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Finding an interior point in the optimal face of linear programs

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We study the problem of finding a point in the relative interior of the optimal face of a linear program. We prove that in the worst case such a point can be obtained in $O(n^3L)$ arithmetic operations. This complexity is the same as the complexity for solving a linear program. We also show how to find such a point in practice. We report and discuss computational results obtained for the linear programming problems in the NETLIB test set.

Key words: Linear programming, primal–dual methods, optimal face, strict complementarity.

1. Introduction

Consider the primal linear program (LP):

$$\begin{array}{ll} \min & c^{\mathrm{T}}x \\ \text{s.t.} & Ax=b, \quad x\geqslant 0, \end{array}$$

and its dual (LD):

$$\begin{array}{ll} \max & b^{\mathrm{T}}y \\ \text{s.t.} & A^{\mathrm{T}}y+s=c, \quad s\geqslant 0, \end{array}$$

where $A\in\mathbb{R}^{m\times n}$, $c\in\mathbb{R}^n$, and $b\in\mathbb{R}^m$. Feasible solutions x^* and (y^*, s^*) are optimal for (LP) and (LD) respectively, if and only if,

$$x_j^* s_j^* = 0 \quad \text{for } j=1,2, ...,n.$$

Let $\sigma(x)$ represent the index set of positive components in $x\geqslant 0$, that is,

$$\sigma(x) = \{i\colon x_i > 0\}. \tag{1}$$

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Among all the optimal solutions for (LP) and (LD), there exists at least one optimal solution pair (x^*, s^*) which is strictly complementary, that is,

$$\sigma(x^*) \cap \sigma(s^*) = \emptyset \quad \text{and} \quad \sigma(x^*) \cup \sigma(s^*) = \{1, 2, \ldots, n\}. \tag{2}$$

This property has been known since the early days of linear programming (Goldman and Tucker [9]). Moreover, $\sigma(x^*)$ and $\sigma(s^*)$ remain invariant for every strictly complementary solution (x^*, s^*). Hence, we can denote $\sigma(x^*)$ by σ^* and let $\bar{\sigma}^* = \{1, \ldots, n\} \setminus \sigma^*$. The partition $\{\sigma^*, \bar{\sigma}^*\}$ of $\{1, 2, \ldots, n\}$ is called the optimal partition. One can further show that

$$\sigma(x^*) \subset \sigma^* \quad \text{and} \quad \sigma(s^*) \subset \bar{\sigma}^* \tag{3}$$

for every complementary solution (x^*, s^*). Thus, the optimal face for the primal is

$$\Theta_{\mathrm{p}} = \{x \colon Ax = b,\ x \geqslant 0,\ x_j = 0 \text{ for } j \in \bar{\sigma}^*\}, \tag{4}$$

and the one for the dual is

$$\Theta_{\mathrm{d}} = \{(y, s) \colon A^{\mathrm{T}} y + s = c,\ s_j = 0 \text{ for } j \in \sigma^*\}. \tag{5}$$

The primal optimal face has a point $x \in \Theta_{\mathrm{p}}$ with $x_j > 0$ for any $j \in \sigma^*$ and the dual optimal face has a point $(y, s) \in \Theta_{\mathrm{d}}$ with $s_j > 0$ for any $j \in \bar{\sigma}^*$. In other words, the relative interior (subsequently called interior) of these two faces is nonempty.

The standard complexity analyses for linear programming are based on finding an optimal primal and dual solution pair, i.e., a solution on Θ_{p} and a solution on Θ_{d}. This pair does not necessarily present enough information to determine the partition σ^* (unless the optimal solution is nondegenerate). Therefore, the identification of σ^* itself is an interesting combinatorial problem associated with (LP).

The ability to identify the optimal partition would lead to practical and more reliable termination criterion for interior-point algorithms. In this context, some of the ideas used in this paper were also used by Gay [7] to perform computational tests. If the LP is the Phase 1 problem to find a feasible solution, the knowledge of σ^* allows us to eliminate all the variables that are zero in every feasible solution. The information of σ^* and an interior point in the optimal face are also useful for the parametric analysis (e.g., see Adler and Monteiro [1]). Furthermore, this knowledge allows us to construct all the distinct optimal vertices by identifying all the feasible vertices on the optimal face. This motivates us to study the problem of finding the optimal partition and a pair of points in the interior of Θ_{p} and Θ_{d}. This work is related to the earlier work of Güler and Ye [11] and Ye [22] on the convergence behavior and finite termination of some interior-point algorithms.

To our knowledge, the previous "best" algorithm for finding the optimal partition is given by Freund et al. [6], who combine the primal and dual problems into a feasibility problem and then use Karmarkar's projective transformation to construct a homogeneous linear system and an artificial linear objective function. After solving this system, one can identify σ^* and a strict complementary solution for the original LP problem. Another approach is given by Tardos [19], who solves a sequence of (at most n) LP problems, where the size of the rational data of c in each problem is relatively small. (Her analysis does not work if the data are real numbers.)

This paper is organized as follows. In the next section we show that the optimal partition can be identified in $O(n^3L)$ arithmetic operations where the data in (LP) are rational and L is their input length. We also develop a practical and (column) scaling-independent criterion for identifying the optimal partition. In Section 3 we give a simplified method for finding an interior solution on the optimal face. Our approach is based on solving a system of linear equations, rather than the projection technique employed in [22]. Section 4 discusses computational results on the problems in the NETLIB test set [8]. Finally, we make some additional remarks in Section 5.

2. Finding the optimal partition in interior-point algorithms

In this section, we show that the optimal partition can be identified in $O(n^3L)$ arithmetic operations by several polynomial interior-point algorithms. We use the result of Güler and Ye [11] and Ye [22] in our analysis. Güler and Ye demonstrated that many $O(n^3L)$ interior-point algorithms (e.g., Gonzaga [10], Kojima et al. [12], Mizuno et al. [15], Monteiro and Adler [16], Renegar [17], Todd [20], Vaidya [21]) generate a sequence of feasible pairs (x^k, s^k) such that

$$\frac{\min(X^k s^k)}{(x^k)^{\mathrm{T}} s^k} > \Omega(1/n), \tag{6}$$

where $X^k = \mathrm{Diag}(x^k)$ and $\min(X^k s^k) = \min_i(x_i^k s_i^k)$.

The following theorem is an enhanced version of a theorem given in Ye [22], where he gave a finite convergence argument but no complexity result.

Theorem 1. *At iteration k of an $O(n^3L)$ interior-point algorithm whose iteration sequence satisfies inequality* (6), *let*

$$\sigma^k = \{j \colon x_j^k \geqslant s_j^k\}. \tag{7}$$

Then, in $O(n^3L)$ arithmetic operations

$$\sigma^k = \sigma^*.$$

Proof. First, for any $j \in \sigma^*$ there exists a complementary solution (x^*, s^*) for (LP) such that

$$x_j^* \geqslant \delta_1 \geqslant 2^{-L} \quad \text{and} \quad s_j^* = 0,$$

and for any $j \in \bar{\sigma}^*$ there exists a complementary solution (x^*, s^*) for (LP) such that

$$s_j^* \geqslant \delta_2 \geqslant 2^{-L} \quad \text{and} \quad x_j^* = 0$$

where δ_1 and δ_2 are fixed positive numbers (e.g., see Schrijver [18]). Second, since

$$(x^k - x^*)^{\mathrm{T}}(s^k - s^*) = 0$$

for any complementary solution (x^*, s^*), we have

$$\sum_{i\in\sigma(x^*)} x_i^* s_i^k + \sum_{i\in\sigma(s^*)} s_i^* x_i^k = (x^k)^{\mathrm{T}} s^k. \tag{8}$$

Thus, if $(x^k)^{\mathrm{T}} s^k < \mathrm{O}(1/n) 2^{-2L}$, then for any $j \in \sigma^*$,

$$s_j^k \leqslant \frac{(x^k)^{\mathrm{T}} s^k}{x_j^*} < \mathrm{O}(1/n) 2^{-2L}/2^{-L} = \mathrm{O}(1/n) 2^{-L}. \tag{9}$$

On the other hand, (8) can be written as

$$\sum_{i\in\sigma(x^*)} \frac{x_i^*}{x_i^k}(x_i^k s_i^k) + \sum_{i\in\sigma(s^*)} \frac{s_i^*}{s_i^k}(x_i^k s_i^k) = (x^k)^{\mathrm{T}} s^k.$$

Thus, we have for any $j \in \sigma^*$,

$$\frac{x_j^*}{x_j^k}(x_j^k s_j^k) \leqslant (x^k)^{\mathrm{T}} s^k,$$

which together with inequality (6) gives

$$x_j^k \geqslant \frac{x_j^k s_j^k}{(x^k)^{\mathrm{T}} s^k} x_j^* > \Omega(1/n) 2^{-L}. \tag{10}$$

Therefore, for any $j \in \sigma^*$,

$$s_j^k < \mathrm{O}(1/n) 2^{-L} \leqslant \Omega(1/n) 2^{-L} < x_j^k.$$

Similarly, for any $j \in \bar{\sigma}^*$ the condition $(x^k)^{\mathrm{T}} s^k \leqslant \mathrm{O}(1/n) 2^{-2L}$ implies

$$x_j^k < s_j^k.$$

Thus, we must have $\sigma^k = \sigma^*$ when $(x^k)^{\mathrm{T}} s^k \leqslant \mathrm{O}(1/n) 2^{-2L}$, which can be achieved in $\mathrm{O}(n^3 L)$ arithmetic operations using an $\mathrm{O}(n^3 L)$ interior-point algorithm satisfying (6). □

Note that σ^k could be defined differently. For example, we could construct $\hat{\sigma}^k$ as

$$\hat{\sigma}^k = \{j \colon x_j^k \geqslant \Omega(1/n)\delta\},$$

where $\delta = \min(\delta_1, \delta_2)$. Whenever $(x^k)^{\mathrm{T}} s^k < \mathrm{O}(1/n)\delta^2$, we have from the proof of Theorem 1 that for any $j \in \sigma^*$,

$$x_j^k > \Omega(1/n)\delta,$$

and for any $j \in \bar{\sigma}^*$,

$$x_j^k \leqslant (x^k)^{\mathrm{T}} s^k / s_j^* < \mathrm{O}(1/n)\delta.$$

Thus, $\hat{\sigma}^k = \sigma^*$. In practice, one can use some heuristic to select δ. Theorem 1 is of theoretical interest only, because it involves the input length. A drawback of the above two criteria for finding σ^* is that the construction of σ^k is (column) scaling-dependent.

We now give a new way to find σ^*, which is (column) scaling-independent for some primal–dual interior-point algorithms. In general, the direction d_x^k and d_s^k at the kth step of the primal–dual algorithms [1, 12, 15] are generated by solving the following system of linear equations:

$$X^k d_s^k + S^k d_x^k = \gamma \frac{(x^k)^{\mathrm{T}} s^k}{n} e - X^k s^k, \tag{11a}$$

$$A d_x^k = 0 \quad \text{and} \quad d_s^k = -A^{\mathrm{T}} d_y^k \tag{11b}$$

where $0 \leqslant \gamma < 1$ is a constant. Then, a fixed step-size $0 < \theta \leqslant 1$ is chosen to generate

$$x^{k+1} = x^k + \theta d_x^k \quad \text{and} \quad s^{k+1} = s^k + \theta d_s^k \tag{11c}$$

such that the iteration sequence satisfies

$$\frac{\min(X^k s^k)}{(x^k)^{\mathrm{T}} s^k} > \frac{1-\beta}{n} \tag{11d}$$

for some constant $0 < \beta < 1$.

Theorem 2. *At iteration k of the primal–dual interior-point algorithm with properties* (11a), (11b), (11c), (11d), *let*

$$\sigma^k = \{j\colon \; |x_j^{k+1} - x_j^k| / x_j^k \leqslant |s_j^{k+1} - s_j^k| / s_j^k\}.$$

Further assume that $\gamma < 1 - \beta$ and $\|d_x^k\|$ and $\|d_s^k\|$ converges to zero for $k \in \tilde{K}$, where $\tilde{K}$ is a subsequence of $\{1, 2, \dots\}$. *Then there exists a finite K such that for all $k \geqslant K$ and $k \in \tilde{K}$,*

$$\sigma^k = \sigma^*.$$

Proof. Let $k \in \tilde{K}$. Then, from system (11a) we have for any j,

$$\frac{(d_s^k)_j}{s_j^k} + \frac{(d_x^k)_j}{x_j^k} = \gamma \frac{(x^k)^{\mathrm{T}} s^k}{n x_j^k s_j^k} - 1.$$

Recall inequality (10), for any $j \in \sigma^*$,

$$x_j^k > \frac{x_j^k s_j^k}{(x^k)^{\mathrm{T}} s^k} x_j^* \geqslant \frac{1-\beta}{n} x_j^* \geqslant \frac{1-\beta}{n} \delta.$$

Thus, we have

$$\lim_{k \to \infty} \frac{|x_j^{k+1} - x_j^k|}{x_j^k} = \theta \lim_{k \to \infty} \frac{|(d_x^k)_j|}{x_j^k} = 0, \quad j \in \sigma^*.$$

Hence, we must have

$$\limsup_{k \to \infty} \frac{(d_s^k)_j}{s_j^k} = \limsup_{k \to \infty} \left(\gamma \frac{(x^k)^{\mathrm{T}} s^k}{n x_j^k s_j^k} - 1 \right) \leqslant \frac{\gamma}{1-\beta} - 1, \quad j \in \sigma^*.$$

Since γ is strictly less than $1 - \beta$, we have

$$\liminf_{k\to\infty}\frac{|s_j^{k+1}-s_j^k|}{s_j^k}=\theta\liminf_{k\to\infty}\frac{|(d_s^k)_j|}{s_j^k}\geqslant\frac{\theta(1-\beta-\gamma)}{1-\beta}>0,\quad j\in\sigma^*.$$

This shows that there is a K such that for all $k\geqslant K$, we have

$$|x_j^{k+1}-x_j^k|/x_j^k<\frac{\theta(1-\beta-\gamma)}{2(1-\beta)}\leqslant|s_j^{k+1}-s_j^k|/s_j^k$$

for any $j\in\sigma^*$.

Similarly, we can prove that for all $k\geqslant K$,

$$|x_j^{k+1}-x_j^k|/x_j^k\geqslant\frac{\theta(1-\beta-\gamma)}{2(1-\beta)}>|s_j^{k+1}-s_j^k|/s_j^k$$

for any $j\in\bar{\sigma}^*$. This concludes the proof of Theorem 2. □

The criterion in Theorem 2 is related to Tapia's indicator described by El-Bakry et al. [4]. The assumptions in Theorem 2 are consistent with some primal–dual interior-point algorithms. In particular, all these assumptions are satisfied in the predictor step of the $O(n^3L)$ predictor–corrector algorithm of Mizuno et al. [15], where

$$\gamma=0,\quad \beta=\tfrac{1}{2},\quad \Omega(1/\sqrt{n})\leqslant\theta<1,$$

and

$$\|d_x^k\|\to 0\quad\text{and}\quad\|d_s^k\|\to 0.$$

The latter was recently proved by Mehrotra [14] and Ye et al. [23]. In fact, they have shown that as long as (11d) holds, the primal–dual affine scaling direction (d_x^k, d_s^k) resulted from (11a) and (11b) with $\gamma=0$ approaches to zero as the primal–dual gap $(x^k)^{\mathrm{T}}s^k$ tends to zero.

3. Finding an interior point in the optimal face

In this section we discuss procedures for finding a feasible primal–dual solution pair in the interior of the optimal face. Our procedure is based on testing the optimality of a partition σ^k at some stage of the algorithm. For simplicity, let those columns in A corresponding to σ^k form matrix B and the remaining columns form matrix N. Let us represent the corresponding variables by x_B and x_N, respectively. Note that we have not made any assumptions on B. To find a point in the interior of the primal face, we solve the system of linear equations

$$B\Delta x_B=b-Bx_B^k=Nx_N^k \tag{12}$$

by the Gaussian elimination for Δx_B. Linearly dependent rows and/or columns of B are deleted during the Gaussian elimination. In other words, we find a largest nonsingular submatrix $\bar{B}$ of B, then solve

$$\bar{B}\Delta x_{\bar{B}}=\bar{N}x_{\bar{N}}^k,$$

where the rows of $\bar{N}$ correspond to the rows of $\bar{B}$. The components of Δx_B corresponding to the linearly dependent columns are set to zero. A primal solution is then generated as $x_B^* = x_B^k + \Delta x_B$ and $x_N^* = 0$.

To find a point in the interior of the dual face, we solve the system of linear equations:

$$B^{\mathrm{T}} \Delta y = c_B - B^{\mathrm{T}} y^k = s_B^k \tag{13}$$

by the Gaussian elimination for Δy. Linearly dependent rows and/or columns of B^{T} are deleted during the Gaussian elimination. The components of Δy corresponding to the linearly dependent columns are set to zero. A dual solution is then generated as $y^* = y^k + \Delta y$ and $s^* = c - A^{\mathrm{T}} y^*$. Note that the factors of $\bar{B}$ used to solve (12) can be used to solve (13). Thus, two systems (12) and (13) can be solved using only one factorization of $\bar{B}$.

Since the procedure for solving systems (12) and (13) does not interfere with the main course of the interior-point algorithms, we can solve the systems at an arbitrary iteration k. If the solutions x_B^* and (y^*, s^*) generated from systems (12) and (13) satisfy

$$Bx_B^* = b \quad \text{and} \quad s_B^* = c_B - B^{\mathrm{T}} y^* = 0$$

and

$$x_B^* > 0 \quad \text{and} \quad s_N^* = c_N - N^{\mathrm{T}} y^* > 0,$$

then, x^* and (y^*, s^*) are strict complementarity solutions for (LP) and (LD), and therefore σ^k must equal σ^*. Otherwise, we repeat the procedure in the next iteration.

The next theorem shows that for the interior-point algorithms described in Theorem 1, the solution x^* and (y^*, s^*) generated above eventually becomes feasible and strictly complementary.

Theorem 3. *Let σ^k be defined in Theorem 1 for the $\mathrm{O}(n^3 L)$ interior-point algorithms whose iteration sequences satisfy inequality (6). Then, in $\mathrm{O}(n^3 L)$ arithmetic operations the solution x^* and (y^*, s^*) generated above satisfies*

$$Bx_B^* = b \quad \textit{and} \quad s_B^* = c_B - B^{\mathrm{T}} y^* = 0,$$

and

$$x_B^* > 0 \quad \textit{and} \quad s_N^* > 0.$$

Proof. From Theorem 1, $\sigma^* = \sigma^k$ after $(x^k)^{\mathrm{T}} s^k < \mathrm{O}(1/n) 2^{-2L}$. Then, systems (12) and (13) are consistent, and we shall have

$$Bx_B^* = b \quad \text{and} \quad s_B^* = c_B - B^{\mathrm{T}} y^* = 0.$$

Moreover, if $(x^k)^{\mathrm{T}} s^k < \mathrm{O}(1/n^2) 2^{-5L}$, then from the proof of Theorem 1 we have

$$x_j^k > \Omega(1/n) 2^{-L} \quad \text{and} \quad s_j^k < \mathrm{O}(1/n^2) 2^{-4L} \quad \text{for } j \in \sigma^k$$

and

$$x_j^k < \mathrm{O}(1/n^2) 2^{-4L} \quad \text{and} \quad s_j^k > \Omega(1/n) 2^{-L} \quad \text{for } j \notin \sigma^k.$$

Let $\bar{B}$ be one of the largest nonsingular submatrices of B used to solve (12) and (13). Since for all nonsingular matrices that can be used to solve (12) and (13), $\|\bar{B}^{-1}\| \leqslant 2^{2L}$ (Schrijver [18]),

$$\begin{aligned}\|\Delta x_B\| &= \|\bar{B}^{-1}\bar{N}x_N^k\| \leqslant \|\bar{B}^{-1}\|\,\|\bar{N}x_N^k\| \leqslant \|\bar{B}^{-1}\| \sum_{j\notin\sigma^k} x_j^k \|A_j\| \\ &\leqslant \|\bar{B}^{-1}\| \mathrm{O}(1/n^2)2^{-4L} \sum_{j\notin\sigma^k} \|A_j\| < 2^{2L}\mathrm{O}(1/n^2)2^{-4L}n2^L = \mathrm{O}(1/n)2^{-L},\end{aligned}$$

where A_j is the jth column of A. This implies that

$$\|\Delta x_B\| \leqslant \mathrm{O}(1/n)2^{-L}$$

since the components of Δx_B corresponding to the linearly dependent columns are set to zero. Thus

$$|\Delta x_j| \leqslant \mathrm{O}(1/n)2^{-L} \leqslant \Omega(1/n)2^{-L} < x_j^k \quad \text{for } j\in\sigma^k,$$

or

$$x_j^k + \Delta x_j > 0 \quad \text{for } j\in\sigma^k.$$

Similarly, we can prove that $(x^k)^{\mathrm{T}}s^k < \mathrm{O}(1/n^2)2^{-5L}$ implies

$$s_j^k + \Delta s_j > 0 \quad \text{for } j\notin\sigma^k.$$

Again, the condition $(x^k)^{\mathrm{T}}s^k < \mathrm{O}(1/n^2)2^{-5L}$ will be satisfied in $\mathrm{O}(n^3L)$ arithmetic operations upon using $\mathrm{O}(n^3L)$ primal–dual algorithms. □

To find an interior point in the optimal face, Ye [22] considers problems

$$\begin{aligned}\min \quad & \|x_B - x_B^k\| \\ \text{s.t.} \quad & Bx_B = b,\end{aligned} \tag{14}$$

and

$$\begin{aligned}\min \quad & \|y - y^k\| \\ \text{s.t.} \quad & B^{\mathrm{T}}y = c_B.\end{aligned} \tag{15}$$

Problem (14) computes the projection of x_B^k onto the affine space $\{x_B\colon Bx_B = b\}$, and (15) projects y^k onto the affine space $\{y\colon B^{\mathrm{T}}y = c_B\}$.

The solution of (12) and (13) can be found more efficiently than the solution of (14) and (15). The use of (14) and (15) requires the solutions of two least-squares problems. In our computational experience we found that solving (12) and (13) is much more efficient and numerically stable than solving (14) and (15), while both approaches find an interior-point in the optimal face at nearly the same iteration. All the computational results reported in the next section were obtained while solving (12) and (13).

4. Computational results

We now present our computational results involving the practical implications of the theory developed in the previous sections. We used a FORTRAN 77 implementation of the algorithm described in Fourer and Mehrotra [5] and the papers referenced in there. All computations were performed using double precision on a Sun 4/110 Workstation.

We used NETLIB test problems for our experiments. We removed variables which are fixed in the MPS file before calling the interior-point solver. No further preprocessing was performed, except for problems *Greenbea* and *Greenbeb*. For these two problems, free variables which are formulated as two non-negative variables in the problem data were replaced by a free variable. No scaling was done.

We called our procedure for finding a point in the interior of a face after

$$|c^{\mathrm{T}}x^k - b^{\mathrm{T}}y^k| / (1 + |b^{\mathrm{T}}y^k|) \leqslant 10^{-8}$$

is satisfied in the algorithm. We used a combination of Theorem 1 and Theorem 2 to partition the variables. In particular, we used

$$\sigma^k = \{j: s_j^k \leqslant 10^{-14} \text{ or } |x_j^{k+1} - x_j^k| / x_j^k \leqslant |s_j^{k+1} - s_j^k| s_j^k\}$$

in all of our experiments. A complementary solution pair x^* and s^* was declared "optimal" if it satisfied

$$x_j^* > 0 \quad \text{for } j \in \sigma^k \quad \text{and} \quad s_j^* > 0 \quad \text{for } j \notin \sigma^k$$

and

$$|c^{\mathrm{T}}x^* - b^{\mathrm{T}}y^*| / (1 + |b^{\mathrm{T}}y^*|) \leqslant \varepsilon^*,$$

$$\|Ax^* - b\| / (1 + \|b\|) \leqslant \varepsilon_x^*,$$

$$\|A^{\mathrm{T}}y^* + s^* - c\| / (1 + \|c\|) \leqslant \varepsilon_s^*.$$

The values of $\varepsilon^* = \varepsilon_x^* = \varepsilon_s^* = 10^{-11}$ were used as a default for all the problems. If the complementary solution obtained at the current iteration failed to satisfy this criterion, this procedure was called at the subsequent iteration.

Table 1 gives the problem data of the NETLIB problems used in our experiments. Problems with bounds and(or) ranges are indicated by B and (or) R in Column 5. The objective values given here are those reported by Bixby [2] using CPLEX. To obtain the objective values for these problems Bixby [2] first solved these problems using default CPLEX settings, and then "reoptimized" with tighter optimality and feasibility tolerances using a threshold pivoting factor equal to 0.99999. Note that for several problems the objective values reported by Bixby [2] are different from the NETLIB objective value currently available. The objective values obtained in our implementation are the same as those reported by Bixby [2].

Table 2 gives computational results on the optimal face problem. The second column of this table gives the iteration number at which we tried to guess the optimal partition by solving system (12) and (13) for the first time. The third column gives the iteration at

Table 1

Problem data summary

Name	Rows	Cols	Nonzeros	BR	Optimal value*
25fv47	822	1571	11127		5.5018458883 E+3
80bau3b	2263	9799	29063	B	9.8722419241 E+5
Adlittle	57	97	465		2.2549496316 E+5
Afiro	28	32	88		−4.6475314286 E+2
Agg	489	163	2541		−3.5991767287 E+7
Agg2	517	302	4515		−2.0239252356 E+7
Agg3	517	302	4531		1.0312115935 E+7
Bandm	306	472	2659		−1.5862801845 E+2
Beaconfd	174	262	3476		3.3592485807 E+4
Blend	75	83	521		−3.0812149846 E+1
Bnl1	644	1175	6129		1.9776295615 E+3
Bnl2	2325	3489	16124		1.8112365404 E+3
Boeing1	351	384	3865	BR	−3.3521356751 E+2
Boeing2	167	143	1339	BR	−3.1501872802 E+2
Bore3d	234	315	1525	B	1.3730803942 E+3
Brandy	221	249	2150		1.5185098965 E+3
Capri	272	353	1786	B	2.6900129138 E+3
Cycle	1904	2857	21322	B	−5.2263930249 E+0
Czprob	930	3523	14173	B	2.1851966989 E+6
D2q06c	2172	5167	35674		1.2278421081 E+5
Degen2	445	534	4449		−1.4351780000 E+3
Degen3	1504	1818	26230		−9.8729400000 E+2
E226	224	282	2767		−1.8751929066 E+1
Etamacro	401	688	2489	B	−7.5571523337 E+2
Fffff800	525	854	6235		5.5567956482 E+5
Finnis	498	614	2714	B	1.7279106560 E+5
Fit1d	25	1026	14430	B	−9.1463780924 E+3
Fit1p	628	1677	10894	B	9.1463780924 E+3
Fit2d	26	10500	138018	B	−6.8464293294 E+4
Fit2p	3001	13525	60784	B	6.8464293294 E+4
Forplan	162	421	4916	BR	−6.6421896127 E+2
Ganges	1310	1681	7021	B	−1.0958573613 E+2
Gfrd-pnc	617	1092	3467	B	6.9022359995 E+6
Greenbea	2393	5405	31499	B	−7.2555248130 E+6
Greenbeb	2393	5405	31499	B	−4.3022602612 E+6
Grow 15	301	645	5665	B	−1.0687094129 E+8
Grow22	441	946	8318	B	−1.6083433648 E+8
Grow7	141	301	2633	B	−4.7787811815 E+7
Israel	175	142	2358		−8.9664482186 E+5
Kb2	44	41	291	B	−1.7499001299 E+3
Lotfi	154	308	1086		−2.5264706062 E+1
Maros	847	1443	10006	B	−5.8063743701 E+4
Nesm	663	2923	13988	BR	1.4076036488 E+7
Perold	626	1376	6026	B	−9.3807552782 E+3
Pilot	1442	3652	43220	B	−5.5740430007 E+2
Pilot.ja	941	1988	14706	B	−6.1131364656 E+3
Pilot.we	723	2789	9218	B	−2.7201075328 E+6
Pilot4	411	1000	5145	B	−2.5811392589 E+3
Pilotnov	976	2172	13129	B	−4.4972761882 E+3
Recipe	92	180	752	B	−2.6661600000 E+2

Table 1 (continued)

Name	Rows	Cols	Nonzeros	BR	Optimal value*
Sc105	106	103	281		−5.2202061212 E+1
Sc205	206	203	552		−5.2202061212 E+1
Sc50a	51	48	131		−6.4575077059 E+1
Sc50b	51	48	119		−7.0000000000 E+1
Scagr25	472	500	2029		−1.4753433061 E+7
Scagr7	130	140	553		−2.3313898243 E+7
Scfxm1	331	457	2612		1.8416759028 E+4
Scfxm2	661	914	5229		3.6660261565 E+4
Scfxm3	991	1371	7846		5.4901254550 E+4
Scorpion	389	358	1708		1.8781248227 E+3
Scrs8	491	1169	4029		9.0429695380 E+2
Scsd1	78	760	3148		8.6666666743 E+1
Scsd6	148	1350	5666		5.0500000078 E+1
Scsd8	398	2750	11334		9.0499999993 E+2
Sctap1	301	480	2052		1.4122500000 E+3
Sctap2	1091	1880	8124		1.7248071429 E+3
Sctap3	1481	2480	10734		1.4240000000 E+3
Seba	516	1028	4874	BR	1.5711600000 E+4
Share1b	118	225	1182		−7.6589318579 E+4
Share2b	97	79	730		−4.1573224074 E+2
Shell	537	1775	4900	B	1.2088253460 E+9
Ship04l	403	2118	8450		1.7933245380 E+7
Ship04s	403	1458	5810		1.7987147004 E+6
Ship08l	779	4283	17085		1.9090552114 E+6
Ship08s	779	2387	9501		1.9200982105 E+6
Ship12l	1152	5427	21597		1.4701879193 E+6
Ship12s	1152	2763	10941		1.4892361344 E+6
Sierra	1228	2036	9252	B	1.5394362184 E+7
Stair	357	467	3857	B	−2.5126695119 E+2
Standata	360	1075	3038	B	1.2576995000 E+3
Standmps	468	1075	3686	B	1.4060175000 E+3
Stocfor1	118	111	474		−4.1131976219 E+4
Stocfor2	2158	2031	9492		−3.9024408538 E+4
Tuff	334	587	4523	B	2.9214776509 E−1
Vtp.base	199	203	914	B	1.2983146246 E+5
Wood1p	245	2594	70216		1.4429024116 E+0
Woodw	1099	8405	37478		1.3044763331 E+0

*The objective value reported here is from Bixby [2].

which the optimal partition was identified. Column 4 gives the total number of guesses. Columns 5 and 6 give primal and dual objective values recorded at the point generated in the optimal face.

Table 3 provides additional information on the quality of primal and dual solutions. Column 2 of this table gives the relative error in the objective value at optimality $|Ty^* - c^{\mathrm{T}}x^*| / (1 + |b^{\mathrm{T}}y^*|)$. Columns 3 and 4 give the relative primal feasibility $\|b - Ax^*\| / (1 + \|b\|)$, and the dual feasibility $\|c - A^{\mathrm{T}}y^* - s^*\| / (1 + \|c\|)$, respectively. The solutions generated in our experiment are strictly complementary for all of the test problems.

Table 2

Results for facet finding problem

Name	it8	itf	try	Objective value	
				Primal	Dual
25fv47	26	26	1	5.5018458882867 E+3	5.5018458882867 E+3
80bau3b	49	51	2	9.8722419240909 E+5	9.8722419240909 E+5
Adlittle	10	10	1	2.2549496316238 E+5	2.2549496316238 E+5
Afiro	7	7	1	−4.6475314285714 E+2	−4.6475314285714 E+2
Agg	26	26	1	−3.5991767286577 E+7	−3.5991767286576 E+7
Agg2	23	23	1	−2.0239252355977 E+7	−2.0239252355977 E+7
Agg3	20	20	1	1.0312115935089 E+7	1.0312115935089 E+7
Bandm	17	17	1	−1.5862801845012 E+2	−1.5862801845012 E+2
Beaconfd	7	7	1	3.3592485807200 E+4	3.3592485807200 E+4
Blend	10	10	1	−3.0812149845828 E+1	−3.0812149845829 E+1
Bnl1	29	29	1	1.9776295615229 E+3	1.9776295615229 E+3
Bnl2	36	36	1	1.8112365403585 E+3	1.8112365403585 E+3
Boeing1	26	26	1	−3.3521356750713 E+2	−3.3521356750712 E+2
Boeing2	19	19	1	−3.1501872801521 E+2	−3.1501872801520 E+2
Bore3d	18	18	1	1.3730803942085 E+3	1.3730803942085 E+3
Brandy	19	19	1	1.5185098964881 E+3	1.5185098964881 E+3
Capri	19	19	1	2.6900129137682 E+3	2.6900129137682 E+3
Cycle	31	31	1	−5.2263930248941 E+0	−5.2263930248941 E+0
Czprob	35	35	1	2.1851966988566 E+6	2.1851966988566 E+6
D2q06c	30	30	1	1.2278421081419 E+5	1.2278421081419 E+5
Degen2	12	12	1	−1.4351780000000 E+3	−1.4351780000000 E+3
Degen3	16	16	1	−9.8729400000000 E+2	−9.8729400000000 E+2
E226	20	20	1	−1.8751929066371 E+1	−1.8751929066371 E+1
Etamacro	30	32	3	−7.5571523337491 E+2	−7.5571523337491 E+2
Fffff800	38	38	1	5.5567956481750 E+5	5.5567956481750 E+5
Finnis	25	27	3	1.7279106559562 E+5	1.7279106559561 E+5
Fit1d	18	18	1	−9.1463780924209 E+3	−9.1463780924209 E+3
Fit1p	18	18	1	9.1463780924209 E+3	9.1463780924209 E+3
Fit2d	23	23	1	−6.8464293293832 E+4	−6.8464293293832 E+4
Fit2p	22	22	1	6.8464293293833 E+4	6.8464293293832 E+4
Forplan	23	23	1	−6.6421896127220 E+2	−6.6421896127220 E+2
Ganges	18	18	1	−1.0958573612928 E+2	−1.0958573612928 E+2
Gfrd-pnc	16	16	1	6.9022359995488 E+6	6.9022359995488 E+6
Greenbea	41	42	2	−7.2555248129846 E+6	−7.2555248129846 E+6
Greenbeb	41	44	3	−4.3022602612066 E+6	−4.3022602612066 E+6
Grow15	12	12	1	−1.0687094129358 E+8	−1.0687094129358 E+8
Grow22	13	13	1	−1.6083433648256 E+8	−1.6083433648256 E+8
Grow7	12	12	1	−4.7787811814712 E+7	−4.7787811814712 E+7
Israel	24	25	2	−8.9664482186305 E+5	−8.9664482186305 E+5
Kb2	20	20	1	−1.7499001299062 E+3	−1.7499001299062 E+3
Lotfi	14	14	1	−2.5264706061882 E+1	−2.5264706061880 E+1
Maros	27	27	1	−5.8063743701126 E+4	−5.8063743701126 E+4
Nesm	35	38	4	1.4076036487563 E+7	1.4076036487562 E+7
Perold	33	34	2	−9.3807552782352 E+3	−9.3807552782352 E+3
Pilot.ja	43	43	1	−6.1131364655813 E+3	−6.1131364655813 E+3
Pilotnov	25	26	2	−4.4972761882189 E+3	−4.4972761882189 E+3
Pilot.we	50	51	2	−2.7201075328450 E+6	−2.7201075328450 E+6
Pilot4	43	44	2	−2.5811392588839 E+3	−2.5811392588839 E+3
Recipe	11	11	1	−2.6661600000000 E+2	−2.6661600000000 E+2

Table 2 (continued)

Name	it8	itf	try	Objective value	
				Primal	Dual
Sc105	9	9	1	−5.2202061211707 E+1	−5.2202061211707 E+1
Sc205	11	11	1	−5.2202061211707 E+1	−5.2202061211707 E+1
Sc50a	8	8	1	−6.4575077058565 E+1	−6.4575077058565 E+1
Sc50b	6	6	1	−7.0000000000000 E+1	−7.0000000000000 E+1
Scagr25	17	17	1	−1.4753433060769 E+7	−1.4753433060769 E+7
Scagr7	13	13	1	−2.3313898243310 E+7	−2.3313898243310 E+7
Scfxm1	18	18	1	1.8416759028349 E+4	1.8416759028349 E+4
Scfxm2	20	20	1	3.6660261564999 E+4	3.6660261564999 E+4
Scfxm3	20	20	1	5.4901254549751 E+4	5.4901254549751 E+4
Scorpion	12	12	1	1.8781248227381 E+3	1.8781248227381 E+3
Scrs8	21	21	1	9.0429695380079 E+2	9.0429695380079 E+2
Scsd1	8	8	1	8.6666666743334 E+1	8.6666666743334 E+1
Scsd6	10	12	3	5.0500000077144 E+1	5.0500000077144 E+1
Scsd8	9	9	1	9.0499999992546 E+2	9.0499999992546 E+2
Sctap1	15	15	1	1.4122500000000 E+3	1.4122500000000 E+3
Sctap2	13	13	1	1.7248071428571 E+3	1.7248071428571 E+3
Sctap3	14	14	1	1.4240000000000 E+3	1.4240000000000 E+3
Seba	18	18	1	1.5711600000000 E+4	1.5711600000000 E+4
Share1b	22	22	1	−7.6589318579186 E+4	−7.6589318579186 E+4
Share2b	12	12	1	−4.1573224074142 E+2	−4.1573224074142 E+2
Shell	20	20	1	1.2088253460000 E+9	1.2088253460000 E+9
Ship04l	12	12	1	1.7933245379704 E+7	1.7933245379704 E+7
Ship04s	13	13	1	1.7987147004454 E+6	1.7987147004454 E+6
Ship08l	14	14	1	1.9090552113891 E+6	1.9090552113891 E+6
Ship08s	14	14	1	1.9200982105346 E+6	1.9200982105346 E+6
Ship12l	18	18	1	1.4701879193293 E+6	1.4701879193293 E+6
Ship12s	16	16	1	1.4892361344061 E+6	1.4892361344061 E+6
Sierra	21	21	1	1.5394362183632 E+7	1.5394362183632 E+7
Stair	16	16	1	−2.5126695119297 E+2	−2.5126695119297 E+2
Standata	14	14	1	1.2576995000000 E+3	1.2576995000000 E+3
Standmps	22	22	1	1.4060175000000 E+3	1.4060175000000 E+3
Stocfor1	16	16	1	−4.1131976219436 E+4	−4.1131976219436 E+4
Stocfor2	24	24	1	−3.9024408537882 E+4	−3.9024408537882 E+4
Tuff	20	20	1	2.92147765093610 E−1	2.92147765093610 E−1
Vtp.base	28	28	1	1.2983146246136 E+5	1.2983146246136 E+5
Wood1p	31	31	1	1.4429024115734 E+0	1.4429024115734 E+0
Woodw	40	40	1	1.3044763330842 E+0	1.3044763330842 E+0

Columns 5 to 8 give information on the range of positive primal and dual variables on the optimal face. In particular, Columns 5 to 8 give $\max\{x_B^*\}$, $\min\{x_B^*\}$, $\max\{s_N^*\}$, and $\min\{s_N^*\}$. We find that the range of primal and dual positive slacks in the optimal face is often a good indicator for the performance of interior-point algorithms and the efficiency with which the optimal face can be found.

We first discuss the quality of solutions we obtained in the interior of the identified face. The results show that accurate solutions were obtained on all the problems. For all the problems the primal and dual objective value matched up to thirteen significant digits.

Table 3

Properties of the interior point obtained on the optimal facet

Name	$\frac{\lvert b^T y^* - c^T x^* \rvert}{1 + \lvert b^T y^* \rvert}$	$\frac{\lVert b - Ax^* \rVert}{1 + \lVert b \rVert}$	$\frac{\lVert c - A^T y^* - s^* \rVert}{1 + \lVert c \rVert}$	$\max_i \{x_i^*\}$	$\min_i \{x_i^{*+}\}$	$\max_i \{s_i^*\}$	$\min_i \{s_i^{*+}\}$
25fv47	3.3 E−16	4.6 E−16	9.4 E−17	1.435 E+5	1.273 E−2	9.741 E+1	1.000 E−4
80bau3b	2.6 E−15	3.5 E−18	2.5 E−14	3.606 E+15	2.063 E−5	5.752 E+9	1.256 E−4
Adlittle	6.4 E−16	7.7 E−17	9.2 E−17	3.131 E+2	3.822 E−1	6.419 E+3	6.185 E−2
Afiro	0.	4.6 E−17	1.0 E−17	5.000 E+2	1.020 E+1	1.000 E+1	1.333 E−2
Agg	1.0 E−15	7.4 E−17	3.2 E−15	4.228 E+6	4.589 E+0	1.866 E+5	1.000 E−2
Agg2	2.2 E−14	1.6 E−16	6.8 E−12	3.545 E+5	1.187 E+1	1.431 E+4	3.988 E−3
Agg3	4.9 E−15	6.0 E−17	3.3 E−12	4.133 E+5	1.183 E+1	1.431 E+4	1.500 E−2
Bandm	3.5 E−16	3.6 E−15	5.6 E−16	1.033 E+3	7.304 E−3	2.459 E+1	1.000 E−4
Beaconfd	2.1 E−16	2.0 E−14	2.8 E−17	4.502 E+3	7.500 E−1	1.000 E+1	3.499 E−4
Blend	1.3 E−15	8.7 E−16	3.0 E−16	8.709 E+1	6.587 E−2	5.863 E+0	1.026 E−2
Bnl1	1.8 E−15	2.2 E−16	7.7 E−15	1.544 E+3	1.512 E−5	2.718 E+3	1.434 E−3
Bnl2	1.2 E−15	8.2 E−12	5.4 E−14	2.645 E+8	1.522 E−3	7.440 E+2	9.304 E−7
Boeing1	1.4 E−14	2.5 E−15	2.2 E−14	2.484 E+4	7.318 E−2	8.617 E+3	1.000 E−5
Boeing2	1.2 E−14	2.9 E−16	1.6 E−12	9.981 E+4	1.346 E−2	6.929 E+1	1.228 E−3
Bore3d	3.1 E−16	2.7 E−15	1.7 E−16	9.396 E+3	2.678 E−2	1.588 E+3	7.852 E−4
Brandy	4.4 E−16	3.6 E−14	3.2 E−15	4.617 E+3	1.908 E−2	5.004 E+1	2.366 E−3
Capri	0.	2.4 E−16	2.9 E−14	5.296 E+3	9.995 E−3	2.618 E+2	1.160 E−3
Cycle	0.	1.7 E−13	1.4 E−13	7.859 E+6	8.603 E−4	2.992 E+3	7.084 E−8
Czprob	1.0 E−16	5.1 E−16	6.5 E−16	1.233 E+4	1.000 E−3	1.905 E+4	1.265 E−2
D2q06c	5.9 E−16	4.0 E−15	7.0 E−16	1.333 E+5	2.029 E−3	1.242 E+4	1.111 E−5
Degen2	1.8 E−15	4.8 E−16	3.8 E−16	5.000 E+0	2.000 E−1	9.379 E+1	2.809 E−3
Degen3	3.4 E−16	5.3 E−16	9.6 E−16	9.800 E+0	4.496 E−2	6.067 E+1	6.612 E−4
E226	1.0 E−15	1.3 E−15	3.0 E−15	4.648 E+2	1.696 E−3	1.349 E+1	2.527 E−4
Etamacro	3.8 E−16	3.3 E−16	3.9 E−16	6.949 E+3	3.000 E−4	1.069 E+7	3.000 E−9
Fffff800	1.4 E−16	2.0 E−16	1.1 E−14	5.025 E+5	9.221 E−2	5.402 E+2	3.153 E−7

Table 3 (continued)

Name	$\frac{\lvert b^{\mathrm{T}}y^* - c^{\mathrm{T}}x^* \rvert}{1+\lvert b^{\mathrm{T}}y^* \rvert}$	$\frac{\lVert b - Ax^* \rVert}{1+\lVert b \rVert}$	$\frac{\lVert c - A^{\mathrm{T}}y^* - s^* \rVert}{1+\lVert c \rVert}$	$\max_i\{x_i^*\}$	$\min_i\{x_i^{*+}\}$	$\max_i\{s_i^*\}$	$\min_i\{s_i^{*+}\}$
Finnis	5.7 E−14	3.0 E−14	7.0 E−15	2.276 E+7	8.999 E−6	3.178 E+5	1.677 E−6
Fit1d	7.9 E−16	1.7 E−13	3.3 E−18	4.359 E+3	5.501 E−3	1.390 E+3	8.872 E−3
Fit1p	3.9 E−16	2.8 E−17	2.7 E−13	1.224 E+3	8.872 E−3	4.359 E+3	5.501 E−3
Fit2d	1.4 E−15	3.2 E−14	2.3 E−18	1.684 E+2	3.426 E−2	8.124 E+3	7.311 E−4
Fit2p	3.1 E−15	1.7 E−17	4.8 E−13	6.504 E+3	7.311 E−4	1.684 E+2	3.426 E−2
Forplan1	1.6 E−15	7.2 E−19	1.8 E−15	9.991 E+6	6.667 E−1	4.734 E+2	9.999 E−6
Ganges	3.7 E−15	1.4 E−16	3.4 E−14	1.713 E+5	1.978 E−4	2.947 E+1	7.589 E−2
Gfrd-pnc	1.4 E−15	7.2 E−17	3.9 E−17	4.481 E+4	2.988 E−1	9.575 E+4	1.955 E−2
Greenbea	2.0 E−16	2.5 E−12	2.9 E−13	3.299 E+8	1.094 E−2	1.219 E+5	5.106 E−3
Greenbeb	0.	3.5 E−16	3.6 E−13	1.906 E+5	5.051 E−3	1.581 E+6	5.020 E−6
Grow22	5.5 E−16	1.0 E−16	2.4 E−16	1.159 E+6	5.273 E+0	8.567 E+1	6.571 E−1
Grow15	8.3 E−16	8.9 E−17	1.1 E−13	1.158 E+6	8.338 E+1	6.153 E+1	1.286 E−1
Grow7	1.5 E−16	7.9 E−17	7.1 E−15	1.114 E+6	1.068 E+2	6.939 E+1	1.317 E−1
Israel	3.8 E−16	3.6 E−16	7.6 E−17	8.966 E+5	1.120 E−3	3.006 E+3	2.460 E−3
Kb2	3.8 E−16	1.3 E−14	1.4 E−16	6.262 E+3	8.118 E−1	1.726 E+1	3.516 E−3
Lotfi	8.0 E−14	1.9 E−14	6.6 E−17	4.624 E+4	7.216 E−2	1.139 E+0	5.103 E−5
Maros	4.0 E−16	6.2 E−14	3.0 E−15	2.929 E+5	2.284 E−2	2.157 E+4	5.646 E−6
Nesm	2.3 E−15	3.2 E−13	3.1 E−12	1.013 E+4	1.096 E−3	2.978 E+4	2.245 E−7
Perold	2.0 E−16	7.4 E−16	4.3 E−14	1.102 E+5	2.419 E−7	3.951 E+3	9.926 E−6
Pilot4	3.5 E−16	3.0 E−15	4.2 E−14	9.613 E+4	1.021 E−7	2.819 E+3	5.510 E−7
Pilot.ja	1.6 E−15	3.4 E−15	8.0 E−13	1.524 E+5	2.000 E−6	1.097 E+5	4.438 E−8
Pilotnov	2.0 E−16	1.9 E−15	6.9 E−17	1.511 E+5	4.027 E−7	1.733 E+2	6.691 E−3
Pilot.we	3.4 E−16	2.3 E−15	5.6 E−17	1.582 E+5	2.000 E−6	1.596 E+6	1.694 E−7
Recipe	1.2 E−15	1.2 E−14	1.4 E−16	4.510 E+4	2.000 E+0	5.383 E+0	4.119 E−4
Sc105	0.	3.5 E−16	1.3 E−17	7.088 E+2	4.881 E+0	1.643 E−1	1.365 E−3
Sc205	1.3 E−16	9.3 E−16	1.3 E−16	2.380 E+3	4.881 E+0	1.643 E−1	3.012 E−4
Sc50a	2.1 E−16	9.5 E−17	1.3 E−16	2.996 E+2	6.842 E+0	3.406 E−1	7.346 E−3
Sc50b	0.	8.9 E−17	1.3 E−16	3.248 E+2	2.800 E+1	3.164 E−1	2.460 E−2
Scagr25	3.7 E−16	1.9 E−16	4.7 E−16	2.293 E+4	1.329 E+0	5.267 E+3	4.999 E−2
Scagr7	1.9 E−16	1.3 E−16	2.9 E−16	4.569 E+3	1.470 E+1	3.225 E+3	1.666 E−3

Table 3 (continued)

Name	$\frac{\lvert b^{\mathrm{T}}y^* - c^{\mathrm{T}}x^*\rvert}{1+\lvert b^{\mathrm{T}}y^*\rvert}$	$\frac{\lVert b-Ax^*\rVert}{1+\lVert b\rVert}$	$\frac{\lVert c-A^{\mathrm{T}}y^*-s^*\rVert}{1+\lVert c\rVert}$	$\max_i\{x_i^*\}$	$\min_i\{x_i^{*+}\}$	$\max_i\{s_i^*\}$	$\min_i\{s_i^{*+}\}$
Scfxm1	1.7 E−16	5.2 E−14	2.8 E−16	2.611 E+4	2.494 E−1	3.271 E+1	1.218 E−3
Scfxm2	2.9 E−15	7.7 E−15	3.8 E−16	5.077 E+4	6.866 E−2	3.943 E+2	1.100 E−3
Scfxm3	1.8 E−15	7.7 E−15	4.4 E−16	4.163 E+4	6.713 E−2	3.915 E+2	1.100 E−3
Scorpion	2.4 E−16	2.9 E−16	3.5 E−16	2.677 E+0	1.410 E−4	4.933 E+2	2.503 E−1
Scrs8	5.0 E−16	2.0 E−15	8.6 E−17	7.909 E+2	7.772 E−4	4.551 E+3	6.028 E−3
Scsd1	5.5 E−16	1.0 E−16	6.9 E−17	5.270 E−1	4.730 E−3	8.000 E+0	6.860 E−3
Scsd6	0.	1.6 E−16	7.9 E−17	1.356 E+0	2.074 E−2	8.000 E+0	8.113 E−10
Scsd8	5.0 E−16	6.5 E−16	3.7 E−16	2.250 E+1	2.503 E−2	7.304 E+0	5.444 E−2
Sctap1	6.4 E−16	4.2 E−16	4.1 E−16	8.000 E+1	5.216 E−3	2.925 E+2	2.225 E−2
Sctap2	7.9 E−16	1.9 E−16	1.1 E−16	4.941 E+1	7.223 E−4	6.100 E+1	4.277 E−2
Sctap3	4.7 E−16	2.6 E−16	9.7 E−17	4.937 E+1	9.991 E−4	7.479 E+1	8.199 E−2
Seba	1.1 E−16	7.7 E−16	1.2 E−15	1.372 E+4	3.000 E−2	1.070 E+4	2.000 E−3
Share1b	0.	4.1 E−14	1.6 E−16	1.277 E+6	2.074 E+0	1.015 E+2	1.191 E−4
Share2b	3.6 E−15	1.3 E−14	7.6 E−15	6.227 E+1	4.564 E−1	4.923 E+1	4.416 E−4
Shell	5.9 E−16	1.5 E−17	0.	2.843 E+5	3.999 E+0	4.023 E+3	8.476 E−1
Ship04l	3.8 E−16	1.8 E−16	4.1 E−16	1.873 E+2	8.104 E−4	1.036 E+4	1.113 E+0
Ship04s	5.1 E−16	4.2 E−16	3.3 E−16	1.909 E+2	7.616 E−3	1.603 E+4	3.677 E+0
Ship08l	4.8 E−16	2.3 E−1	4.6 E−16	1.260 E+2	1.572 E−2	2.182 E+4	7.075 E−1
Ship08s	9.7 E−16	3.4 E−16	4.1 E−16	1.260 E+2	1.572 E−2	2.649 E+4	3.828 E−1
Ship12l	7.9 E−16	1.1 E−15	1.0 E−15	1.359 E+2	9.184 E−3	3.711 E+4	1.445 E−1
Ship12s	3.1 E−15	1.1 E−15	7.7 E−16	1.342 E+2	9.184 E−3	3.653 E+4	2.453 E−1
Sierra	7.2 E−16	3.2 E−17	7.0 E−17	1.000 E+5	5.000 E−4	4.511 E+4	3.000 E−7
Stair	0.	3.9 E−15	7.5 E−16	1.570 E+4	1.096 E−3	8.831 E+0	2.547 E−3
Standata	0.	1.2 E−16	2.3 E−17	1.337 E+3	1.000 E+0	1.000 E+2	1.217 E−1
Standmps	3.2 E−16	1.1 E−16	6.0 E−17	1.337 E+3	3.329 E−2	1.000 E+2	1.656 E−1
Stochfor1	1.7 E−16	4.0 E−14	3.6 E−16	6.271 E+3	9.652 E−2	1.796 E+2	5.730 E−1
Stochfor2	1.6 E−15	1.0 E−13	1.7 E−15	5.164 E+3	3.921 E−2	7.793 E+1	1.342 E−3
Tuff	4.2 E−17	8.2 E−16	1.6 E−17	4.091 E+3	3.033 E−6	3.925 E+1	4.257 E−6
Vtp.base	2.4 E−15	1.4 E−17	1.1 E−13	1.027 E+5	2.438 E−1	1.934 E+3	2.875 E−2
Wood1p	3.6 E−16	3.3 E−14	1.6 E−14	1.442 E+0	2.470 E−6	1.535 E+1	1.089 E−1
Woodw	1.9 E−16	8.4 E−14	9.2 E−16	7.080 E−1	1.145 E−6	1.384 E+1	1.367 E−5

Futhermore, for all the problems these objective values were same (to all eleven significant digits reported) as those reported by Bixby [2].

Although the accuracy of primal and dual solutions varied, all problems had approximately twelve digits of accuracy in both primal and dual solutions. For a majority of the problems the accuracy was close to fourteen digits. (The FORTRAN double precision on Sun 4/110 uses 11 bits for exponent, and it has 53 significant bits. This is an equivalent of 15 to 17 digit precision on one floating point operation.)

We now discuss the work required to identify the optimal partition. First note that for 74 out of the 86 tested problems the partition was declared optimal at the first attempt. We needed more than two attempts only for problems *Etamacro*, *Finnis*, *Greenbeb*, *Nesm*, and *Scsd6*. For all of these problems, positive dual slacks with relatively small values are present. The influence due to the presence of small dual slacks is best seen when we consider the results on *Scsd* problems. For problems *Scsd1* and *Scsd8* the optimal partition was found at the first attempt. The smallest possible dual slack for these problems is greater than 10^{-3}. However, for problem *Scsd6* we needed three attempts. The smallest positive dual slack for this problem is less than 10^{-9}. For problem *Scsd6* the smallest dual slack variable gave no indication of staying positive until ten digits of accuracy was achieved in the solution.

Since, an attempt to identify the optimal partition only requires us to factor one basic matrix, the cost is equivalent to, or even less than, one iteration of interior-point algorithms. The results given in this section clearly indicate that the number of attempts we need is typically small. Obviously, if we wait long enough we will always be able to identify the optimal partition in one attempts, but this would cost extra iterations of interior-point algorithms. The question about the choice of iteration at which we should attempt to find the optimal partition remains to be addressed.

5. Additional remarks

Computations on a finite precision machine

We first discuss the importance of feasibility tolerances while testing for the optimality of a partition. We need these tolerances because of finite machine precision. The machine precision becomes important in at least two ways. At each iteration of interior-point algorithms the search direction is computed by inverting a matrix. This matrix may become ill-conditioned. As a result of this ill-conditioning, the computed direction may not be accurate.

Even if the matrix inverted at each iteration remains well conditioned, it is possible that some primal and/or dual slacks may have very small positive values at solutions in the optimal face. The partition for such variables may never become clear on a finite precision machine, because it could require accuracy in the iterates which may not be possible.

Obtaining an optimal vertex solution

Once a solution on the optimal face is available, a primal optimal basic solution (vertex solution) can be obtained in strongly polynomial time, e.g., see Megiddo [13]. In fact, it can be obtained in no more than $n-m$ pivots of the simplex method. For example, if the cardinality $|\sigma^*| = |\sigma(x_B^*)| \leqslant m$, then x_B^* generated from our procedure is already a basic solution. In the following, we assume that $|\sigma(x_B^*)| > m$ and B has full row rank, which is without loss of generality. We emphasize that a basic matrix $\bar{B}$ of B was already factorized when solving systems (12) and (13) in our procedure to identify the optimal partition. Furthermore, we have

$$Bx_B^* = b \quad \text{and} \quad x_B^* > 0.$$

Let B be written as $B = (\bar{B}, H)$ and let $\bar{B}$ be the initial basis for the (dual) simplex method. If $\bar{b} = \bar{B}^{-1} b \geqslant 0$, then $x_{\bar{B}} = \bar{b}$ is a basic feasible solution and $\bar{B}$ is an optimal basis. Otherwise, let

$$\bar{x}_B = \alpha x_B^* + (1-\alpha)\begin{pmatrix} \bar{b} \\ 0 \end{pmatrix}.$$

Therefore, there must exist an $0 < \alpha < 1$ such that

$$B\bar{x}_B = b \text{ and } \bar{x}_B \geqslant 0,$$

and at least one component of $\bar{x}_{\bar{B}}$ is zero. Thus, the corresponding column can be deleted, permanently in this case, from the basis $\bar{B}$. Select an incoming column from H to form a new basis, and continue this process. Thus, in no more than $|\sigma(x_B^*)| - m$ pivot steps, we shall obtain a basic feasible solution, since the system $\{x_B\colon Bx_B = b, x_B \geqslant 0\}$ is known feasible.

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