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# Dines-Fourier-Motzkin quantifier elimination and an application of corresponding transfer principles over ordered fields

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A constructive procedure using Dines-Fourier-Motzkin elimination is given for eliminating quantifiers in a linear first order formula over ordered fields. An ensuing transfer principle is illustrated by showing that a locally one-to-one affine map is globally one-to-one and onto all over ordered fields.

*Key words*: Quantifier elimination, ordered fields, Fourier-Motzkin elimination, linear inequalities.

## 1. Introduction

Let $\varphi$ be a statement constructed by variables, integers, the functions "addition" and "multiplication", the predicates "equality" and "greater than", the logical connectives "negation", "and" and "or" and the logical quantifiers "for all" and "for some". Such a statement is called a *first order formula for ordered fields*, or more briefly, herein, a formula. We call a formula *quantifier-free* or *linear* if there are no quantifiers, or if all quantified variables are linear with respect to each other, respectively.

A constructive, short and elementary proof of quantifier elimination for linear formulae over ordered fields using Dines-Fourier-Motzkin elimination is given (cf. Dines, 1918-1919; Fourier, 1826; Motzkin, 1936). Also, an example illustrating the use of an ensuing transfer principle is developed. In the example we require, in addition, the use of Tarski's (1951) quantifier elimination and transfer principle over real closed fields.

Van den Dries (1981) has shown, using a model theoretic test, that validity of a linear formula over one ordered field implies its validity over all ordered fields. He

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then concludes, again using model theoretic arguments, that linear first order formulae over ordered fields are equivalent to quantifier-free formulae. Weispfenning (1988), "substituting Skolem terms," has shown that quantifier elimination for linear formulae can be executed in doubly exponential time and space. For a survey of recent results in quantifier elimination see Grigor'ev (1986).

The development herein parallels the classical avenue of Tarski. The proof of Tarski's Theorem is based on Sturm's Theorem, whereas ours is based on Dines-Fourier-Motzkin elimination. In turn, Sturm's Theorem is based on mean value existence for polynomials which is valid in real closed fields whereas Dines-Fourier-Motzkin elimination is based on mean value existence for linear functions which is valid in all ordered fields. Our construction is weaker than Tarski's in that we eliminate the quantifiers only when the quantified variables are linear with respect to each other, on the other hand we do so over all ordered fields including those that are not real closed. Further, Dines-Fourier-Motzkin elimination is constructive, in that when one eliminates a quantifier "for some" one can get an explicit representation of an element satisfying the concerned statement, when such exists. However, the conclusion of Sturm's Theorem is existential, and such explicit identification is in general not available; for example, Sturm's Theorem will assure that the statement

$$\exists x \quad (x^7 - 5x^6 + 2x^5 + x^4 - 2x^3 + x^2 - 10x - 19 = 0)$$

is equivalent to the statement $1 = 1$ over all real closed fields, i.e. it is true over all real closed fields, but neither Sturm's nor Tarski's method can be used to obtain an element that satisfies the above polynomial equation. However, Sturm's Theorem does suggest a procedure for approximating a root of the equation.

Our efforts here are motivated by our study of piecewise affine maps over the rational field denoted $Q$ and over the real field augmented to include an infinitesimal, denoted $R(\omega)$ (see Eaves and Rothblum, 1987, 1989). In particular, our interest focuses on computation with piecewise affine functions and on drawing conclusions for restrictions and extensions of real piecewise affine functions to $Q$ and $R(\omega)$, respectively. We note that $Q$ and $R(\omega)$ are neither complete nor real closed. For example, Eaves and Rothblum (1987) show how to construct invariant polynomial curves of certain piecewise affine maps; Kohlberg (1980) had obtained sufficient conditions for existence of invariant rays, a special type of invariant polynomial curves. Other typical questions occuring regarding piecewise affine maps over $R(\omega)$ are as follows: Given a piecewise affine map $f: R^n \to R^n$, can $f$ be extended to a piecewise affine function $\bar{f}: R(\omega)^n \to R(\omega)^n$? Can $\bar{f}$ be constructed so that it inherits from $f$ properties like locally one-to-one, globally one-to-one or onto? These questions can be cast as statements about the data of the map $f$ where all the quantified variables are linear with respect to each other. So, these statements can be converted to quantifier-free statements and their validity over $R$ implies their validity over $R(\omega)$. Hence these questions are answered in the affirmative (see Eaves and Rothblum, 1989, for further details). Here we consider one example of the use of the transfer principle by showing that a piecewise affine map that is locally one-to-one must be globally one-to-one and onto over all ordered fields. This fact is known for the real field, and we use the transfer principle together with Tarski's to prove the result for any ordered field. Further applications of this type of the transfer principles can be found in Eaves and Rothblum (1989). In a broader sense, given a fact true over a real vector space there is always the question of does it remain true over, say, a rational vector space. The transfer principle for linear formulae offers an important device for addressing such questions.

We introduce background material in Section 2 and state and prove the quantifier elimination result and transfer principles in Section 3. Finally, in Section 4 we illustrate the use of the ensuing transfer principle.

## 2. Preliminaries

Let $\mathscr{L}$ be a first order language with equality symbol $=$ whose parameters are three constant symbols $-1$,0 and 1, two two-place function symbols $\cdot$ and $+$ and a two-place predicate symbol $>$. Let the set of variables for the language be $\{x, y, z, x_1, x_2, \ldots, y_1, y_2, \ldots, z_1, z_2, \ldots)$. This language is called *the language of ordered fields.* As usual, for readability, we write $x+y$ for $+(x, y)$ and $x \cdot y$ or $xy$ for $\cdot(x, y)$. We also use the notation $x^n$ to denote $x$ when $n=1$ and (inductively) $xx^{n-1}$ when $n=2, 3, \ldots$.

Terms of the language $\mathscr{L}$ are called *polynomials*, i.e. polynomials are expressions generated from the constant symbols and variables by the functions $\cdot$ and $+$. *Atomic formulae of* $\mathscr{L}$ are obtained via the application of the predicate symbols $=$ and $>$ to pairs of polynomials, e.g. given polynomials $p$ and $q$, $(p=q)$ and $(p>q)$ are atomic formulae. Finally, (*well formed*) *formulae of* $\mathscr{L}$ are built up from atomic formulae via the finite application of the connectives $\neg$ (negation), $\wedge$ (and), $\vee$ (or) and the quantifiers $\forall$ (for every) and $\exists$ (there exists) as follows. All atomic formulae of $\mathscr{L}$ are formulae of $\mathscr{L}$, and if $\varphi$ and $\psi$ are formulae of $\mathscr{L}$ then so are $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\forall x\varphi$ and $\exists x\ \varphi$, where $x$ is any variable. For example, $\varphi$ defined by

$$\varphi: \quad \forall y\ \exists x\ \ ((y<0) \vee ((x^2=y) \vee (z>0))) \tag{2.1}$$

is a formula of $\mathscr{L}$. Since the language $\mathscr{L}$ is the only one we consider in the paper, we will simply refer throughout to *atomic formulae* and *formulae* without reference to $\mathscr{L}$. Variables occuring in a formula captured by a quantifier $\forall$ or $\exists$ are called *quantified variables*, otherwise *free variables.* In the formula $\varphi$ in (2.1), $x$ and $y$ are quantified variables and $z$ is a free variable. If a formula has no variables, no free variables, or no quantifiers we call it *variable-free*, *closed*, or *quantifier-free*, respectively.

A *structure* $\mathscr{F}$ for $\mathscr{L}$ is a tuple consisting of a nonempty set $F$, called the *universe* of $\mathscr{F}$ three elements in $F$, two binary functions on $F$ and a binary predicate where these objects correspond to the parameters $(-1, 0, 1, \cdot, +, >)$ of $\mathscr{L}$. The elements in the universe of a structure corresponding to $-1$, 0 and 1 are called *negative unity*, *zero* and *unity*, respectively, the functions corresponding to $\cdot$ and $+$ are called *multiplication* and *addition*, respectively, and the predicate corresponding to $>$ is called *greater than*. As usual, we do not distinguish between the notation for symbols of the language and the corresponding assignments of the structure, indeed, we denote a structure for $\mathscr{L}$ by $\mathscr{F}=(F,-1,0,1,\cdot,+,>)$. Also, we sometimes identify a structure with its universe. Finally, the outcomes of the applications of successive addition operations or successive multiplication operations are called, respectively, *sums* or *products*.

Given a formula $\varphi$, a structure $\mathscr{F}$ for $\mathscr{L}$ and an assignment of elements in the universe of $\mathscr{F}$ to the variables, $\varphi$ is either *satisfied* or *not satisfied* in $\mathscr{F}$. A formal definition of satisfiability requires an inductive argument. In particular, given a structure $\mathscr{F}$, a closed formula $\sigma$ is satisfied under all assignments of elements of the universe of $\mathscr{F}$ to the variables if and only if $\sigma$ is satisfied by one such assignment and in this case we say that the closed formula $\sigma$ *is true over* $\mathscr{F}$. Of course, a closed formula can be true over one ordered field but false over another. For example, the closed formula $\forall y\, \exists x\, ((y<0) \vee (x^2=y))$ is true over the reals but false over the rationals, and vice versa for the negation. Given two formulae $\varphi$ and $\psi$ and a structure $\mathscr{F}$ for $\mathscr{L}$, we say that $\varphi$ and $\psi$ are *equivalent over* $\mathscr{F}$ if $((\neg\varphi \vee \psi) \wedge (\varphi \vee \neg\psi))$ is satisfied by all assignments of elements of the universe of $\mathscr{F}$ to the variables.

Let $\Gamma$ be a set of formulae and let $\varphi$ be a formula. We say that $\Gamma$ *logically implies* $\varphi$, written $\Gamma \vDash \varphi$, if for every structure for the language $\mathscr{L}$ and every assignment of elements of the universe of that structure to the variables we have that $\varphi$ is satisfied whenever all the members of $\Gamma$ are satisfied. We say that two formulae $\varphi$ and $\psi$ are *logically equivalent* if and only if

$$\varphi \vDash \psi \quad \text{and} \quad \psi \vDash \varphi,$$

where, as usual, a singleton is represented by its element.

By renaming quantified variables in a formula, if necessary, we may and do assume that the variables corresponding different quantifiers are (pairwise) disjoint and that the sets of free and quantified variables are also disjoint. For example, given the formula $(x>0) \vee (\exists x\, (x^2>2)) \vee (\forall x\, (x^2<2)))$ we rename the quantified $x$'s to $y$ and $z$, respectively, to get the logically equivalent formula $((x>0) \vee (\exists y\, (y^2>2)) \vee (\forall z\, (z^2<2)))$. Henceforth, we drop the formality in the use of parenthesis and use them only to enhance readability. Also, sometimes, when a finite set of formulae $\{\varphi_i\colon i \in \mu\}$ is given, we write $\bigwedge_{i\in\mu} \varphi_i$ or $\bigvee_{i\in\mu} \varphi$ for the corresponding formula obtained from successive use of $\wedge$ or $\vee$, respectively.

A structure $\mathscr{F}$ for $\mathscr{L}$ with universe $F$ is called an *ordered field* if 0 and 1 are distinct, 0 is the addition identity, 1 is the multiplication identity, $-1$ is the additive inverse of 1, addition and multiplication are both associative and commutative, multiplication is distributive over addition, all elements in $F$ have an additive inverse, all nonzero elements in $F$ have a multiplicative inverse, and greater than is a total order that is preserved under addition and under multiplication by positive elements, the latter means that for $x$, $y$ and $z$ in $F$ with $x>y$ one has that $x+z>y+z$ and if $z>0$ then $xz>yz$. These requirements can be cast as a closed formula in the language $\mathscr{L}$ and we denote such a closed formula by $\Pi$. We say that two formulae $\varphi$ and $\psi$ are *ordered field equivalent* if and only if

$$\Pi, \varphi \models \psi \quad \text{and} \quad \Pi, \psi \models \varphi.$$

Of course, two formulae that are logically equivalent are also ordered field equivalent.

We say that an ordered field $\mathscr{H}$ is an *extension* of an ordered field $\mathscr{F}$ if the universe of $\mathscr{H}$ contains the universe of $\mathscr{F}$ and addition, multiplication and greater than in $\mathscr{F}$ are, respectively, the restriction of addition, multiplication and greater than in $\mathscr{H}$. We note that every ordered field is an extension of an (isomorphic) copy of the rational ordered field $Q$.

We say that an ordered field $\mathscr{F}$ with universe $F$ is *real closed* if the following two conditions hold (see Jacobson, 1964, pp. 273–277):

(a) Every positive element in $F$ has a square root.

(b) Any polynomial of odd degree with coefficients in $F$ has a root in $F$.

We observe that the statement that an ordered field is real closed can be cast as a countable set of closed formulae in the language $\mathscr{L}$ and we denote by $\Sigma$ such a (countable) set of closed formulae. We say that two formulae $\varphi$ and $\psi$ are *real closed field equivalent* if and only if

$$\Sigma, \Pi, \varphi \models \psi \quad \text{and} \quad \Sigma, \Pi, \psi \models \varphi.$$

Call a polynomial *simple* if it is the sum of products of constants and variables, i.e. if one can first execute all multiplications and then execute the additions. An atomic formula is defined to be *simple* if it has the form $(p = q)$ or $(p > q)$ where $p$ and $q$ are simple polynomials. Clearly, every atomic formula is ordered field equivalent to a simple atomic formula. Next observe that as $\neg((p > q)$ is ordered field equivalent to $(((-1)p > q) \vee (p = q))$ and $\neg(p = q)$ is ordered field equivalent to $((p > q) \vee ((-1)p > q))$, we get from standard arguments, e.g. Tarski (1951), that every quantifier-free formula can be rearranged to obtain an ordered field equivalent formula having the form

$$\bigvee_{i \in \lambda} \bigwedge_{j \in \eta_i} \varphi_j$$

where $\lambda$ and the $\eta_i$'s, $i \in \lambda$, are finite (index) sets and where each $\varphi_j$ is a simple atomic formula. A formula having the latter form is said to be in *disjunctive-conjunctive-simple form.* A formula $\varphi$ of form $Q_1 x_1, \ldots, Q_n x_n\ \psi$ where each $Q_i$ represents $\forall$ or $\exists$ and where $\psi$ is in disjunctive-conjunctive-simple form is defined to be in *quantifier-disjunctive-conjunctive-simple form.* Standard arguments (e.g. Tarski, 1951) show that any formula can be rearranged to obtain an ordered field equivalent formula which is in quantifier-disjunctive-conjunctive-simple form, with the same set of quantifiers and corresponding quantified variables.

Let $W$ be a finite set of variables, say $W = \{x_1, \ldots, x_n\}$. A polynomial $p$ is defined to be *linear in* $W$ if it has the form $p_1 x_1 + \cdots + p_n x_n$ where $p_1, \ldots, p_n$ are polynomials that do not contain the variables $x_1, \ldots, x_n$. An atomic formula is defined to be *linear in* $W$ if it has the form $p = q$ or $p > q$ where $p$ is a polynomial that is linear in $W$ and $q$ is a polynomial that does not contain the variables of $W$. For example, the atomic formula $x_1x_2+x_2x_3>1$ is linear in $\{x_1, x_3\}$ and in $\{x_2\}$ but not in $\{x_1, x_2, x_3\}$. A formula is defined to be *linear in W* if all of its atomic formulae are linear in $W$. We observe that the transformation of a formula into an ordered field equivalent formula that is in quantifier-disjunctive-conjunctive-simple form can be executed while preserving linearity in $W$. Thus, we conclude that if $\varphi$ is a formula that is linear in $W$, then there exists a formula $\psi$ that is in quantifier-disjunctive-conjunctive-simple form, has the same set of quantifiers and corresponding quantified variables as $\varphi$, is linear in $W$ and is ordered field equivalent to $\varphi$. Finally, a formula $\varphi$ is defined to be *linear* (without reference to a set of variables) if $\varphi$ is linear in the set of its quantified variables. A formula $\varphi$ is defined to be *universal-linear* if it has the form $\forall x_1 \cdots \forall x_n\, \varphi$ where $\varphi$ is a linear formula.

## 3. Quantifier elimination

In this section we obtain our main result concerning the elimination of quantifiers. For readability we use, as usual, the notation $(\varphi \to \psi)$ for $((\neg\varphi) \vee \psi)$ when $\varphi$ and $\psi$ are formulae. Also, given polynomials $p$ and $q$, we write $(p \neq q)$ for $(p>q) \vee (q>p)$ and we write $(p<q)$ for $(q>p)$. Finally, for every finite set $\mu$, let $\mu^*$ denote the set of all partitions of $\mu$ into three pairwise disjoint subsets, denoted $\mu_-$, $\mu_0$ and $\mu_+$, i.e.

$$\mu^* \equiv \{(\mu_-, \mu_0, \mu_+)\colon \mu_- \cup \mu_0 \cup \mu_+ = \mu,\ \mu_- \cap \mu_0 = \mu_0 \cap \mu_+ = \mu_- \cap \mu_+ = \emptyset\}.$$

The next lemma, based on Dines–Fourier–Motzkin elimination, represents the critical step for our quantifier elimination procedure.

**Lemma.** *Let $\varphi$ be the formula*

$$\varphi\colon\quad \exists x \left(\left(\bigwedge_{i\in\mu} (p_i x = q_i)\right) \wedge \left(\bigwedge_{i\in\nu} (p_i x > q_i)\right)\right),$$

*where $\mu$ and $\nu$ are finite disjoint sets and where each $p_i$ and each $q_i$ is a polynomial in which the variable $x$ does not occur, and let $\psi$ be the formula*

$$\begin{aligned}
\psi\colon \bigwedge_{(\mu_-,\mu_0,\mu_+)\in\mu^*,(\nu_-,\nu_0,\nu_+)\in\nu^*} &\left[\left(\bigwedge_{i\in\mu_-\cup\nu_-} (p_i<0)\right) \wedge \left(\bigwedge_{i\in\mu_0\cup\nu_0} (p_i=0)\right) \wedge \left(\bigwedge_{i\in\mu_+\cup\nu_+} (p_i>0)\right)\right] \\
\to &\left[\left(\bigwedge_{i\in\mu_0} (q_i=0)\right) \wedge \left(\bigwedge_{i\in\nu_0} (q_i<0)\right)\right. \\
&\wedge \left(\bigwedge_{i\in\mu_+, j\in\nu_-\cup\nu_+} (p_j q_i > p_i q_j)\right) \wedge \left(\bigwedge_{i\in\mu_-, j\in\nu_-\cup\nu_+} (p_j q_i < p_i q_j)\right) \\
&\left.\wedge \left(\bigwedge_{i\in\nu_-, j\in\nu_+} (p_j q_i < p_i q_j)\right) \wedge \left(\bigwedge_{i,j\in\mu_-\cup\mu_+} (p_j q_i = p_i q_j)\right)\right]
\end{aligned}$$

*which does not contain the variable x. Then $\varphi$ and $\psi$ are ordered field equivalent.*

**Proof.** We establish the ordered field equivalence of $\varphi$ and $\psi$ by using the Dines-Fourier-Motzkin elimination that characterizes solvability of a system of linear equations and inequalities in a single variable, here $x$, via necessary and sufficient conditions on the coefficients. Our proof is somewhat informal and its main purpose is to explain why the conditions of $\psi$ characterize solvability of the linear system that appears in $\varphi$ rather than exercise the formalities of logical implications.

Throughout this proof let $\mathscr{F}$ be a (fixed) ordered field, i.e. a structure for the language $\mathscr{L}$ that satisfies $\Pi$. Also, let a (fixed) assignment of variables to elements of the universe $F$ of $\mathscr{F}$ be given. We will show that $\varphi$ is satisfied by this assignment if and only if $\psi$ is also. Of course, once the specific ordered field and the specific assignment are given we may evaluate polynomials and divide by nonzero elements. Henceforth, in this proof, we let $p_i$ and $q_i$, $i \in \mu \cup \nu$, represent the evaluations of the polynomials $p_i$ and $q_i$ under the given assignment, no confusion should occur. Further, for $\lambda = \mu$ or $\lambda = \nu$, let

$$\lambda'_- = \{i \in \lambda : p_i < 0\}, \qquad \lambda'_0 = \{i \in \lambda : p_i = 0\} \quad \text{and} \quad \lambda'_+ = \{i \in \lambda : p_i > 0\}.$$

First assume that $\varphi$ is satisfied and that $\alpha$ is an element in $F$ that satisfies $p_i\alpha = q_i$ for all $i \in \mu$ and $p_i\alpha > q_i$ for all $i \in \nu$. Now, if $i \in \mu'_0$ then $p_i = 0$ and therefore

$$0 = p_i\alpha = q_i, \tag{3.1}$$

and if $i \in \nu'_0$ then $p_i = 0$ and therefore

$$0 = p_i\alpha > q_i. \tag{3.2}$$

Also, for $i \in \mu$ and $j \in \nu'_- \cup \nu'_+$ we have from $p_i\alpha = q_i$ that $p_jp_i\alpha = p_jq_i$; hence, if $i \in \mu'_-$ then from $p_j\alpha > q_j$ and $p_i < 0$ we have that $p_ip_j\alpha < p_iq_j$ and therefore

$$p_jq_i = p_jp_i\alpha = p_ip_j\alpha < p_iq_j, \tag{3.3}$$

and if $i \in \mu'_+$ then from $p_j\alpha > q_j$ and $p_i > 0$ we have that $p_ip_j\alpha > p_iq_j$ and therefore

$$p_jq_i = p_jp_i\alpha = p_ip_j\alpha > p_iq_j. \tag{3.4}$$

Next, for $i \in \nu'_-$ and $j \in \nu'_+$ we have from $p_i\alpha > q_i$ and $p_j > 0$ that $p_jp_i\alpha > p_jq_i$ and from $p_j\alpha > q_j$ and $p_i < 0$ we also have that $p_ip_j\alpha < p_iq_j$, so,

$$p_jq_i < p_jp_i\alpha = p_ip_j\alpha < p_iq_j. \tag{3.5}$$

Finally, for $i, j \in \mu'_- \cup \mu'_+$ we have from $p_i\alpha = q_i$ and $p_j\alpha = q_j$ that $p_jp_i\alpha = p_jq_i$ and $p_ip_j\alpha = p_iq_j$, implying that

$$p_jq_i = p_jp_i\alpha = p_ip_j\alpha = p_iq_j. \tag{3.6}$$

Evidently, (3.1)-(3.6) establishes the one implication of $\psi$ with $(\mu_-, \mu_0, \mu_+, \nu_-, \nu_0, \nu_+) = (\mu'_-, \mu'_0, \mu'_+, \nu'_-, \nu'_0, \nu'_+)$. Of course, all other implications of $\psi$ are satisfied vacuously as the premise of this implications is not satisfied. So, all of the implications of $\psi$ are satisfied, completing the proof that $\psi$ is satisfied whenever $\varphi$ is.

Next assume that $\psi$ is satisfied and we will show that in this case $\varphi$ is satisfied as well, i.e. for some $\alpha$ in $F$, $p_i\alpha = q_i$ for all $i \in \mu$ and $p_i\alpha > q_i$ for all $i \in \nu$. The definition of the set $\mu'_-$, $\mu'_0$, $\mu'_+$, $\nu'_-$, $\nu'_0$ and $\nu'_+$ and the fact that $\psi$ is satisfied assure that

$$q_i = 0 \quad \text{for } i \in \mu'_0, \tag{3.7}$$

$$q_i < 0 \quad \text{for } i \in \nu'_0, \tag{3.8}$$

$$p_j q_i > p_i q_j \quad \text{for } i \in \mu'_+ \text{ and } j \in \nu'_- \cup \nu'_+, \tag{3.9}$$

$$p_j q_i < p_i q_j \quad \text{for } i \in \mu'_- \text{ and } j \in \nu'_- \cup \nu'_+, \tag{3.10}$$

$$p_j q_i < p_i q_j \quad \text{for } i \in \nu'_- \text{ and } j \in \nu'_+, \tag{3.11}$$

$$p_j q_i = p_i q_j \quad \text{for } i, j \in \mu'_- \cup \mu'_+. \tag{3.12}$$

Our proof of the fact that $\varphi$ is satisfied is separated into two cases.

*Case I.* $\mu'_- \cup \mu'_+ \neq \emptyset$: we consider only the case where $\mu'_- \neq \emptyset$ as the alternative case where $\mu'_- = \emptyset$ has $\mu'_+ \neq \emptyset$, and similar arguments apply. So, let $j \in \mu'_-$. Let $\alpha = q_j/p_j$. Then, for each $i \in \mu'_- \cup \mu'_+$ (3.12) implies that $p_j q_i = p_i q_j$ and therefore

$$p_i\alpha = p_i q_j / p_j = p_j q_i / p_j = q_i.$$

Also, (3.7) implies that for $i \in \mu'_0$,

$$p_i\alpha = 0\alpha = 0 = q_i, \tag{3.13}$$

and (3.8) implies that for $i \in \nu'_0$,

$$p_i\alpha = 0\alpha = 0 > q_i. \tag{3.14}$$

Finally, for $i \in \nu'_- \cup \nu'_+$, (3.10) and the fact that $p_j < 0$ (as $j \in \mu'_-$) imply that

$$p_i\alpha = p_i q_j / p_j > p_j q_i / p_j = q_i.$$

So, indeed $\varphi$ is satisfied.

*Case II.* $\mu'_- \cup \mu'_+ = \emptyset$: In this case (3.11) assures that $q_i/p_i > q_j/p_j$ for each $i \in \nu'_-$. and $j \in \nu'_+$. Thus, there exists an $\alpha$ in $F$ with

$$q_i/p_i > \alpha > q_j/p_j \quad \text{for all } i \in \nu'_- \text{ and } j \in \nu'_+.$$

In particular, for $i \in \nu'_-$ we conclude that $q_i < p_i\alpha$ and for $j \in \nu'_+$ we have that $p_j\alpha > q_j$. Also, (3.13) and (3.14) still apply and show that $p_i\alpha > q_i$ for $i \in \nu'_0$ and $p_i\alpha = q_i$ for $i \in \mu'_0$. As $\mu'_- \cup \mu'_+ = \emptyset$ we conclude that $\varphi$ is satisfied. □

Notice that $\varphi$ in the lemma may contain variables other than $x$, that is, one might regard the system, and the solution, as parametric. We have eliminated the existential quantifier of $x$ using the Dines–Fourier–Motzkin variable elimination method, and note, for example, that one could equally well base the quantifier elimination on the use of Cramer's rule via reasoning found in the theory of linear programming (e.g. Dantzig, 1963). On the other hand, it is doubtful that elipsoidal methods such as Kachian (1979) or Karmarkar (1984) can be adopted for the quantifier elimination.

**Theorem.** *Let $\varphi$ be a linear formula. Then one can construct, using Dines–Fourier–Motzkin elimination, a quantifier-free formula $\psi$ which is ordered field equivalent to $\varphi$.*

**Proof.** If a formula is linear and is in quantifier-disjunction-conjunctive-simple form, perhaps with a negation in front, we shall define it to be in *normal* form. Without loss of generality, we may assume that our initial formula is in normal form (see Tarski, 1951); then as we eliminate quantifiers we shall maintain the normal form, that is, the generated intermediate formulae will be in normal form. Let $\varphi$ be in normal form, that is, in the form

$$\text{(N)} \qquad \varphi\colon \quad Q_0 Q_1 x_1 \cdots Q_m x_m \bigvee_{i=1}^{k} \bigwedge_{j\in\eta_i} \varphi_j$$

where $Q_0$ is $\neg$ or nothing, where $Q_1, \ldots, Q_m$ are quantifiers, the $\eta_i$'s are finite index sets and each $\varphi_j$ is a simple atomic formula. Our procedure for eliminating quantifiers consists of repeated application of two operations, namely $F_\exists$ and $F_\forall$ defined below:

$F_\exists$ – *Innermost existential elimination*: Given a formula $\varphi$ in normal form whose inner most quantifier is existential, that is, $Q_n$ is $\exists$, then $F_\exists$ eliminates that quantifier as in the Lemma and puts the formula into normal form. The new formula is denoted $F_\exists(\varphi)$.

$F_\forall$ – *Innermost universal conversion*: Given a formula $\varphi$ in normal form whose inner most quantifier is universal, that is, $Q_n$ is $\forall$, then, $F_\forall$ converts this formula into a formula in normal form whose innermost quantifier is existential. The new formula is denoted $F_\forall(\varphi)$.

We are now ready to define $F_\exists(\varphi)$ where $\varphi$ is a formula in normal form where $Q_n$ is $\exists$. First $\exists x_m \bigvee_{i=1}^{k} \bigwedge_{j\in\eta_i} \varphi_j$ is exchanged with the logically equivalent formula $\bigvee_{i=1}^{k} \exists x_m \bigwedge_{j\in\eta_i} \varphi_j$. The Lemma is then applied to each $\exists x_m \bigwedge_{j\in\eta_i} \varphi_j$ to obtain an ordered field equivalent formula having the form

$$\bigwedge_{j\in\eta_i^*} (\lambda_j \to \xi_j),$$

where as before $\eta_i^*$ is the set of partitions of $\eta_i$ into three pairwise disjoint subsets. Let $\#(\eta_i)$ indicate the number of elements in the set $\eta_i$. Then, each $\lambda_j$ is the conjunction of $\#(\eta_i)$ simple atomic formula and each $\xi_j$ is the conjunction of at most $\#(\eta_i)^2$ simple atomic formulae. The formula $\lambda_j \to \xi_j$ is logically equivalent to $\xi_j \wedge (\neg\lambda_j)$. Each $\neg\lambda_j$ is ordered field equivalent to the disjunction of $2\#(\eta_i)$ simple atomic formula. Thus $\lambda_j \to \xi_j$ is ordered field equivalent to a formula having the form

$$\rho_i: \quad \left( \bigwedge_{h=1}^{\#(\eta_i)^2} \xi_h^i \right) \vee \left( \bigvee_{h=1}^{2\#(\eta_i)} \lambda_h^i \right)$$

where each $\lambda_h^i$ and $\xi_h^i$ are simple atomic formula. These formulae $\rho_i$ are next converted to ordered field equivalent quantifier-free formulae $\rho_i'$ in disjunctive-conjunctive-simple-form and $F_\exists(\varphi)$ is defined to be $Q_0 Q_1, \ldots, Q_{m-1} \bigvee_{i=1}^k \rho_i'$.

We next define $F_\forall(\varphi)$ for a formula $\varphi$ that is in normal form where $Q_n$ is $\forall$. For a quantifier $Q$ define $\bar{Q}$ by $\bar{\exists} = \forall$ and $\bar{\forall} = \exists$ The formula $\varphi$ is logically equivalent to $\neg\neg\varphi$ and therefore also to

$$\bar{Q}_0 \bar{Q}_1 x_1 \cdots \bar{Q}_m x_m \bigwedge_{i=1}^{k} \bigvee_{j \in \eta_i} \neg\varphi_j,$$

where $\bar{Q}_0$ is nothing or $\neg$ as $Q_0$ is $\neg$ or nothing, respectively. Of course, $\bar{Q}_m$ is $\exists$. Let

$$\tau: \quad \bigwedge_{i=1}^{k} \bigvee_{j \in \eta_i} \neg\varphi_j.$$

This formula is next converted to ordered field equivalent quantifier-free formula $\tau'$ in disjunctive-conjunctive-simple form and $F_\forall(\varphi)$ is defined to be $\bar{Q}_0 \bar{Q}_1 x_1 \cdots \bar{Q}_m x_m \tau'$. In particular, $\bar{Q}_m$ is $\exists$.

Given $\varphi$ in normal form let $n$ be the number of switches in $Q_1, \ldots, Q_m$. Then, applying $F_\exists$ $m$ times and $F_\forall$ $n$ or $n+1$ times, we arrive at an ordered field equivalent quantifier-free formula $\psi$. □

The proof of the theorem shows that if a formula in quantifier-disjunctive-conjunctive-simple-form is linear in the variables corresponding to the innermost quantifiers then these quantifiers can be eliminated (see Corollary 3 below). However, the theorem and its proof do not imply that the formula $\forall y \, \exists x \, ((x^2 + x + y = 0) \vee (y > 0))$, which is linear in $\{y\}$, is ordered field equivalent to a formula having a single quantifier $\exists$ and no quantifier $\forall$.

We next obtain bounds on the size of the quantifier-free formula generated via the procedure outlined in the proof of the theorem. Perhaps the most natural measure of the size of a formula is its length where each variable, constant, multiplication, addition, equality, inequality, connective, quantifier and, maybe parenthesis, contributes a unit. However, for brevity and clarity we shall measure the size of a formula $\varphi$ by the following four parameters:

$\#_q(\varphi)$ = the number of quantifiers of $\varphi$,

$\#_a(\varphi)$ = the number of simple atomic formulae in $\varphi$,

$\#_t(\varphi)$ = the largest number of terms in a simple atomic formula of $\varphi$, and

$\#_p(\varphi)$ = the largest sum of the powers of variables and constants in terms of $\varphi$.

For a formula in normal form the number of connectives and pairs of parentheses is essentially the number of simple atomic formulae, and therefore, it is not necessary to count them separately.

Let $\varphi$ be a formula in normal form. We obtain bounds for the parameters of $F_\exists(\varphi)$ and $F_\forall(\varphi)$ in terms of the parameters of $\varphi$. First, the formulae $\rho_i'$ obtained in the construction of $F_\exists(\varphi)$ has

$$\#_a(\rho_i') \leq (2\#(\eta_i)+1)^{3^{\#(\eta_i)}}(\#(\eta_i))^2 3^{\#(\eta_i)}.$$

As $\sum_{i=1}^k \#(\eta_i) = \#_a(\varphi)$ we have

$$\#_a(F_\exists(\varphi)) \leq \sum_{i=1}^{k} \#_a(\rho_i') \leq (2\#_a(\varphi)+1)^{3\#_a(\varphi)} \#_a(\varphi)^3 \, e^{\#_a(\varphi)}.$$

Also, trivially, we have $\#_q(F_\exists(\varphi)) = \#_q(\varphi) - 1$, $\#_t(F_\exists(\varphi)) \leq (\#_t(\varphi))^2$, and $\#_p(F_\exists(\varphi)) \leq 2\#_p(\varphi)$. Next consider $F_\forall(\varphi)$. As each $\neg\varphi_j$ is ordered field equivalent to the disjunction of two simple atomic formula, we have that the quantifier-free formula $\tau'$ which is in disjunctive-conjunctive-simple form and is ordered field equivalent to $\tau: \bigvee_{i=1}^k \bigwedge_{j\in\eta_i} \neg\varphi_j$ has

$$\#_a(\tau') = k \prod_{i=1}^{k} (2\#(\eta_i)) = k2^k \prod_{i=1}^{k} \#(\eta_i).$$

As $\max\{\prod_{i=1}^k x_i : \sum_{i=1}^k x_i = a, x_i \geq 0\}$ is attained for $x_i = a/k$ for $i = 1, \ldots, k$ we have

$$\#_a(\tau') \leq k2^k[\#_a(\varphi)/k]^k \leq \#_a(\varphi)[2\#_a(\varphi)/k]^k.$$

A simple calculation shows that this expression is maximized by letting $k = 2\#_a(\varphi)\, e^{-1}$ resulting in the bound

$$\#_a(F_\forall(\varphi)) \leq \#_a(\varphi)\, e^{2\#_a(\varphi)/e} \leq \#_a(\varphi)\, e^{\#_a(\varphi)}.$$

Of course, we have that $\#_q(F_\forall(\varphi)) = \#_q(\varphi)$, $\#_t(F_\forall(\varphi)) = \#_t(\varphi)$, and $\#_p(F_\forall(\varphi)) = \#_p(\varphi)$. We conclude that given a linear formula $\varphi$, the ordered field equivalent quantifier-free formula $\psi$ constructed in the proof of the Theorem has $\#_a(\psi)$ bounded by a tower of exponentials in $\#_q(\varphi)$ and $\#_a(\varphi)$. The parameters $\#_t(\psi)$ and $\#_p(\psi)$ are exponential in $\#_q(\varphi)$ and polynomial in $\#_t(\varphi)$ and $\#_p(\varphi)$, respectively.

We next draw a number of transferability results from the Theorem. These results allow one to conclude validity or satisfiability in one ordered field from the knowledge of a corresponding fact in another ordered field.

**Corollary 1.** *Let $\sigma$ be a linear closed formula. Then $\sigma$ is true over one ordered field if and only if $\sigma$ is true over all ordered fields.*

**Proof.** By the Theorem, there exists a quantifier-free formula $\psi$ that is ordered field equivalent to $\varphi$. In particular, as $\varphi$ has no free variables neither does $\psi$. So, $\psi$ has no variables and the assertion whether or not it is satisfied can be determined independently of the underlying ordered field and the assignments of elements to variables. In particular, we conclude that $\psi$ is true over one ordered field if and only if it is true over all ordered fields. By the ordered field equivalence of $\varphi$ and $\psi$ the same conclusion holds for $\varphi$. □

**Corollary 2.** *Let $\varphi$ be a linear formula, let $\mathscr{F}$ be an ordered field and let $\mathscr{H}$ be an extension of $\mathscr{F}$. Suppose that $\varphi$ is satisfied under $\mathscr{F}$ and a given assignment of elements of the universe of $\mathscr{F}$ to the variables. Then $\varphi$ is satisfied under $\mathscr{H}$ and that assignment.*

**Proof.** By the Theorem, there exists a quantifier-free formula $\psi$ that is ordered field equivalent to $\varphi$. As $\psi$ is quantifier-free it immediately follows that if $\varphi$ is satisfied under $\mathscr{F}$ and a given assignment of elements of the universe of $\mathscr{F}$ to the variables, then $\psi$ is satisfied under $\mathscr{H}$ and that assignments. By the ordered field equivalence of $\varphi$ and $\psi$ the same conclusion holds for $\varphi$. □

**Corollary 3.** *Let $\sigma$ be a universal-linear closed formula. If $\sigma$ is true over one real closed field then $\sigma$ is true over all ordered fields.*

**Proof.** Let $\sigma$ have the form $\forall x_1 \cdots \forall x_n\, \varphi$, where $\varphi$ is a linear formula. The Theorem shows that $\varphi$ is ordered field equivalent to a quantifier-free formula, say $\psi$. Let $\xi$ be the closed formula $\forall x_1 \cdots \forall x_n\, \psi$. Then $\sigma$ and $\xi$ are ordered field equivalent. In particular, it suffices to show that if $\xi$ is true over one real closed field then $\xi$ is true over all ordered fields.

Suppose $\xi$ is true over one real closed field and let $\mathscr{F}$ be any ordered field. There exists a real closed field $\mathscr{H}$ which is an extension of $\mathscr{F}$ (see Jacobson, 1964, 285). As Tarski's Transfer Principle (e.g. Tarski, 1951, or Seidenberg, 1954) assures that $\xi$ is true over all real closed fields, we conclude that $\xi$ is true over $\mathscr{H}$. This conclusion immediately implies that $\xi$ is true over $\mathscr{F}$ since $\xi$ is the closed formula $\forall x_1 \cdots \forall x_n\, \psi$ where $\psi$ is quantifier-free and the domain of the quantifiers $\forall$ is smaller when considered over $\mathscr{F}$ than when considered over $\mathscr{H}$ (see Eaves and Rothblum, 1987a, Theorem 7.3, for details). □

**Corollary 4.** *Let $\varphi$ and $\psi$ be linear formulae which are equivalent over some real closed field. Then they are ordered field equivalent.*

**Proof.** Apply Corollary 3 to the closed formula $\forall x((\varphi \to \psi) \wedge (\psi \to \varphi))$ where $x$ is the vector of free variables of $\varphi$ and $\psi$. □

Consider any procedure, such as Tarski's (1951) or Grigor'ev (1986), for example, for eliminating quantifiers over real closed fields. Given a linear formula $\varphi$ such a procedure would generate a quantifier-free formula $\psi$. From Corollary 4 we see that $\varphi$ and $\psi$ are ordered field equivalent.

The transfer principles permit one to determine validity of a statement in one ordered field by examining it in a more convenient ordered field as the reals where completeness is available, that is, where every set bounded from above has a least upper bound. We recall that the real field is the only complete ordered field.

## 4. An application

In this section we illustrate an application of the transfer principles. Specifically, we show that if a piecewise affine map is locally one-to-one over an ordered field then it is necessarily globally one-to-one and onto over the ordered field. We start with definitions of the concepts used in the above assertion.

Throughout this section we let $\mathcal{G}$ be an arbitrary ordered field with universe $G$. We will use matrix notation and matrix operations for expressions that involve elements from $G$ and variables. In particular, for brevity, we substitute elements of $G$ into formulae rather than considering a formula under particular assignments, e.g., we talk about validity of "$Ax \leq a$" rather than of "$Yx \leq y$ under assignments which map the elements of $A$ and $a$, to the variables of $Y$ and $y$ respectively".

A set $\sigma \subseteq G^n$ is called a *cell* if it has the representation $\sigma = \{u \in G^n\colon Au \leq a\}$ for some matrix $A \in G^{p\times n}$ and $a \in G^p$ where $p$ is some positive integer. In this case we say that $(p, A, a)$ is a *representation* of $\sigma$.

Let $k$ be a positive integer and let $\langle k\rangle$ denote the set of integers $\{1, \ldots, k\}$. A finite collection of cells $\{\sigma_1, \ldots, \sigma_k\}$ is called a *k-cover* of $G^n$ if $\bigcup_{i\in\langle k\rangle} \sigma_i \supseteq G^n$. We define $\{(p_i, A_i, a_i)\colon i \in \langle k\rangle\}$ to be a *representation* of such a $k$-cover if for each $i \in \langle k\rangle$, $(p_i, A_i, a_i)$ is a representation of $\sigma_i$. Evidently, given a vector $q \equiv (k, p_1, \ldots, p_k)$ of positive integers, matrices $A_i \in G^{p_i\times n}$ and vectors $a_i \in G^{p_i}$, $i \in \langle k\rangle$, the set $\{(p_i, A_i, a_i)\colon i \in \langle k\rangle\}$ is a representation of a $k$-cover if and only if

$$\varphi_1(q)\colon\quad \forall x \bigvee_{i\in\langle k\rangle} (A_i x \leq a_i)$$

is satisfied over $\mathcal{G}$.

A function of $f\colon G^n \to G^n$ is called a *piecewise affine map*, abbreviated a PA map, if for some positive integer $k$ there is a $k$-cover $\{\sigma_1, \ldots, \sigma_k\}$ of $G^n$, matrices $B_1, \ldots, B_k$ in $G^{n\times n}$ and vectors $b_1, \ldots, b_k$ in $G^n$ such that for each $u \in G^n$ contained in $\sigma_i$, where $i \in \langle k\rangle$, we have $f(u) = B_i u + b_i$. If $\{(p_i, A_i, a_i)\colon i \in \langle k\rangle\}$ is a representation of the $k$-cover $\{\sigma_1, \ldots, \sigma_k\}$ we say that $\{(p_i, A_i, a_i, B_i, b_i)\colon i \in \langle k\rangle\}$ is a *representation* of $f$. Of course, each PA map has many representations. Further, given a vector $q \equiv (k, p_1, \ldots, p_k)$ of positive integers, matrices $A_i \in G^{p_i\times n}$, $B_i \in G^{n\times n}$ and vectors $a_i \in G^{p_i}$ and $b_i \in G^n$, $i \in \langle k\rangle$, the set $\{(p_i, A_i, a_i, B_i, b_i)\colon i \in \langle k\rangle\}$ is a representation of some PA map if and only if $\{(p_i, A_i, a_i)\colon i \in \langle k\rangle\}$ is a $k$-cover of $G^n$ and

$$\varphi_2(q)\colon\quad \forall x \bigwedge_{i,j\in\langle k\rangle} (((A_i x \leq a_i) \wedge (A_j x \leq a_j)) \to (B_i x + b_i = B_j x + b_j))$$

is satisfied over $\mathcal{G}$. Of course, the PA map corresponding to such a representation is unique and is defined by $f(u) = B_i u + b$, for any vector $u \in G^n$ which is contained in $\sigma_i \equiv \{x \in G^n\colon A_i x \leq a_i\}$, $i \in \langle k\rangle$.

A PA map $f\colon G^n \to G^n$ is defined to be *one-to-one* if $f(u) \neq f(v)$ for distinct vectors $u$ and $v$ in $G^n$. Let $\| \;\|_\infty$ denote the $\ell_\infty$ norm over $G^n$, i.e. for $u \in G^n$, $\|u\|_\infty = \max_{i\in\langle n\rangle|} |u_i|$, where $|\;|$ denotes the usual absolute value. The map $f$ is defined to be *locally one-to-one* if for every $u \in G^n$ there is a positive $\varepsilon$ such that $f(v) \neq f(w)$ for every distinct pair $v$ and $w$ in $G^n$ that satisfy $\|v-u\|_\infty<\varepsilon$ and $\|w-u\|_\infty<\varepsilon$. The map $f$ is defined to be *onto* if for each vector $v$ in $G^n$ there is a vector $u$ in $G^n$ with $f(u)=v$. Given a vector $q\equiv(k, p_1, \ldots, p_k)$ of positive integers and a representation $\{(p_i, A_i, a_i, B_i, b_i): i\in\langle k\rangle\}$ of a PA map $f: G^n\to G^n$, then $f$ is one-to-one if and only if

$$\varphi_3(q):\quad \forall x\,\forall y\,((x\neq y)\to\left(\bigwedge_{i,j\in\langle k\rangle}((A_ix\leqslant a_i)\wedge(A_jy\leqslant a_j))\to(B_ix+b_i\neq B_jy+b_j)\right)$$

is satisfied over $\mathscr{G}$. Further $f$ is locally one-to-one if and only if

$$\varphi_4(q):\quad \forall x\,\exists\varepsilon\,\forall y\,\forall z\,[((\|y-x\|_\infty<\varepsilon)\wedge(\|z-x\|_\infty<\varepsilon)\wedge(y\neq z)]$$
$$\to\left[\bigwedge_{i,j\in\langle k\rangle}((A_iy\leqslant a_i)\wedge(A_jz\leqslant a_i))\to(B_iy+b_i\neq B_jy+b_j)\right].$$

is satisfied over $\mathscr{G}$. Finally, $f$ is onto if and only if

$$\varphi_5(q):\quad \forall y\,\exists x\left(\bigvee_{i\in\langle k\rangle}((A_ix\leqslant a_i)\wedge(B_ix+b_i=y))\right)$$

is satisfied over $\mathscr{G}$.

We cannot express by a single closed formula the assertion that a PA map which is locally one-to-one must be one-to-one and onto. However, we can express this assertion by a countable collection of linear closed formulae $\{\psi(q)\}$, where $q$ ranges over all tuples $(k, p_1, \ldots, p_k)$ of positive integers and $\psi(q)$ is given by

$$\psi(q):\quad \forall\{(A_i, a_i, B_i, b_i): i\in\langle k\rangle\}\,(\varphi_1(q)\wedge\varphi_2(q)\wedge\varphi_4(q))\to(\varphi_3(q)\wedge\varphi_5(q))$$

(here the $A_i$'s, $a_i$'s, $B_i$'s, and $b_i$'s have dimension $p_i\times n$, $p_i\times 1$, $n\times n$ and $n\times 1$, respectively, and denote variables rather than elements of the universe of the underlying ordered field). The explicit expressions for the $\varphi_j(q)$'s that we obtained earlier show that each $\psi(q)$ is a universal-linear sentence; thus, Corollary 3 implies that in order to establish that the $\psi(q)$'s are satisfied over all ordered fields it suffices to show that they are satisfied over the reals (which is a real closed field).

Recall that a PA map on the reals is globally one-to-one and onto if it is locally one-to-one. For example, Ortega and Rheinbolt (1970, Theorem 5.3.8, p. 136) and Artin and Braun (1969, Corollary 14.8, p. 156) imply that a map $f: R^n\to R^n$ is globally one-to-one and onto if it is continuous, is locally one-to-one and is coercive ($f$ is coercive if for each $\alpha\in R$ there exists some $\beta\in R$ such that $\|f(x)\|_\infty\geqslant\alpha$ whenever $\|x\|_\infty\geqslant\beta$). Now, a PA map is clearly continuous and the fact that it is coercive if it is locally one-to-one is elementary. So, the $\psi(q)$'s are true over the reals and therefore must be true over all ordered fields. Note that the proofs of cited results in Ortega and Rheinbolt and Artin and Braun rely on the completeness of the reals.

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