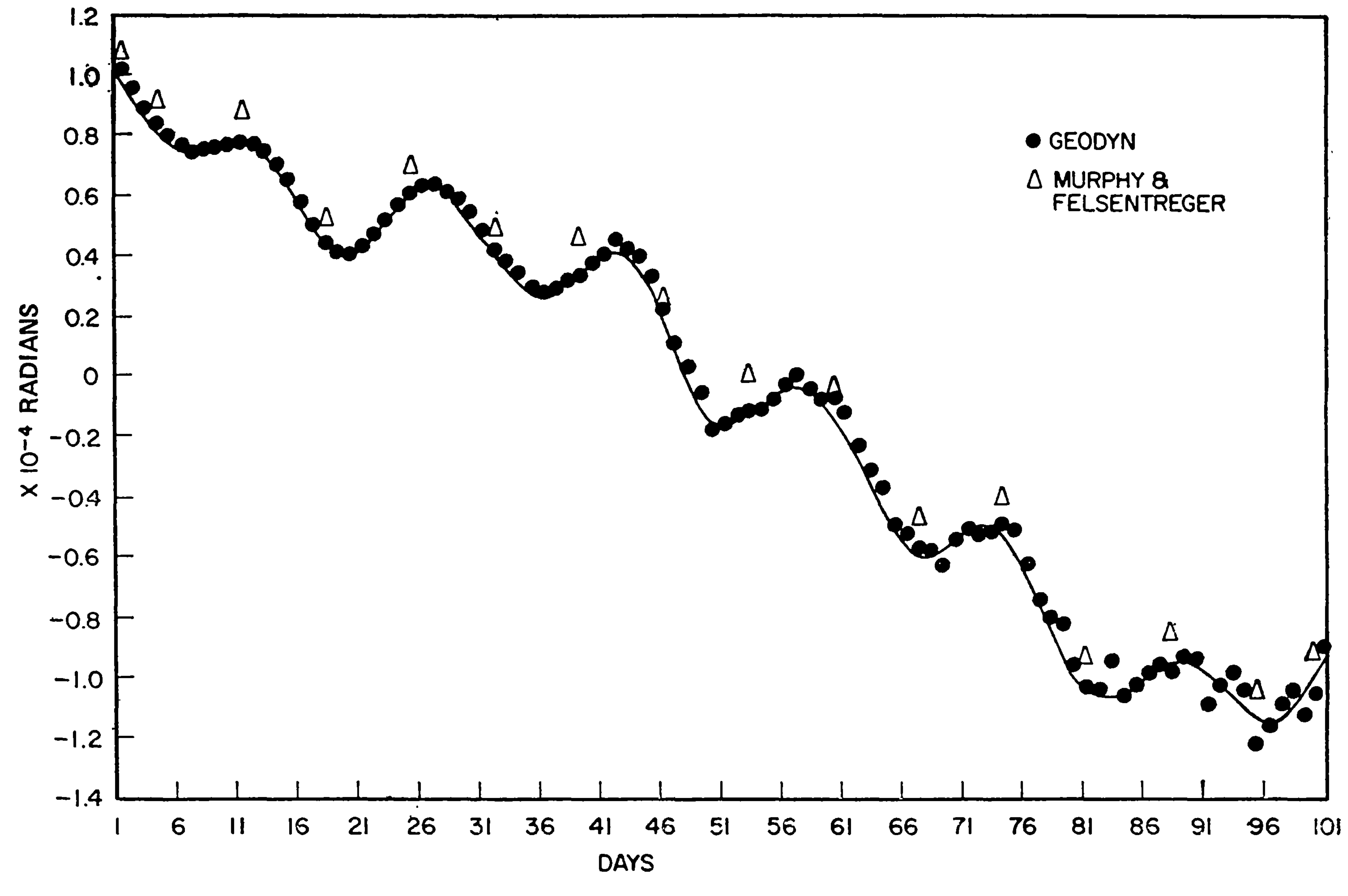

Fig. 2. δe at the epoch March 17.5075, 1958 for the Vanguard I satellite.

$$F = 11\overset{\circ}{.}250\,889 + 13\overset{\circ}{.}229\,350\,449(t - t_0) -$$
$$- 0\overset{\circ}{.}000\,240\,7((t - t_0) \times 10^{-4})^2,$$

$$D = 350\overset{\circ}{.}737\,486 + 12\overset{\circ}{.}190\,749\,191\,4(t - t_0) -$$
$$- 0\overset{\circ}{.}000\,107\,6((t - t_0) \times 10^{-4})^2,$$

$$\Gamma = 281\overset{\circ}{.}220\,833 + 0\overset{\circ}{.}000\,047\,068\,4(t - t_0) +$$
$$+ 0\overset{\circ}{.}000\,033\,9((t - t_0) \times 10^{-4})^2,$$

and the obliquity is given by

$$\varepsilon = 23\overset{\circ}{.}452\,294 - 0\overset{\circ}{.}003\,562\,6(t - t_0) \times 10^{-4} -$$
$$- 0\overset{\circ}{.}000\,000\,123((t - t_0) \times 10^{-4})^2.$$

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References

Brouwer, D.: 1959, *Astron. J.* **64**, 378–397.
Kaula, W. M.: 1962, *Astron. J.* **67**, 300.
Kozai, Y.: 1959, *Smithsonian Inst. Astrophys. Obs. Rept.*, No. 22, 7–10.
Murphy, J. and Felsentreger, T.: 1966, NASA TN D-3559.
Musen, P., Bailie, A., and Upton, E.: 1961, NASA TN D-494.
Musen, P. and Estes, R.: 1971, NASA-X-550-71-342.
Musen, P. and Felsentreger, T.: 1972, NASA-X-550-72-192.