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SHORT COMMUNICATION

ON THE MAXIMAL FLOW PROBLEM WITH REAL ARC CAPACITIES

J. PONSTEIN

Department of Econometrics, University of Groningen, The Netherlands

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A counter example is given to the assertion that a certain algorithm will yield the maximal flow of a finite network in a finite number of steps, even if the arc capacities are real numbers. It is shown that an addition to the algorithm will make it finite.

In [1], the following algorithm is given for the maximal flow problem of finite networks with real arc capacities:

Step 1: Ignore all arcs that are saturated with flow. Go to Step 2.

Step 2: If possible, find a flow augmenting path by the labeling method, using only arcs that are not ignored, and send as much flow as is possible along that path. If such a path can indeed be found, go to Step 1; otherwise go to Step 3.

Step 3: No longer ignore any arc. If possible, find a flow augmenting path by the labeling method, and send as much flow as is possible along that path. If such a path can indeed be found, go to Step 1; otherwise, stop.

Then it is shown that this algorithm is finite if the network is *undirected*. Finally, it is stated that this is also true for any *directed* network, although the proof "is a bit tedious". In fact, this statement is not true, as follows from the following counterexample: Consider a directed network with nodes s (source), 1, 2, 3, 4, 5, 6, and t (sink); and the following directed arcs and their capacities as shown in table 1.

Table 1

arc	capacity	"*a*-arcs" arc	"*a*-arcs" capacity	"*b*-arcs" arc	"*b*-arcs" capacity	"*c*-arcs" arc	"*c*-arcs" capacity
(1, 2)	>1	$(s, 6)$	$a > \frac{1}{2}(1+\sqrt{2})$	$(s, 6)$	$b > \frac{1}{2}$	$(s, 2)$	$c > \frac{1}{2}\sqrt{2}$
(3, 4)	$>\sqrt{2}$	(5, 4)	a	(5, 3)	b	(1, 5)	c
(5, 6)	>1	(3, 1)	a	(4, 2)	b	(6, 4)	c
$(s, 3)$	$\sqrt{2}$	$(2, t)$	a	$(1, t)$	b	$(3, t)$	c
$(4, t)$	$\sqrt{2}$						
$(s, 5)$	1						
$(6, t)$	1						

Applying the above algorithm, first we find the path $(s, 3, 4, t)$ giving the flow $\sqrt{2}$ and saturating $(s, 3)$ and $(4, t)$. Then we find $(s, 5, 6, t)$ increasing the flow to $1+\sqrt{2}$ and saturating $(s, 5)$ and $(6, t)$. From then on, each time we find a path containing two of the arcs (1, 2), (3, 4) and (5, 6) as backward arcs, such that the flow in one of them becomes zero. Starting from the flow $1+\sqrt{2}$, we get table 2.

The first four flow increments are $1, -1+\sqrt{2}, -1+\sqrt{2}$ and $3-2\sqrt{2}$, adding up to 2. Since the flows through the arcs (1, 2), (3, 4) and (5, 6) given in the fifth line of the table are $3-2\sqrt{2}$ times those given in the first line, we can follow a cyclic pattern. As $r = 3-2\sqrt{2} < 1$, the flow will then converge to $1+\sqrt{2}+2(1-r)^{-1} = 2+2\sqrt{2}$; and the flows through the *a*-, *b*- and *c*-arcs will never decrease and converge to $\frac{1}{2}(1+\sqrt{2})$, $\frac{1}{2}$ and $\frac{1}{2}\sqrt{2}$, respectively, as can be verified by a similar com-

Table 2

Flows in (1, 2)	(3, 4)	(5, 6)	*a*-arcs	*b*-arcs	*c*-arcs	path
0	$\sqrt{2}$	1	0	0	0	$(s, 6, 5, 4, 3, 1, 2, t)$
1	$-1+\sqrt{2}$	0	1	0	0	$(s, 2, 1, 5, 6, 4, 3, t)$
$2-\sqrt{2}$	0	$-1+\sqrt{2}$	1	0	$-1+\sqrt{2}$	$(s, 6, 5, 3, 4, 2, 1, t)$
$3-2\sqrt{2}$	$-1+\sqrt{2}$	0	1	$-1+\sqrt{2}$	$-1+\sqrt{2}$	$(s, 2, 1, 5, 6, 4, 3, t)$
0	$-4+3\sqrt{2}$	$3-2\sqrt{2}$	1	$-1+\sqrt{2}$	$2-\sqrt{2}$	$(s, 6, 5, 4, 3, 1, 2, t)$

putation. Nevermore an arc is saturated, hence Step 2 is executed an infinite number of times, and the limit of the flow obtained is lower than the maximal flow, because of the paths (s, 2, t) and (s, 6, 4, 2, t).

In order to make sure that the algorithm is finite, simply change Step 1 into:

Step 1′: Ignore all arcs that are saturated with flow or that become flowless after being used as backward arcs in Step 2.

To show that the revised algorithm is finite, consider the cut $(X, \bar{X})$ that is obtained in Step 2, immediately before entering Step 3. Any arc connecting a node of X with a node of $\bar{X}$ in either direction is either saturated or flowless, and is at the same time either ignored or not, although some of the combinations cannot occur (namely, saturated arcs that are not ignored and certain flowless arcs). Given any X and any combination of all arcs connecting X and $\bar{X}$, the flow value is uniquely determined. Hence, since each time Step 3 is entered, the flow will be increased or the maximal flow is reached, and since the network is finite, Step 3 is entered a finite number of times. Hence, if the algorithm were infinite, after a number of steps, Steps 1′ and 2 would be executed for ever, which, however, is impossible because of the fact that each time the flow is increased, some arc becomes saturated or flowless.

Reference

[1] T.C. Hu, *Integer programming and network flows* (Addison-Wesley Publ. Co., Inc., Reading, Mass., 1969) 115.