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A Discrepancy Theorem Concerning Polynomials of Best Approximation in $L^p_w[-1,1]$

By

H.-P. Blatt, Eichstätt, and **H. N. Mhaskar***, Los Angeles, CA

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Abstract. Let w be a suitable weight function, $B_{n,p}$ denote the polynomial of best approximation to a function f in $L^p_w[-1,1]$, ν_n be the measure that associates a mass of $1/(n+1)$ with each of the $n+1$ zeros of $B_{n+1,p}-B_{n,p}$ and μ be the arcsine measure defined by $d\mu := (\pi\sqrt{1-x^2})^{-1}dx$. We estimate the rate at which the sequence ν_n converges to μ in the weak-* topology. In particular, our theorem applies to the zeros of monic polynomials of minimal L^p_w norm.

1. Introduction

Let $C[-1,1]$ denote the class of all continuous real functions on $[-1,1]$ and $\mathscr{P}_n$ denote the class of all polynomials of degree at most n. Let $f \in C[-1,1]$ and $B_{n,\infty} := B_{n,\infty}(f) \in \mathscr{P}_n$ be the polynomial of best approximation to f in the sense that

$$\max_{-1 \le x \le 1} |f(x) - B_{n,\infty}(x)| = E_{n,\infty}(f) := \min_{P \in \mathscr{P}_n} \max_{-1 \le x \le 1} |f(x) - P(x)|. \quad (1.1)$$

It is well known (cf. [5, p. 75]) that there exist at least $n+2$ *alternation points* y_i^* in $[-1,1]$ and a number $\delta = \pm 1$ such that

$$f(y_i^*) - B_{n,\infty}(y_i^*) = \delta(-1)^i E_{n,\infty}(f), \quad i = 1, \dots, n+2. \quad (1.2)$$

Let μ_n be the measure that associates the mass $1/(n+2)$ with each of the alternation points and μ be the arcsine measure defined by

$$d\mu := \frac{dx}{\pi\sqrt{1-x^2}}, \quad x \in [-1,1]. \quad (1.3)$$

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In [8], KADEC showed in particular that a subsequence of $\{\mu_n\}$ converges to μ in the weak-* topology. A major idea in the proof of his theorem was to consider the zeros of the polynomials $B_{n+1,\infty} - B_{n,\infty}$. These zeros interlace the alternation points y_i^*. If $\nu_{n,\infty}$ denotes the measure that associates the mass $1/(n+1)$ with each of these zeros, then Kadec showed that a subsequence of $\{\nu_{n,\infty}\}$ converges in the weak-* topology to μ. The interlacing property then easily gives the result about the alternation points themselves.

In [9], KROÓ and SWETITS obtained the L^p analogue of this result. If g is a Lebesgue measurable function and $w \geqslant 0$ is an integrable function on $[-1, 1]$ then we define for $1 \leqslant p < \infty$

$$\|g\|_{p,w} := \left(\int_{-1}^{1} |g(t)|^p w(t) dt \right)^{1/p} \tag{1.4a}$$

and denote by L_w^p the class of all functions g for which $\|g\|_{p,w} < \infty$, where two functions are identified if they are equal a.e. (almost everywhere with respect to the Lebesgue measure). We also define

$$\|g\|_\infty := \operatorname*{ess\,sup}_{t \in [-1,1]} |g(t)|. \tag{1.4b}$$

Let $1 < p < \infty$ be fixed, $w > 0$ a.e. and $f \in L_w^p$. We define the polynomial $B_{n,p} := B_{n,p,w}(f)$ of best approximation to f from $\mathscr{P}_n$ by

$$\|f - B_{n,p}\|_{p,w} = E_{n,p,w} = \min_{P \in \mathscr{P}_n} \|f - P\|_{p,w}. \tag{1.5}$$

KROÓ and SWETITS showed that all the zeros of the polynomials $B_{n+1,p} - B_{n,p}$ are simple and in $[-1, 1]$ if $B_{n+1,p} \neq B_{n,p}$. We denote the zeros of $B_{n+1,p} - B_{n,p}$ by

$$-1 < y_{n+1,n+1,p} < \cdots < y_{1,n+1,p} < 1 \tag{1.6}$$

and let $\nu_{n,p}$ denote the measure that associates the mass $(1/(n+1)$ with each of these zeros. KROÓ and SWETITS proved that a subsequence of $\{\nu_{n,p}\}$ converges to μ in the weak-* topology.

In this paper, we obtain a quantitative estimate on the rate of convergence of these measures. To this end we define, following ERDÖS and TURÁN [6], the *discrepancy* of a signed measure σ on $[-1, 1]$ by the expression

$$D[\sigma] := \sup_{-1 \leqslant a \leqslant b \leqslant 1} |\sigma([a, b])|. \tag{1.7}$$

Various estimates for the discrepancy $D[\nu_{n,\infty} - \mu]$ were given by many authors [8, 3, 2]. Using lower estimates on the derivatives of the polynomials $B_{n+1,\infty} - B_{n,\infty}$, it is proved in [2] that for a subsequence of positive integers,

$$D[\nu_{n,\infty} - \mu] \leqslant c\frac{(\log n)^2}{n} \tag{1.8}$$

where c is an absolute constant independent of f and n. Here and in the sequel, we adopt the convention that $c, c_1, \ldots$ denote positive constants depending only on p and w but their value may be different at different occurrences, even within the same formula. The main theorem of this paper is the following.

Theorem 1.1. *Let* $-1 = t_1 < \cdots < t_m = 1$ *be fixed points,* $a_1, \ldots, a_m > -1$ *be fixed numbers,* and w *be a weight function that satisfies the following conditions*:

$$c_1 w(x) \leqslant c \prod_{i=1}^{m} |x - t_i|^{a_i} \leqslant w(x), \quad x \in [-1, 1]. \tag{1.9}$$

Let $1 < p < \infty$ *and* $f \in L^p_w$, f *not a polynomial. Then for any integer* n *such that*

$$E_{n,p,w}(f) + E_{n+1,p,w}(f) \leqslant n^2 (E_{n,p,w}(f) - E_{n+1,p,w}(f)) \tag{1.10}$$

we have

$$D[\nu_{n,p} - \mu] \leqslant c\frac{(\log n)^2}{n}. \tag{1.11}$$

In particular, there exists a subsequence Λ *of positive integers such that* (1.11) *holds for all* $n \in \Lambda$. *Because of* (1.9) *there exists a monic polynomial* $P^* \in \mathscr{P}_n$ *with all zeros in* $[-1, 1]$ *such that*

$$|w(x) P^*(x)| \leqslant c < \infty, \quad x \in [-1, 1] \tag{1.12}$$

The typical examples of the weight functions considered in Theorem 1.1 are the Jacobi weights and certain generalized Jacobi weights [cf. 11]. If $1 < p < \infty$ and f satisfies certain additional conditions, then applying the results of PINKUS and ZIEGLER in [12], we see that the zeros of $B_{n+1,p} - B_{n,p}$ interlace the points where $f - B_{n,p}$ changes sign. Therefore, in this case, Theorem 1.1 gives the discrepancy between the measure associating these points of sign changes and the arcsine measure. If f is a 2π-periodic continuous function and s_n is the n-th partial sum of its Fourier series, then BINEV et al. [1] have obtained, in particular, the following result. For any $0<\delta<(2/3)(2-\sqrt{3})$, there exists a subsequence $\{n_v\}$ and points $\{x_j^{(v)}\}_{j=1,\ldots,2n_v,v=1,2,\ldots}$ such that $f-s_{n_v}$ vanishes at $x_j^{(v)}, j=1,\ldots,2n_v$ and

$$\left|x_j^{(v)}-\frac{j\pi}{n_v}\right|<n_v^{-\delta}, \quad j=1,\ldots,2n_v,\ v=1,2,\ldots .$$

As a corollary of Theorem 1.1 and its proof, we discuss the zeros of polynomials of minimal L^p-norm. Let $T_{n,p,w}\in\mathcal{P}_n$ denote the monic polynomial satisfying the equation

$$\|T_{n.p}\|_{p,w}=\varepsilon_{p,w}:=\min_{P\in\mathcal{P}_{n-1}}\|x^n-P(x)\|_{p,w}. \tag{1.13}$$

It is easy to see that $T_{n,p,w}$ has n simple zeros on $[-1,1]$. Let $\mu_{n,p,w}$ be the measure that associates the mass $1/(n+1)$ with each of the zeros of $T_{n+1,p,w}$.

Corollary 1.2. *We have*

$$D[\mu_{n,p,w}-\mu]\leqslant c\frac{(\log n)^2}{n}, \quad n=2,3,\ldots . \tag{1.14}$$

When $p=2$ then $T_{n,p,w}$ is the monic orthogonal polynomial with respect to w. Corollary 1.2 thus generalizes the corresponding result in [2] about orthogonal polynomials.

2. Proofs

We observe that if $f(x)=x^{n+1}$ then $B_{n+1,p,w}(f)=x^{n+1}$ and consequently, $T_{n+1,p,w}=B_{n+1,p,w}(f)-B_{n,p,w}(f)$ for this choice of f. Since the constants involved in Theorem 1.1 are independent of the function, f, and the integer n clearly satisfies (1.11), Corollary 1.2 follows directly from Theorem 1.1. Thus, we need to prove only Theorem 1.1. In the sequel, $1<p<\infty$ will be fixed, w will be a fixed function satisfying (1.9) and f will also be a fixed function. We will omit their mention from the notation. For example, $\|g\|$ will denote $\|g\|_{p,w}$, E_n will denote $E_{n,p,w}(f)$ etc. Since $E_n\to 0$ as $n\to\infty$, standard arguments (cf. [2]) show that there exists a subsequence Λ of integers such that (1.10) is satisfied for all $n \in \Lambda$. Our starting point is the following consequence of Theorem 2.1 in [4].

Theorem 2.1. *Let ν be any Borel, positive unit measure supported on $[-1,1]$. For $\alpha \geqslant 1$ let $\varepsilon(\alpha)$ be found such that*

$$\varepsilon(\alpha) \geqslant \max\{|U(\mu-\nu, z)| : |z+\sqrt{z^2-1}|=\alpha\} \tag{2.1a}$$

where

$$U(\mu-\nu, z) := \int_{-1}^{1} \log\frac{1}{|z-t|}\, d(\mu-\nu)(t). \tag{2.1b}$$

Then

$$D[\nu-\mu] \leqslant c\varepsilon(\alpha)\log(1/\varepsilon(\alpha)) \tag{2.2}$$

for all α with $\alpha \leqslant 1+\varepsilon(\alpha)^3$ and $\varepsilon(\alpha) < 1/e$.

We write

$$Q_{n+1}(x) := a_{n+1}x^{n+1} + \cdots = B_{n+1}(x) - B_n(x), \quad \bar{Q}_{n+1} := a_{n+1}^{-1} Q_{n+1}. \tag{2.3}$$

In order to apply Theorem 2.1, we obtain estimates of the form

$$\|\bar{Q}_{n+1}\|_\infty \leqslant c_1 n^c 2^{-n}, \qquad \min_{1 \leqslant i \leqslant n+1} \left|\frac{\bar{Q}'_{n+1}(y_{i,n+1})}{P^*(y_{i,n+1})}\right| \geqslant c_1 n^{-c} 2^{-n} \tag{2.4}$$

for all n in a subsequence of integers. We will then use the ideas applied frequently in [2] and [4]. We choose $\alpha = n^{-4}$. Using the maximum principle for logarithmic potentials the upper estimate in (2.4) gives an upper estimate on $U(\mu-\nu, z)$. Next we use the Lagrange interpolation formula based on the zeros of Q_{n+1} for the polynomial $P^*(x)T_m(x)$, where $m := n - \deg(P^*)$ and T_m is the Chebyshev polynomial of degree m, to get a lower estimate on $U(\mu-\nu, z)$. These estimates will yield (1.11).

The following Proposition 2.2 summarizes certain known facts about Q_{n+1}. Let

$$\Phi_n := |f-B_n|^{p-1}\operatorname{sign}(f-B_n) - |f-B_{n+1}|^{p-1}\operatorname{sign}(f-B_{n+1}), \tag{2.5}$$

and

$$w_n := \left|\frac{\Phi_n}{Q_{n+1}}\right| w. \tag{2.6}$$

Proposition 2.2. (KROÓ-SWETITS [9]) (a) *If* $x, y \in \mathbf{R}$, $\alpha > 0$,

$$\phi(x, y) := |x|^\alpha \operatorname{sign} x - |y|^\alpha \operatorname{sign} y, \quad M := |x| + |y|$$

then $\operatorname{sign} \phi(x, y) = \operatorname{sign}(x - y)$ *and*

$$\alpha 2^{-\alpha} M^{\alpha - 1} |x - y| \leqslant |\phi(x, y)| \leqslant 2|x - y|^\alpha, \quad \alpha \leqslant 1, \tag{2.7a}$$

$$2^{-\alpha} |x - y|^\alpha \leqslant |\phi(x, y)| \leqslant 2\alpha M^{\alpha - 1} |x - y|, \quad \alpha \geqslant 1. \tag{2.7b}$$

(b) *The polynomial* Q_{n+1} *has* $n + 1$ *simple zeros in* $(-1, 1)$ *if* $B_n \neq B_{n+1}$. *Moreover,*

$$\operatorname{sign} Q_{n+1} = \operatorname{sign} \Phi_n$$

and

$$\int_{-1}^{1} P(x) Q_{n+1}(x) w_n(x) dx = 0 \quad P \in \mathscr{P}_n. \tag{2.8}$$

We have

$$A_n^2 := \int_{-1}^{1} |Q_{n+1}(x)|^2 w_n(x) dx = \int_{-1}^{1} |Q_{n+1}(x) \Phi_n(x)| w(x) dx \geqslant$$

$$\geqslant \begin{cases} 2^{-p+1} \| Q_{n+1} \|^p \geqslant c(E_n - E_{n+1})^p, & \text{if } p \geqslant 2, \\ c \| Q_{n+1} \|^2 (E_n + E_{n+1})^{p-2}, & \text{if } 1 < p < 2. \end{cases} \tag{2.9}$$

For the convenience of notation, we will adopt the convention that the symbol M_n will denote a quantity of the form $c_1 n^c$ or $c_1 n^{-c}$. The values of c_1 and c will be different in different occurrences of M_n, even within the same formula.

Lemma 2.3. (a) *For any polynomial* $P \in \mathscr{P}_n$,

$$|P(x)| \leqslant M_n \| P \|, \quad |x| \leqq 1. \tag{2.10}$$

(b) *The leading coefficients* a_{n+1} *of* Q_{n+1} *satisfy*

$$c 2^n (E_n - E_{n+1}) \leqslant |a_{n+1}| \leqslant M_n (E_n + E_{n+1}) 2^n. \tag{2.11}$$

Proof. Using (1.9) and Theorem 28 in [11, p. 120], we get for $|x| \leqslant 1$

$$\min_{P \in \mathscr{P}_n} |P(x)|^{-p} \int_{-1}^{1} |P(t)|^p w(t) dt \geqslant$$

$$\geqslant c \min_{P \in \mathscr{P}_n} |P(x)|^{-p} \int_{-1}^{1} |P(t)|^p \prod_{i=1}^{m} |t - t_i|^{\alpha_i} dt \geqslant$$

$$\geqslant M_n. \tag{2.12}$$

This proves part (a). Let T_{n+1} be the Chebyshev polynomial of degree $n+1$ defined, as usual, by

$$T_{n+1}(\cos\theta) = \cos((n+1)\theta).$$

We get the first inequality in (2.11) exactly as in [2] by observing that

$$Q_{n+1} - 2^{-n} a_{n+1} T_{n+1} \in \mathscr{P}_n$$

and using the definitions of best approximation. To obtain the second inequality, we use the extremal property of the Chebyshev polynomials. Using (2.10), we get

$$2^{-n}|a_{n+1}| = 2^{-n}\|a_{n+1}T_{n+1}\|_\infty \leqslant \|Q_{n+1}\|_\infty \leqslant$$
$$\leqslant M_n\|Q_{n+1}\| \leqslant M_n(E_n + E_{n+1}).$$

This gives the second inequality in (2.11). ∎

Using the definition of Q_{n+1} and (2.11) it is easy to get the upper estimate in (2.4).

Corollary 2.4. *If n is any integer satisfying* (1.11) *then*

$$\|\bar{Q}_{n+1}\|_\infty \leqslant 2^{-n} M_n. \tag{2.13}$$

Next, we turn to the lower estimate in (2.4). Our proof involves one step where it is easier to assume that all the moments of w_n are finite; so that one can construct orthogonal polynomials with respect to w_n. In view of the estimates (2.7), this is certainly the case when $p \geqslant 2$ and also in the case when $1 < p < 2$ provided that none of the zeros of Q_{n+1} coincide with the points $t_2, \ldots, t_{m-1}$ described in the hypothesis of Theorem 1.1. Therefore, it is convenient to assume first that these points "do not exist" or equivalently, that each of the exponents $a_2, \ldots, a_{m-1}$ in (1.9) is zero. Thus, we first give the lower estimate in (2.4) which is valid under this assumption and later indicate the changes required in the proof so as to remove this assumption.

Proposition 2.5. *Let all the moments of w_n be finite. If* $\deg P^*(x) =$ $= d$ *and $n \geqslant d$ is an integer which satisfies* (1.11), *then*

$$\bar{m}_n := \min_{1 \leqslant i \leqslant n+1} \left|\frac{\bar{Q}'_{n+1}(y_{i,n+1})}{P^*(y_{i,n+1})}\right| =: \left|\frac{\bar{Q}'_{n+1}(y_{l,n+1})}{P^*(y_{l,n+1})}\right| \geqslant 2^{-n} M_n. \tag{2.14}$$

Proof. In this proof, it is convenient to write

$$m_n := |a_{n+1}\bar{m}_n| = \min_{1 \leqslant i \leqslant n+1} \left| \frac{\bar{Q}'_{n+1}(y_{i,n+1})}{P^*(y_{i,n+1})} \right| =: \left| \frac{\bar{Q}'_{n+1}(y_{l,n+1})}{P^*(y_{l,n+1})} \right|. \tag{2.15}$$

Since all the moments of w_n are assumed to be finite, there exists (cf. [7, Chapter I]) a system of polynomials

$$\tilde{Q}_k = \gamma_k x^k + \cdots, \quad \gamma_k > 0, \tag{2.16}$$

orthonormal with respect to w_n on $[-1, 1]$. Further, (2.9) and (2.8) imply that

$$A_n^{-1} Q_{n+1} = \tilde{Q}_{n+1}.$$

Let λ_j be the Cotes number corresponding to $y_{j,n+1}$ (cf. [7]). Using the Christoffel–Darboux formula, we get

$$\frac{A_n P^*(y_{l,n+1})}{Q'_{n+1}(y_{l,n+1})} = \frac{\gamma_n}{\gamma_{n+1}} \lambda_l P^*(y_{l,n+1}) \tilde{Q}_n(y_{l,n+1}). \tag{2.17}$$

Since w_n is supported on $[-1, 1]$, an application of Schwarz inequality gives

$$\frac{\gamma_n}{\gamma_{n+1}} = \left| \int_{-1}^{1} x \tilde{Q}_n(x) \tilde{Q}_{n+1}(x) w_n(x) dx \right| \leqslant 1.$$

Using this estimate, the Schwarz inequality and the quadrature formula, we get

$$\begin{aligned} \frac{A_n}{m_n} &\leqslant \lambda_l |P^*(y_{l,n+1}) \tilde{Q}_n(y_{l,n+1})| \leqslant \\ &\leqslant \sum_{j=1}^{n+1} \lambda_j |P^*(y_{j,n+1}) \tilde{Q}_n(y_{j,n+1})| \leqslant \\ &\leqslant \left\{ \int_{-1}^{1} P^{*2}(x) w_n(x) dx \right\}^{1/2}. \end{aligned} \tag{2.18}$$

This ends the part of the proof where the assumption that all moments of w_n should be finite is necessary. The rest of the proof does not use this assumption.

We now distinguish between two cases. First, let $p \geqslant 2$. Using (2.7b) with $\alpha = p - 1 \geqslant 1$ we see that

$$|\Phi_n| \leqslant 2(p-1)\{|f - B_n| + |f - B_{n+1}|\}^{p-2} |Q_{n+1}|.$$

So, using Hölder's inequality with exponents $p/(p-2)$ and $p/2$, we get

$$\int_{-1}^{1} P^{*2}(x)w_n(x)dx =$$
$$= \int_{-1}^{1} P^{*2}(x)\left|\frac{\Phi_n}{Q_{n+1}}\right| w(x)dx \leqslant$$
$$\leqslant 2(p-1)\int_{-1}^{1} \{|f-B_n|+|f-B_{n+1}|\}^{p-2}P^{*2}(x)w(x)dx \leqslant$$
$$\leqslant 2(p-1)\| |f-B_n| + |f-B_{n+1}| \|^{p-2} \| P^* \|^2 \leqslant$$
$$\leqslant c(E_n + E_{n+1})^{p-2}. \tag{2.19}$$

From (2.11),

$$(E_n + E_{n+1})^{-2} \leqslant M_n |a_{n+1}|^{-2} 2^{2n}.$$

The estimate (2.18) now gives

$$\int_{-1}^{1} P^{*2}(x)w_n(x)dx \leqslant M_n |a_{n+1}|^{-2} 2^{2n}(E_n + E_{n+1})^p. \tag{2.20}$$

Again, from (2.9) and (1.10), we have

$$A_n^2 \geqslant c(E_n - E_{n+1})^p \geqslant M_n(E_n + E_{n+1})^p. \tag{2.21}$$

From (2.18), (2.20) and (2.21), it follows that

$$m_n^{-2} \leqslant A_n^{-2} \int_{-1}^{1} P^{*2}(x)w_n(x)dx \leqslant M_n |a_{n+1}|^{-2} 2^{2n},$$

which leads to (2.14) in the case when $p \geqslant 2$.

Next, we consider the case when $1 < p < 2$. Using (2.7a) with $\alpha = p-1$ and (1.12) we get

$$\int_{-1}^{1} P^{*2}(x)w_n(x)dx \leqslant 2\int_{-1}^{1} |Q_{n+1}(x)|^{p-2}|P^*(x)|^{2-p}|P^*(x)|^p w(x)dx \leqslant$$
$$\leqslant c\int_{-1}^{1} \left|\frac{P^*(x)}{Q_{n+1}(x)}\right|^{2-p} dx. \tag{2.22}$$

Since $n \geqslant d$, we may express P^* using the Lagrange interpolation formula based on the zeros of Q_{n+1} and deduce that

$$\frac{P^*(x)}{Q_{n+1}(x)} = \sum_{j=1}^{n+1} \frac{P^*(y_{j,n+1})}{(x - y_{j,n+1})Q'_{n+1}(y_{j,n+1})}.$$

Using the fact that $0 < 2 - p < 1$ and the definition (2.15) of m_n, we get

$$\left|\frac{P^*(x)}{Q_{n+1}(x)}\right|^{2-p} \leqslant m_n^{p-2} \sum_{j=1}^{n+1} |x - y_{j,n+1}|^{p-2}$$

and hence that

$$\int_{-1}^{1} \left|\frac{P^*(x)}{Q_{n+1}(x)}\right|^{2-p} dx \leqslant m_n^{p-2} \sum_{j=1}^{n+1} \int_{-1}^{1} |x - y_{j,n+1}|^{p-2} dx \leqslant cnm_n^{p-2}.$$

Along with (2.22) and (2.18) this gives

$$m_n^{-2} \leqslant cnA_n^{-2} m_n^{p-2},$$

i.e.,

$$m_n \geqslant M_n A_n^{2/p}. \tag{2.23}$$

From (2.9), (1.10) and (2.11) we have

$$\begin{aligned} A_n^2 &\geqslant c\,\| Q_{n+1} \|^2 (E_n + E_{n+1})^{p-2} \geqslant \\ &\geqslant c(E_n - E_{n+1})^2 (E_n + E_{n+1})^{p-2} \geqslant \\ &\geqslant cn^{-4}(E_n + E_{n+1})^p \geqslant M_n(|a_{n+1}| 2^{-n})^p. \end{aligned}$$

Together with (2.23) this gives

$$m_n \geqslant M_n |a_{n+1}| 2^{-n}$$

and hence (2.14) also in the case when $1 < p < 2$. ■

Next, we show that the estimate (2.18) holds even without the assumption that all moments of w_n be finite. This will be done by an approximation argument.

Lemma 2.6. *Let E_k be a neighborhood of the point set $\{t_2, \ldots, t_{m-1}\}$ such that*

$$\int_{E_k} w(t)dt \leqslant 1/k, \tag{2.24}$$

$E_{k+1} \subseteq E_k$, $k = 1, 2, \ldots$, and $[-1, 1] \backslash E_1$ have a positive Lebesgue

measure. Let

$$W_k(x) := \begin{cases} w(x), & \text{if } x \in [-1,1] \setminus E_k, \\ 0, & \text{if } x \in E_k, \end{cases}$$

and let

$$\| f - B_{p,n,k} \|_{p,W_k} = \min_{P \in \mathscr{P}_n} \| f - P \|_{p,W_k}.$$

Then the sequence $\{B_{n,p,k}\}$ converges uniformly on compact subsets of the complex plane to $B_{n,p}$.

Proof. In this proof, p and n are fixed quantities and therefore, it is convenient to denote $\|\cdot\|_{p,W_k}$ by $\|\cdot\|_k$, $B_{n,p,k}$ by B_k, $\|\cdot\|_{p,w}$ by $\|\cdot\|$ and $B_{n,p}$ by B. We have

$$\limsup_{k\to\infty} \| f - B_k \|_1 \leqslant \limsup_{k\to\infty} \| f - B_k \|_k \leqslant$$
$$\leqslant \limsup_{k\to\infty} \| f - B \|_k \leqslant \| f - B \|. \tag{2.25}$$

Therefore, the sequence $\{B_K\}$ is uniformly bounded on $F_1 := [-1,1] \setminus E_1$. Since each B_k is in the finite dimensional space $\mathscr{P}_n$, for any subsequence Λ of integers, there exists a subsequence Λ_1 of Λ and a polynomial $\tilde{B} \in \mathscr{P}_n$ such that $\{B_k\}_{k\in\Lambda_1}$ converges uniformly to $\tilde{B}$ on F_1. Necessarily, the subsequence $\{B_k\}_{k\in\Lambda_1}$ converges uniformly to $\tilde{B}$ also on $[-1,1]$. Then the dominated convergence theorem shows that

$$\| f - B \| \leqslant \| f - \tilde{B} \| = \lim_{k\in\Lambda_1} \| f - B_k \|_k. \tag{2.26}$$

Together with (2.24), this shows that $\| f - B \| = \| f - \tilde{B} \|$. Since the best approximating polynomial from $\mathscr{P}_n$ in the L^p_w norm is unique, we conclude that $B = \tilde{B}$. Thus, the whole sequence $\{B_k\}$ converges uniformly on $[-1,1]$, and hence on compact subsets of the complex plane, to B. ■

Now, we are in a position to obtain the estimate (2.18) without w having to satisfy the condition that all moments of w_n be finite. Let $A_{n,k}$, $m_{n,k}$, $Q_{n+1,k}$ and $w_{n,k}$ etc. denote the quantities A_n, m_n, Q_{n+1} and w_n corresponding to W_k rather than the original weight function w. We observe that we can choose the same polynomial P^* for all k. Since all moments of each $w_{n,k}$ are easily seen to be finite, we have

$$\frac{A_{n,k}}{m_{n,k}} \leqslant \left\{ \int_{-1}^{1} P^{*2}(x) w_{n,k}(x) dx \right\}^{1/2}. \tag{2.27}$$

In view of Lemma 2.6, the zeros of $Q_{n+1,k}$ converge to the zeros of Q_{n+1}. Therefore, we arrive at (2.18) by letting $k \to \infty$ in (2.27). As remarked earlier, this completes the proof of (2.14) even without the assumption that all moments of w_n are finite.

Proof of Theorem 1.1. We observe that

$$\int_{-1}^{1} \log\frac{1}{|z-t|}\,d\mu(t) = \begin{cases} \log 2, & x \in [-1,1], \\ \log 2 - \log|z+\sqrt{z^2-1}|, & z \in \mathbf{C}\setminus[-1,1]. \end{cases} \tag{2.28}$$

The estimate (2.13) can now be written in the form

$$U(\mu - v, z) \leqslant c\frac{\log n}{n}, \quad z \in [-1,1]. \tag{2.29}$$

In view of the maximum principle for potentials ([10, Theorem 1.10]) the estimate (2.29) holds for all $z \in \mathbf{C}$. Next, let $n \geqslant 2d$ and (1.10) be satisfied. We express the polynomial $T_{n-d}P^*$ using the Lagrange interpolation formula and deduce that for any z with $|z+\sqrt{z^2-1}| = 1 + n^{-4}$,

$$|T_{n-d}(z)P^*(z)| \leqslant M_n 2^n |\bar{Q}_{n+1}(z)|.$$

Using well known formulas for the Chebyshev polynomials in the complex domain and (2.28) this leads to

$$U(\mu - v, z) \geqslant -c\frac{\log n}{n}, \quad |z+\sqrt{z^2-1}| = 1 + n^{-4}. \tag{2.30}$$

In view of (2.29) and (2.30), we may apply Theorem 2.1 with $\alpha = 1 + n^{-4}$ and $\varepsilon(\alpha) = c \log n/n$ to arrive at (1.12). ∎

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H. P. BLATT
Mathem.-Geogr. Fakultät
Katholische Universität
D-85072 Eichstätt
Federal Republic of Germany

and

H. N. MHASKAR
Department of Mathematics
California State University
Los Angeles, CA 90032
USA