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EXCHANGE PRICE EQUILIBRIA AND VARIATIONAL INEQUALITIES

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The aim of this paper is to show the relevance of the concept and the theory of variational inequalities in the study of economic equilibria.

Key words: Walrasian equilibria, variational inequalities.

Introduction

Principal objectives of the theory of equilibrium in economics (as well as regional science, transportation theory, game theory etc.) are to establish existence of equilibrium, to discuss conditions under which the equilibrium is unique, to perform sensitivity analysis (comparative statics) and, finally, to develop algorithms for the efficient computation of equilibria. There is voluminous literature on the subject in which the above issues are being addressed by means of a variety (often ad hoc) techniques. Over the last few years the author has been pursuing a research program whose goal is to demonstrate that the theory of variational inequalities provides the most natural, direct, simple and efficient framework for a *unifying* treatment of all equilibrium problems.

It is by now familiar that the equilibrium conditions of virtually every equilibrium problem may be formulated as a variational inequality: This has been observed in the context of numerous specific examples (see, e.g., Dafermos, 1980, 1982; Gabay and Moulin, 1980; Florian and Los, 1982, Dafermos and Nagurney, 1984, 1987; Border, 1985). To the above list we should add all cases in which the equilibrium conditions have been reduced to a complementarity problem which in turn is a special case of a variational inequality (see, e.g., Lemke, 1965; Karamardian, 1969; Eaves, 1972; Scarf, 1973; Todd, 1976). This reduction provides the opportunity for developing a *unifying* and *integrated* theory of equilibria that encompasses qualitative study as well as computations. It seems that this potential has not been yet fulfilled, especially in economics. In this paper, an attempt is being made to illustrate this approach by means of the simple and familiar example of a pure exchange

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economy. Other, more elaborate models will be discussed in the same vein in future publications.

In Section 1 we show that Walrasian equilibrium price vectors can be characterized as solutions of a remarkably simple variational inequality (see also Border, 1985, Section 8). As a first indication of the usefulness of this observation, we show how the theory of variational inequalities provides a simple direct proof of the existence of Walrasian equilibrium price vectors under quite weak assumptions on the aggregate demand function.

The situation in which the power of the theory of variational inequalities becomes particularly obvious is when we impose on the aggregate excess demand function certain monotonicity assumptions which, though restrictive, are in the spirit of the 'law of demand'. To demonstrate this, we derive, under the monotonicity assumption, an alternative, even simpler, characterization of Walrasian equilibrium from which we immediately infer that the set of Walrasian equilibrium price vectors is convex. Under strict monotonicity, we show that the Walrasian price equilibrium vector is unique and depends continuously on the aggregate demand function. Some of our results are slight generalizations of propositions that can be found in the literature; however, our derivations, based on the theory of variational inequalities, are more direct, short, unifying and, I believe, conceptually simpler. Furthermore, the characterization of Walrasian price equilibrium vectors as solutions of a simple variational inequality induces efficient algorithms for their computation (see Dafermos, 1983; Zhao, 1988). The issue of local uniqueness and sensitivity of equilibrium can also be settled effectively through the theory of variational inequalities (see Dafermos, 1988; Zhao, 1988).

As stated above, the goal of this paper is only to provide an example of the application of an approach which is very general. Other applications will follow. For instance, the general equilibrium problem with production economies has recently been studied via the present approach (Zhao, 1988; Dafermos and Zhao, 1989).

1. Exchange price equilibrium and variational inequalities

We consider here an exchange economy with price vector $p=(p_1, \dots, p_k)$, taking values in the positive orthant $\mathbb{R}^k_+$, and induced aggregate excess demand function $z(p)$. In order to take into consideration the possibility that aggregate excess demand may become unbounded when the price of a certain good vanishes, we assume that $z(p)$ is generally defined on a subcone C of $\mathbb{R}^k_+$ which contains the interior $\mathbb{R}^k_{++}$ of $\mathbb{R}^k_+$, i.e. $\mathbb{R}^k_{++} \subset C \subset \mathbb{R}^k_+$. As usual $z(p)$ will be homogeneous of degree zero in p on C and will satisfy Walras' law , namely $p \cdot z(p)=0$, on C. By virtue of homogeneity, we may normalize prices so that they take values on the simplex

$$S^k=\left\{p \in \mathbb{R}^k_+ \colon \sum_{i=1}^{k} p_i=1\right\} \tag{1.1}$$

and, accordingly, we restrict the aggregate demand function $z(p)$ to the intersection D of S^k with C. Note that $S^k_+ \subset D \subset S^k$, where S^k_+ denotes the intersection of S^k with $\mathbb{R}^k_{++}$. We assume tht the function

$$z(p): D \to \mathbb{R}^k$$

is continuous. By Walras' law, we have in particular

$$p \cdot z(p) = 0, \quad p \in D. \tag{1.2}$$

A price vector $p^* \in D$ is called a *Walrasian equilibrium* if

$$z(p^*) \leqslant 0. \tag{1.3}$$

We show in the next theorem that Walrasian equilibrium price vectors can be characterized as solutions of a variational inequality (see, e.g., Kinderlehrer and Stampacchia, 1980; Border, 1985).

Theorem 1.1. *A price vector $p^* \in D$ is a Walrasian equilibrium if and only if it satisfies the variational inequality*

$$z(p^*) \cdot (p - p^*) \leqslant 0, \quad p \in S^k. \tag{1.4}$$

Proof. First note that, by virtue of Walras' law (1.2), inequality (1.4) is equivalent to

$$z(p^*) \cdot p \leqslant 0, \quad p \in S^k. \tag{1.5}$$

Assume now that $p^* \in D$ is a Walrasian equilibrium price vector, i.e., it satisfies (1.3). Then (1.5) obviously holds. Conversely, assuming (1.5) holds for all $p \in S^k$ and choosing $p = (0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 located at the ith position, we deduce $z_i(p^*) \leqslant 0$, $i = 1, \ldots, m$. The proof is complete. □

The above characterization allows us to utilize the well-developed theory of variational inequalities in order to establish the existence and other properties of Walrasian equilibria and to formulae efficient algorithms for their computation.

2. Existence of Walrasian equilibria

In general, the existence of solutions of a variational inequality over a convex and compact set is established by the following standard theorem (see, e.g., Kinderlehrer and Stampacchia, 1980, Theorem 3.1).

Theorem 2.1. *Let $K \subset \mathbb{R}^k$ be compact and convex and let*

$$f: K \to \mathbb{R}^k$$

be continuous. Then there is at least one $p^* \in K$ such that

$$f(p^*) \cdot (p - p^*) \geqslant 0, \quad p \in K. \quad □ \tag{2.1}$$

We observe that, in particular, if the aggregate excess demand function $z(p)$ is defined and is continuous on all of S^k, i.e., $D=S^k$, then the existence of at least one Walrasian equilibrium price vector in S^k follows directly from Theorems 1.1 and 2.1.

Since D is not necessarily compact, Theorem 2.1 is not directly applicable to variational inequality (1.4). However, we may still use this theorem to deduce the existence of an equilibrium, provided $z(p)$ exhibits the proper behavior near the boundary of S^k, namely that at least some components of $z(p)$ become 'large' as p approaches points on the boundary of S^k that are not contained in D. Several existence theorems of this type are recorded in the literature (see, e.g., Border, 1985). We choose here to prove an existence theorem under a hypothesis on the boundary behavior of $z(p)$ proposed by Grandmont (1977), which is perhaps the weakest considered so far in the literature. The reader will realize, of course, that the pattern of our proof is very general and may be adapted easily when alternative assumptions are imposed on $z(p)$.

Theorem 2.2. *Assume that the aggregate excess demand function $z(p)$ satisfies the following assumption: if $S^k \backslash D$ is nonempty, then with any sequence $\{p_n\}$ in S^k_+ which converges to a point of $S^k \backslash D$ there is associated a point $\bar{p} \in S^k_+$ (generally dependent on $\{p_n\}$) such that the sequence $\{z(p_n) \cdot \bar{p}\}$ contains infinitely many positive terms. Then there is a Walrasian equilibrium price vector $p^* \in D$.*

Proof. Let K_n denote the set of all $p \in S^k_+$ whose distance from the boundary $S^k \backslash S^k_+$ is greater than or equal to $1/n$. Note that for n sufficiently large K_n is a nonempty, compact, convex subset of S^k_+ and $\bigcup_n K_n = S^k_+$. By virtue of Theorem 2.1, the variational inequality

$$z(p_n^*) \cdot (p - p_n^*) \leq 0, \quad p \in K_n, \tag{2.2}$$

has at least one solution $p_n^* \in K_n$. Because of (1.2),

$$z(p_n^*) \cdot p \leq 0, \quad p \in K_n. \tag{2.3}$$

The sequence $\{p_n^*\}$ contains a convergent subsequence denoted again by $\{p_n^*\}$, say

$$p_n^* \to p^*, \quad n \to \infty, \tag{2.4}$$

where $p^* \in S^k$. We claim $p^* \in D$. Indeed, if $p^* \in S^k \backslash D$ then, by hypothesis, there is $\bar{p} \in S^k_+$ such that the sequence $\{z(p_n^*) \cdot \bar{p}\}$ has infinitely many positive terms. However, $\bar{p} \in K_n$ for n sufficiently large and so, by virtue of (2.3), $z(p_n^*) \cdot \bar{p} \leq 0$, for n sufficiently large, and this is a contradiction.

Since $p^* \in D$ and $z(p)$ is continuous on D, we may pass to the limit in (2.2), to infer

$$z(p^*) \cdot (p - p^*) \leq 0, \quad p \in \bigcup_n \bar{K}_n = S^k, \tag{2.5}$$

and so, by Theorem 1.1, p^* is a Walrasian equilibrium price vector. This completes the proof. □

3. Monotone aggregate excess demand

Here we consider a special class of aggregate excess demand functions for which the theory of variational inequalities gives particularly strong results (see, e.g., Kinderlehrer and Stampacchia, 1980).

We say that $-z(p)$ is *monotone* on D if

$$(z(p)-z(q))\cdot(p-q)\leqslant 0, \quad p, q \in D. \tag{3.1}$$

Though restrictive, assumption (3.1) is in the spirit of the 'law of demand'. Hildenbrand (1983), for example, showed that if the income of the individual does not depend on the price system, then individual demand functions which satisfy the weak axiom of revealed preference induce monotone aggregate demand functions. Conditions in the same spirit on the aggregate demand function have frequently been resorted to in the economics literature (cf. Mas-Colell, 1985) in order to establish uniqueness and other desirable properties of equilibrium price vectors.

Theorem 3.1. *Assume $-z(p)$ is continuous and monotone on D. Then $p^* \in D$ is a Walrasian equilibrium price vector if and only if*

$$z(p)\cdot(p-p^*)\leqslant 0, \quad p\in D, \tag{3.2}$$

or, equivalently, if and only if

$$z(p)\cdot p^* \geqslant 0, \quad p \in D. \tag{3.3}$$

Proof. Assume first $p^* \in D$ is a Walrasian equilibrium vector, in which case, in view of Theorem 1.1, it satisfies the variational inequality

$$z(p^*)\cdot(p-p^*)\leqslant 0, \quad p\in S^k. \tag{3.4}$$

We note,

$$z(p)\cdot(p-p^*) = z(p^*)\cdot(p-p^*) + (z(p)-z(p^*))\cdot(p-p^*), \quad p\in D. \tag{3.5}$$

The right-hand side of (3.5) is nonpositive, by virtue of (3.4) and the monotonicity assumption (3.1). Hence (3.2) holds.

Conversely, assume (3.2) holds. Fix any $q \in S^k_+$ and set $p=(1-\lambda)p^*+\lambda q$, where $\lambda \in (0, 1)$. It follows that $p \in S^k_+ \subset D$ and therefore (3.2) holds, i.e.,

$$z((1-\lambda)p^*+\lambda q)\cdot(q-p^*)\leqslant 0. \tag{3.6}$$

Since $z(p)$ is continuous on D, letting $\lambda \to 0$, (3.6) yields

$$z(p^*)\cdot(q-p^*)\leqslant 0$$

for all $q\in S^k_+$ and so, by completion, for all $q \in S^k$. Hence, by Theorem 1.1, p^* is a Walrasian equilibrium price vector. The proof is complete. □

As an immediate corollary of Theorem 3.1, we obtain the following.

Corollary 3.1. *Assume $-z(p)$ is continuous and monotone on D and D is convex. Then the set of Walrasian equilibrium price vectors is a convex subset of D.* □

Remark 3.1. When the economy is *regular*, i.e. $z(\bar{p})=0$ implies $\operatorname{rank}[\partial z(\bar{p})/\partial p]=k-1$, the set of equilibrium price vectors is finite (see, e.g., Mas-Colell, 1985), and hence an immediate consequence of the above corollary is that in a *regular* economy with $-z(p)$ *monotone* there is at most one interior equilibrium price vector.

We now proceed to strengthen the monotonicity assumption (3.1). We say that $-z(p)$ is *strictly monotone* on D if

$$(z(p)-z(q))\cdot(p-q)<0, \quad p, q \in D,\ p \neq q. \tag{3.7}$$

Theorem 3.2. *Assume $-z$ is strictly monotone on D. Then there exists at most one Walrasian equilibrium price vector.*

Proof. Assume $p^*, q^* \in D$ are Walrasian equilibrium price vectors. They satisfy, by Theorem 1.1, the variational inequalities

$$z(p^*)\cdot(p-p^*)\leqslant 0, \quad p \in S^k, \tag{3.8}$$

and

$$z(q^*)\cdot(p-q^*)\leqslant 0, \quad p \in S^k. \tag{3.9}$$

We write (3.8) for $p=q^*$ and (3.9) for $p=p^*$ and combine to obtain

$$(z(p^*)-z(q^*))\cdot(p^*-q^*)\geqslant 0. \tag{3.10}$$

By virtue of (3.7), (3.10) cannot hold unless $p^*=q^*$. The proof is complete. □

Remark 3.2. We say that $-z(p)$ is strictly monotone about a particular point $p^* \in D$ if

$$(z(p^*)-z(q))\cdot(p^*-q)<0, \quad q \in D,\ q \neq p^*.$$

It is clear from the proof of Theorem 3.2 that the following weaker statement holds: If $p^* \in D$ is a Walrasian equilibrium and $-z(p)$ is strictly monotone about p^*, then p^* is the only equilibrium point. At this point it is appropriate to recall that in the economics literature uniquess is usually derived from the weak axiom of revealed preference, namely,

$$q\cdot z(p)\leqslant 0 \quad \text{implies} \quad p\cdot z(q)>0, \qquad p, q \in D,\ p \neq q.$$

For a fixed point p^* in D, we say that the weak axiom of revealed preference holds about p^* if

$$q\cdot z(p^*)\leqslant 0 \quad \text{implies} \quad p^*\cdot z(q)>0, \qquad q \in D,\ q \neq p^*.$$

Because of Walras' law (1.2), it is clear that strict monotonicity implies the weak axiom of revealed preference and strict monotonicity about $p^* \in D$ implies the weak axiom of revealed preference about p^*. In particular, if p^* is a Walrasian equilibrium such that $p^* > 0$, in which case $z(p^*) = 0$, it is easily seen that, conversely, the weak axiom of revealed preference about p^* implies strict monotonicity about p^*. We thus see that for interior equilibrium points our uniqueness result derived from strict monotonicity about the equilibrium point p^* is completely equivalent with the standard uniqueness result derived from the weak axiom of revealed preference about p^*.

We now turn to the question of comparing equilibria. The following lemma is useful in analyzing the effects of a change in an economy.

Lemma 3.1. *Consider two aggregate excess demand functions z and z'. Let p and p' be Walrasian equilibrium price vectors associated, respectively, with z and z'. Then*

$$(z'(p') - z(p)) \cdot (p' - p) \geqslant 0. \tag{3.11}$$

Furthermore, when $-z$ is strictly monotone (without any monotonicity restriction on z'), then

$$(z'(p') - z(p')) \cdot (p' - p) \geqslant 0, \tag{3.12}$$

with equility holding only when $p = p'$.

Proof. Being Walrasian equilibrium price vectors, p and p' must satisfy, by Theorem 1.1, the variational inequalities

$$z(p) \cdot (q - p) \leqslant 0, \quad q \in S^k, \tag{3.13}$$

and

$$z'(p') \cdot (q - p') \leqslant 0, \quad q \in S^k. \tag{3.14}$$

We write (3.13) for $q = p'$ and (3.14) for $q = p$ and sum the resulting inequalities thus obtaining (3.11).

From (3.11),

$$(z'(p') - z(p) + z(p') - z(p')) \cdot (p' - p) \geqslant 0. \tag{3.15}$$

When $-z(p)$ is strictly monotone, (3.15) yields

$$(z'(p') - z(p')) \cdot (p' - p) \geqslant -(z(p') - z(p)) \cdot (p' - p) \geqslant 0 \tag{3.16}$$

and hence (3.12) follows with equality holding only when $p = p'$. □

Applying Walras' law to (3.11) and (3.12), above, we deduce the following corollary.

Corollary 3.2. *Consider two aggregate excess demand functions z and z'. Let p and p' be Walrasian equilibrium price vectors associated with z and z', respectively. Then*

$$p\cdot z'(p')+p'\cdot z(p)\leqslant 0. \tag{3.17}$$

Furthermore, assuming that $-z$ is strictly monotone,

$$p\cdot z'(p')\leqslant p\cdot z(p') \tag{3.18}$$

with equality only where $p=p'$. □

In particular, if z' is induced by a binary change in z (in which case $z(q)-z'(q)$ has only two nonzero components) and if, for definiteness, $z_1'(q)>z_1(q)$ and $z_2'(q)<z_2(q)$ for all $q\in D$, then (3.18) and Walras' law yield

$$p_1/p_1'\leqslant p_2/p_2'$$

with equality holding only if $p=p'$.

When the assumption of monotonicity is strengthened even further, we deduce that Walrasian price equilibria depend continuously upon the aggregate excess demand function.

We will say that $-z(p)$ is *strongly monotone* on D if

$$(z(p)-z(q))\cdot(p-q)\leqslant -\alpha|p-q|^2, \quad p,q\in D, \tag{3.19}$$

where α is a positive number.

Theorem 3.3. *Let z and z' be aggregate excess demand functions. Let p and p' be corresponding Walrasian equilibrium price vectors. Assume z satisfies the strong monotonicity assumption* (3.19). *Then*

$$|p'-p|\leqslant\frac{1}{\alpha}|z'(p')-z(p')|. \tag{3.20}$$

Proof. By Lemma 3.1, (3.11) holds. From (3.11),

$$(z'(p')-z(p)+z(p')-z(p'))\cdot(p'-p)\geqslant 0. \tag{3.21}$$

On account of the strong monotonicity condition (3.19), (3.21) yields

$$(z'(p')-z(p'))\cdot(p'-p)\geqslant -(z(p')-z(p))\cdot(p'-p)\geqslant\alpha|p'-p|^2. \tag{3.22}$$

By virtue of Schwarz's inequality, (3.22) gives

$$\alpha|p'-p|^2\leqslant|z'(p')-z(p')|\,|p'-p|, \tag{3.23}$$

whence (3.20) follows and the proof is complete. □

Being nonlocal conditions, (3.1), (3.7) and (3.19) do not generally provide an easy test for determining whether a given aggregate excess demand function is monotone, strictly monotone or strongly montone. However, when $z(p)$ is continuously differentiable on S_+^k, the issue of monotonicity may be resolved through a straightforward computation. Specifically, consider the Jacobian matrix

$$J(p)=[\partial z_i(p)/\partial p_j], \quad p \in S_+^k. \tag{3.24}$$

It is known that a necessary and sufficient condition for $-z(p)$ to be monotone on C is that $-J(p)$ be positive semi-definite on $\mathbb{R}_{++}^k$, and a sufficient condition for $-z(p)$ to be strictly (strongly) monotone on C would be that $-J(p)$ be positive (uniformly positive) definite on $\mathbb{R}_{++}^k$. This last condition, however, can never hold because no function which is homogeneous of degree zero may be strictly (strongly) monotone on a cone C. This has discouraged researchers from using monotonicity techniques in the study of the general equilibrium problem (see, e.g., Mathiesen, 1987). We should emphasize, however, that all the results obtained above require only monotonicity, strict monotonicity, or strong monotonicity on D. It is clear that $-z(p)$ may be strictly (strongly) monotone on D even though it is not strictly monotone on C.

In order to test the monotonicity of $-z(p)$ on D, we let V be the $(k-1)$-dimensional subspace of $\mathbb{R}^k$ which is parallel to S^k, i.e.,

$$V=\left\{v \in \mathbb{R}^k: \sum_{i=1}^{k} v_i=0\right\}.$$

Then a sufficient condition for strict monotonicity on D is that

$$-v \cdot J(p) v>0, \quad v \in V,\ v \neq 0,\ p \in S_+^k. \tag{3.25}$$

A sufficient condition for strong monotonicity on D is that there is a positive number α such that

$$-v \cdot J(p) v \geqslant \alpha, \quad v \in V,\ |v|=1,\ p \in S_+^k. \tag{3.26}$$

The above may be nicely illustrated by means of the aggregate demand function obtained from a population of n individuals, each with a Cobb-Douglas utility function. The aggregate demand function is

$$z(p)=\sum_{j=1}^{n} z^j(p) \tag{3.27}$$

where

$$z_i^j(p)=\frac{1}{p_i} \alpha_i^j(w^j \cdot p)-w_i^j, \quad i=1, \ldots, k.$$

To show that $-z(p)$ is strongly monotone it suffices to show that for each fixed j, $-z^j(p)$ is strongly monotone. Therefore, without loss of generality, we may consider the case of a single consumer and examine the monotonicity of the function $z(p)$ with components

$$z_i(p)=\frac{a_i(w \cdot p)}{p_i}-w_i, \quad i=1, \ldots, k, \tag{3.28}$$

where $a \in S_+^k$ and $w \in \mathbb{R}_{++}^k$ are given vectors.

The Jacobian $J(p)$ is here given by

$$J_{ij}(p)=\frac{a_i w_j}{p_i}-\frac{a_i(w\cdot p)}{p_i^2}\delta_{ij} \tag{3.29}$$

where $\delta_{ij}=1$ if $i=j$, and $\delta_{ij}=0$ if $i\neq j$. Then

$$-v\cdot J(p)v=(w\cdot p)\sum_{i=1}^{k}\frac{a_i}{p_i^2}v_i^2-(w\cdot v)\sum_{i=1}^{k}\frac{a_i}{p_i}v_i. \tag{3.30}$$

With an eye on (3.26), we estimate from below the right-hand side of (3.30) for $v\in V$ such that $|v|=1$. In the first place, we note that

$$w\cdot p=\sum_{i=1}^{k}w_i p_i\geqslant \min_{1\leqslant i\leqslant k} w_i,\quad p\in S^k. \tag{3.31}$$

Next, we introduce the vector $e=(1,\ldots,1)$ in $\mathbb{R}^k$ and scalar

$$\omega=\frac{1}{k}w\cdot e=\frac{1}{k}\sum_{i=1}^{k}w_i. \tag{3.32}$$

Since $v\in V$ and $|v|=1$,

$$|w\cdot v|=|(w-\omega e)\cdot v|\leqslant|w-\omega e|. \tag{3.33}$$

Using Schwarz's inequality and since $a\in S_+^k$,

$$\sum_{i=1}^{k}\frac{a_i}{p_i}v_i\leqslant\left[\sum_{i=1}^{k}\frac{a_i}{p_i^2}v_i^2\right]^{1/2}. \tag{3.34}$$

Moreover,

$$\left[\sum_{i=1}^{k}\frac{a_i}{p_i^2}v_i^2\right]^{1/2}\geqslant\min_{1\leqslant i\leqslant k}\sqrt{a_i}. \tag{3.35}$$

Combining the above estimates, we obtain

$$-v\cdot J(p)v\geqslant\left[\sum_{i=1}^{k}\frac{a_i}{p_i^2}v_i^2\right]^{1/2}\left\{\left(\min_{1\leqslant i\leqslant k}\sqrt{a_i}\right)\left(\min_{1\leqslant i\leqslant k}w_i\right)-|w-\omega e|\right\}. \tag{3.36}$$

Therefore, if

$$\min_{1\leqslant i\leqslant k}a_i>\left[\sum_{i=1}^{k}(w_i-\omega)^2\right]\Big/\min_{1\leqslant i\leqslant k}w_i^2, \tag{3.37}$$

then it follows from (3.36) and (3.26) that the excess demand function given by (3.28) is strongly monotone on S_+^k. We note that (3.37) will hold when the endowment vector w forms a small angle with the vector e. In particular, (3.37) always holds when $w=\omega e$, i.e., when $w_1=\cdots=w_k=\omega$. As we shall see below, this 'symmetric' case has other special properties as well. Of course, (3.37) is just a sufficient condition for strong monotonicity.

It is interesting to consider the problem of whether the Walrasian equilibrium price vector minimizes a scalar function $\phi(p)$ on S^k. This will be the case when the projection of the Jacobian matrix $J(p)$ on the subspace V is symmetric. (As it was pointed out by one of the referees, when the aggregate demand function $z(p)$ is homogeneous of degree zero and satisfies Walras' law, the Jacobian $J(p)$ itself cannot be symmetric, unless $z(p)$ is identically zero.) The precise result is stated in the following theorem whose proof is standard.

Theorem 3.4. *Assume $z(p)$ is a continuously differentiable function on S^k whose Jacobian matrix $J(p)$ satisfies the symmetry condition*

$$s\cdot J(p)r = r\cdot J(p)s, \quad p\in S^k, \; s, r\in V. \tag{3.38}$$

Then there is a real-valued, twice continuously differentiable function $\phi(p)$ on S^k such that

$$\partial_r\phi(p) = -r\cdot z(p), \quad p\in S^k, \; r\in V, \tag{3.39}$$

where ∂_r denotes differentiation in the direction r. Any minimum p^* of $\phi(p)$ on S^k is a Walrasian price equilibrium. Moreover, $-z(p)$ is (strictly, strongly) monotone on S^k if and only if $\phi(p)$ is (strictly, uniformly) convex on S^k. When $\phi(p)$ is convex, then $p^*\in S^k$ is a Walrasian price equilibrium if and only if p^* is a minimum of $\phi(p)$ on S^k. □

As an illustration of the above, consider again the aggregate excess demand function given by (3.27). Again, it is clear that (3.38) will hold for the Jacobian of $z(p)$ if it holds for the Jacobians of $z^j(p)$, $j=1,\ldots,n$. Therefore, as before, it suffices to consider, without loss of generality, the case of a single consumer (3.28). Using the expression (3.29) for $J(p)$ we find

$$s\cdot J(p)r = (w\cdot r)\left[\sum_{i=1}^{k}\frac{a_i}{p_i}s_i\right] - (w\cdot p)\left[\sum_{i=1}^{k}\frac{a_i}{p_i^2}s_i r_i\right]. \tag{3.40}$$

We now note that the second term on the right-hand side of (3.40) is symmetric in (s, r), for all $p\in S_+^k$. On the other hand, the first term on the right-hand side of (3.40) may be symmetric in (s, r) only if it is identically zero, i.e., only if $w\cdot r=0$ for all $r\in V$, i.e., only if the components of w are equal, $w_1=\cdots=w_k=\omega$. So, it is only in that case that (3.39) may hold for some function $\phi(p)$. In fact, it holds for

$$\phi(p) = -w\sum_{i=1}^{k} a_i \log p_i + \omega p\cdot e. \tag{3.41}$$

In this case the unique Walrasian equilibrium p^* is given by

$$p_i^* = a_i, \quad i=1,\ldots,k. \tag{3.42}$$

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