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ALGORITHMS FOR THE VECTOR MAXIMIZATION PROBLEM *

Johan PHILIP
Royal Institute of Technology, Stockholm, Sweden

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We consider a convex set B in R^n described as the intersection of halfspaces $a_i^{\mathrm{T}} x \leqq b_i$ $(i \in I)$ and a set of linear objective functions $f_j = c_j^{\mathrm{T}} x$ $(j \in J)$. The index sets I and J are allowed to be infinite in one of the algorithms. We give the definition of the *efficient* points of B (also called functionally efficient or Pareto optimal points) and present the mathematical theory which is needed in the algorithms. In the last section of the paper, we present algorithms that solve the following problems:

I. To decide if a given point in B is efficient.
II. To find an efficient point in B.
III. To decide if a given efficient point is the only one that exists, and if not, find other ones.
IV. The solutions of the above problems do not depend on the absolute magnitudes of the c_j. They only describe the relative importance of the different activities x_i. Therefore we also consider

$$\max_{x \text{ efficient}} G^{\mathrm{T}} x$$

for some vector G.

1. Introduction

We shall consider the vector maximization (VM) problem in R^n. Thus, we have, as in linear programming (LP), a convex constraint set B of activities $x_1, \ldots, x_n$ (forming a vector x). We also have a set of n-vectors c_j defining linear objective functions $f_j = c_j^{\mathrm{T}} x$. The optimality in LP is replaced by efficiency:

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Definition 1a. A point $x \in B$ is *efficient* if and only if there exists no other point $y \in B$ such that

$$c_j^{\mathrm{T}} y \geqq c_j^{\mathrm{T}} x \qquad \text{for all } j$$

$$c_j^{\mathrm{T}} y > c_j^{\mathrm{T}} x \qquad \text{for at least one } j\,.$$

In general, a VM problem has a solution set containing several points. As to the use and interpretation of vector maximization, we refer to references [3], [4] and [6].

We shall consider the following problems:

I. To decide if a given point in B is efficient.

II. To find an efficient point in B.

III. To decide if a given efficient point is the only one that exists, and if it is not, find other ones.

IV. The solutions of the above problems do not depend on the absolute magnitudes of the c_j. They only describe the relative importance of the different activities $x_1, ..., x_n$. Therefore, we also consider

$$\max_{x \text{ efficient}} \; G^{\mathrm{T}} x$$

for some vector G.

In [2], Bod gives an algorithm that solves problem I for a polyhedral set B. Our method I.1 does the same thing in a different way and method I.2 solves probelm I even for a general (nonpolyhedral) set B. The duality theory of VM (see below) immediately gives a method to solve problem II. We describe it briefly as method II.1. The same method is also in the papers by Benayoun and Tergny [1] and Bod [2]. We give also a more elaborate method (II.2) which hopefully requires less computation then method II.1, and upon which we can base a method to solve problem III. Problem IV seems not to have been considered before. Alternative ways of treating a VM problem, other than by solving any of the four problems that we have listed, are described by Benayoun and Tergny [1]. The mathematical theory, in particular the duality theory, of VM is described already by Kuhn and Tucker in their fundamental paper on nonlinear programming [7]. Roughly speaking, Kuhn and Tucker show that a given point in the constraint set is efficient if and only if its dual vector p, which can be interpreted as a price vector, is positive. Since LP algorithms are adapted to nonstrict inequalities, they cannot be used directly to find this strictly positive p. In the corollary of theorem 3 we make a simple observation which produces the missing link and makes it possible to establish the existence or nonexistence of a positive p by phase I of the simplex method. We have collected the theory needed for our algorithms in the next section. Except, possibly, for the corollary we have just mentioned, it is certainly not new. The algorithms have not been tested on large problems.

2. Mathematical preliminaries

A useful concept for the description of duality is that of the polar set:

Definition 2. The polar set of a set $A \in R^n$ is

$$A^* = \{u : u \in R^n, u^{\mathrm{T}} a \geqq 0 \text{ for all } a \in A\} .$$

A polar set is always a closed convex cone. If C is the set of all the c_j defining the objective functions, C^* is the cone of "good directions" in the VM problem. The formulation of efficiency in terms of polar sets is

Definition 1b. A point $x \in B$ is efficient if and only if $y \in B$ and $y - x \in C^*$ implies $x - y \in C^*$.
We write cone (A) for the convex cone with vertex at the origin that is generated by a set A, and denote (topological) closure by superbar. The well-known Farkas lemma then reads

Proposition 1. If A is nonempty,

$$A^{**} = \overline{\text{cone}\,(A)} ,$$

(where A^{**} is the polar of the polar of A). If A is a finite set, the closure bar is superfluous.
In the following theorem we deal with different sets of objective functions. We denote a VM problem with constraint set B and objective function set C by (B, C^*). The translation of a set A by a vector z is denoted A_z.

Theorem 1. Let P and Q be two closed convex cones with vertices at the origin in R^n. If x is efficient in (B, P), $P_x \cap B = \{x\}$ and $Q \subset P$, the x is efficient in (B, Q).

Proof: $Q \subset P$ implies $Q_x \subset P_x$ which implies $Q_x \cap B \subset P_x \cap B = \{x\}$. This means that the only y satisfying the conditions in definition 1b is $y = x$, so $x - y \in Q$.

Remark. This theorem can be used to establish the efficiency of x in (B, C^*) by finding a subset D of C such that x is efficient in (B, D^*).

Corollary. A point x^0 that is a *unique* solution of the LP problem

$$\max_{x \in B} c_j^{\mathrm{T}} x ,$$

where $c_j^{\mathrm{T}} x$ is any of the objective functions in the VM problem, is efficient.

Remark. This corollary also follows directly from definition 1a.
We shall present the duality theory of VM in the following two theorems.

Theorem 2. Let $C = \{c_j\}_{j=1}^{m}$ so that C^* is a polyhedral cone and let

$$w = \sum_{j=1}^{m} \lambda_j c_j \quad \text{with all} \quad \lambda_j > 0 .$$

A point x^0 that solves the LP problem

$$\max_{x \in B} w^{\mathrm{T}} x$$

is efficient.

Proof: Suppose that x^0 is not efficient. Then, by definition 1a, there exists a $y \in B$ such that

$$c_j^{\mathrm{T}} y \geqq c_j^{\mathrm{T}} x^0 \quad \text{for all } j$$

$$c_j^{\mathrm{T}} y > c_j^{\mathrm{T}} x^0 \quad \text{for at least one } j .$$

By multiplying inequality j by $\lambda_j > 0$ and adding we get

$$w^{\mathrm{T}} y = \sum_{j=1}^{m} \lambda_j c_j^{\mathrm{T}} y > \sum_{j=1}^{m} \lambda_j c_j^{\mathrm{T}} x^0 = w^{\mathrm{T}} x^0 ,$$

which contradicts the maximality of x^0 and thus proves the theorem.

Remark. The hyperplane $w^{\mathrm{T}} x = w^{\mathrm{T}} x^0$ (= constant) separates B and $(C^*)_{x^0}$.

Proof: By the construction, we have $w^{\mathrm{T}} x \leqq w^{\mathrm{T}} x^0$ for $x \in B$ and since $w \in \mathrm{cone}(C) = C^{**}$, we have $w^{\mathrm{T}}(x - x^0) \geqq 0$ for $x - x^0 \in C^*$.

Theorem 3. Let B be a polyhedron and x^0 a point on its boundary. Let

$$a_i^{\mathrm{T}} x \leqq b_i \qquad (1 \leqq i \leqq p)$$

be the inequalities in the description of B that are tight at x^0. (The inequalities of the form $-x_k \leqq 0$, which are tight, are also included.) Let $A = \{a_i\}_{i=1}^{p}$ and $C = \{c_j\}_{j=1}^{m}$. Then, x^0 is efficient if and only if there exist $\kappa_i \geqq 0$ $(1 \leqq i \leqq p)$ and $\lambda_j > 0$ $(1 \leqq j \leqq m)$ such that

$$\sum_{i=1}^{p} \kappa_i a_i = \sum_{j=1}^{m} \lambda_j c_j \quad (= w) . \tag{1}$$

Proof: To simplify the notations assume that x^0 is the origin (note that this implies $b_i = 0$, $i = 1, \ldots, p$). If there exist $\kappa_i \geqq 0$ satisfying (1), we have $w \in \mathrm{cone}(A) = A^{**}$, so $w^{\mathrm{T}} x \leqq 0$ for $x \in (-A)^* \supset B$. Thus, the "if"-part of the theorem follows from theorem 2, and we can proceed to the "only if"-part. Since B is a polyhedron, there exists a $\delta > 0$ to every $y \in (-A)^*$ such that $\delta y \in B$. This means that $B = (-A)^*$ in an open neighbourhood of x^0 (= origin), so by definition 1b, x^0 is efficient only if $c_k^{\mathrm{T}} y \leqq 0$ for all k $(1 \leqq k \leqq m)$ and all $y \in (-A)^* \cap C^*$. This can be written

$$-c_k \in ((-A)^* \cap C^*)^* = ((-A) \cup C)^{**} = \mathrm{cone}((-A) \cup C) .$$

Thus, there exist $\mu_{kj} \geqq 0$ and $\nu_{ki} \geqq 0$ such that

$$-c_k = \sum_{j=1}^{m} \mu_{kj} c_j - \sum_{i=1}^{p} \nu_{ki} a_i \quad (1 \leqq k \leqq m) .$$

Adding these m equations, we get

$$\sum_{i=1}^{p} \kappa_i a_i = \sum_{j=1}^{m} \lambda_j c_j \quad (= w) ,$$

where

$$\kappa_i = \sum_{k=1}^{m} \nu_{ki} \geqq 0 \qquad \text{and} \qquad \lambda_j = 1 + \sum_{k=1}^{m} \mu_{kj} > 0 .$$

Remark. If B is not a polyhedron, x^0 can be efficient without $-c_k \in ((-A) \cup C)^{**}$, since there may exist a y in $(-A)^* \cap C^*$ for which there is no $\delta > 0$ such that $\delta y \in B \cap C^*$. Let e.g.

$$B = \{(x_1, x_2) : (x_1 + 1)^2 + x_2^2 \leqq 1\} ,$$

$$c_1^{\mathrm{T}} = (1, 0) \qquad \text{and} \qquad c_2^{\mathrm{T}} = (0, 1) .$$

Then the origin is efficient and the hyperplane supporting B there is unique with $w = 1 \cdot c_1 + 0 \cdot c_2$ (fig. 1).

Cf. Kuhn and Tucker [7], who introduce the concept of *proper* efficiency to be able to state a corresponding theorem for nonpolyhedral sets B. See also Geoffrion [5], who introduces another kind of efficiency to deal with the same kind of problem. Our algorithms give efficient points which are proper in both these senses.

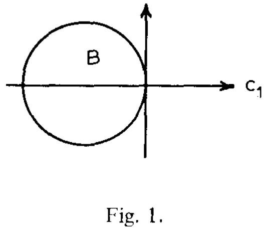

Fig. 1.

Corollary. Let B and C be as in theorem 3 and let $\alpha_j > 0$ $(1 \leqq j \leqq m)$ be m given positive numbers. Then, x^0 is an efficient point if and only if there exist $\kappa_i \geqq 0$ $(1 \leqq i \leqq p)$ and $\lambda_j \geqq \alpha_j$ $(1 \leqq j \leqq m)$ such that (1) holds.

Proof: The conditions in the theorem and the corollary are equivalent since (1) is a homogeneous expression.

3. Algorithms

We describe points as column-vectors and let C denote the $(n \times m)$-matrix of the c_j and A the $(n \times p)$-matrix of the a_i. We let e stand for a one-column matrix of suitable length with all elements equal to 1. Let b and c be two vectors, M a matrix, and c_0 a constant. We exhibit the simplex program

$$y = b + Mx, \quad x \geqq 0\,, \quad y \geqq 0\,,$$

$$\text{minimize } z = c_0 + c^{\mathrm{T}} x\,,$$

with the vector y of basic variables and the vector x of nonbasic variables in the tableau

	1	x_1 x_2 . . . x_r
y_1 =		
y_2 =		
.	b	M
.		
y_n =		
z =	c_0	c^{T}

A tableau of this kind with the y_i's replaced by zeros, denotes a program where a basis has not yet been found.

Problem I. To decide if a given point x^0 in B is efficient.

Method I.1. (This method can only be used when C^* is polyhedral.) We apply the corollary of theorem 3 with $\alpha_j = 1$ $(1 \leqq j \leqq m)$. Put

$$\lambda_j - \alpha_j = \lambda_j - 1 = u_j \quad (1 \leqq j \leqq m) ,$$

so that we have to find $u_j \geqq 0$ $(1 \leqq j \leqq m)$ and $\kappa_i \geqq 0$ $(1 \leqq i \leqq p)$ satisfying (1). We put $\kappa_i = v_i$ (to get rid of greek letters). Formula (1) becomes

$$Av = Cu + Ce . \tag{1'}$$

To decide if there exist vectors $u \geqq 0$ and $v \geqq 0$ satisfying (1′), we use phase I of the simplex method. Thus, we add vectors $s \geqq 0$ and $t \geqq 0$ and solve

$$\begin{aligned} &0 = Ce + Cu - Av + s - t \\ &\text{minimize } g = e^{\mathrm{T}} s + e^{\mathrm{T}} t . \end{aligned} \tag{2}$$

The point x^0 is efficient if and only if $g_{\min} \neq 0$.

Numerical example 1 (fig. 2)

$$A = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

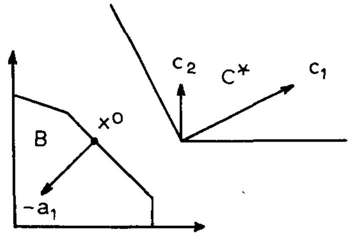

Fig. 2.

	1	u_1	u_2	v_1	s_1	s_2	t_1	t_2
$0 =$	2	2		-1	1		-1	
$0 =$	2	1	1	-1		1		-1
$g =$					1	1	1	1

A primal feasible solution is

	1	u_1	u_2	v_1	s_1	s_2
$t_1 =$	2	2		-1	1	
$t_2 =$	2	1	1	-1		1
$g =$	4	3	1	-2	2	2

Make v_1 basic!

	1	u_1	u_2	s_1	s_2	t_1
$v_1 =$	2	2		1		-1
$t_2 =$	0	-1	1	-1	1	
$g =$	0	-1	1		2	2

We have reached $g = 0$ so x^0 is efficient.

Method I.2. (This method can be used even when there are infinitely many a_i and c_j.) The ideas in the methods I.1 and I.2 are the same. Here, however, we try to establish the efficiency of x^0 by using finite subsets of $C = \{c_j\}$ and $A = \{a_i\}$. If we can find subsets $C_0 \subset C$ and $A_0 \subset A$ such that C_0^* and A_0^* have no point $y \neq 0$ in common, the sets $C^* \subset C_0^*$ and $A^* \subset A_0^*$ certainly have no such y in common and x^0 is efficient. (Cf. theorem 1 and the remark following it.) We construct the sets A_0 and C_0 step by step by adding one element, i.e. an a_i or c_j, in each step until we get (1) satisfied or find that x^0 is not efficient. First, we describe the calculations involved in adding an element and describe the choice of the element later. Thus, suppose we have subsets $\underline{A}$ and $\underline{C}$ and the corresponding $g_{\min} > 0$. We describe the simplex iterations that already have been done as a premultiplication of program (2) by a $(n+1) \times (n+1)$-matrix

$$\bar{M} = \left[\begin{array}{c|c} M & 0 \\ \hline z^{\mathrm{T}} & 1 \end{array}\right]$$

The equation of program (2) then has the form

$$0 = M\underline{C}e + M\underline{C}u - M\underline{A}v + Ms - Mt \tag{3}$$

and $M\underline{C}e \geqq 0$, since this is the way M is chosen in the simplex method. If it is an $a_i \in A$ that is to be added, we simply add $Ma_i v_q$, where v_q is the new element in v, to the equation. From the choice of a_i, to be described later, it follows that v_q can be made basic and we get a new $g_{\min}$. If it is a c_j that is to be added, we first note that we can add αc_j, $\alpha > 0$, instead of c_j. If u_q stands for the new element in u, (3) becomes

$$0 = M\underline{C}e + \alpha M c_j + M\underline{C}u + \alpha M c_j u_q - M\underline{A}v + Ms - Mt.$$

For this to be a primal feasible simplex tableau we must have

$$M\underline{C}e + \alpha M c_j \geqq 0\,. \tag{4}$$

We have three cases:

Case A. The inequality (4) can be satisfied if $\alpha > 0$ is sufficiently small. Choose in this case preferably such an α that (4) is an equality in a row corresponding to an s_k or t_k still in the basis.

Case B. The inequality (4) is only violated in rows corresponding to s_k or t_k in the basis. Exchange in this case these s_k with the t_k and vice versa and choose α as in case A. (The expression for g must be altered in connection with such a change!)

Case C. If neither case A nor B is applicable, add αc_j anyhow. First make the variables in the rows of (4) that become negative, nonbasic and the corresponding s_k or t_k basic (cf. the dual simplex method). Then change signs in these rows as in case B. For the practical calculations, we suggest "the inverse matrix method" (also called the revised simplex method). In this method only the matrix $\overline{M}$ is computed at each pivot and the multiplications MC and MA are not performed. Let z^{T} be the last row (except the last element) corresponding to the objective function in the $\overline{M}$-matrix solving the present LP problem. Then the simplex criterion says $z^{\mathrm{T}} \underline{C} \geq 0$ and $z^{\mathrm{T}} \underline{A} \leq 0$ meaning that $z \in \underline{C}^* \cap (-\underline{A})^*$. To find an a_i or c_j to add to $\underline{A}$ or $\underline{C}$ if $g_{\min} > 0$, compute

$$h = \max \left[\max_{c_j \in C} - z^{\mathrm{T}} c_j \,,\; \max_{a_i \in A} z^{\mathrm{T}} a_i \right] .$$

(This computation can of course be a great numerical problem.) Again, we have three cases:

(a) If $h < 0$, z is strictly inside both C^* and $(-A)^*$ so x^0 is not efficient.
(b) If $h = 0$, $z \in C^* \cap (-A)^*$, but not strictly. In general, x^0 is not efficient in this case either, except in the particular situation when $z^{\mathrm{T}} c_j = 0$ for all c_j (i.e. $z \in C^* \cap (-C)^*$). In the latter situation, a special investigation must be undertaken to decide whether $x^0 + \beta z$ is, or is not, in B for any $\beta \neq 0$.
(c) If $h > 0$, we add an a_i or c_j for which h is attained to $\underline{A}$ or $\underline{C}$.

Take a c_j (and not an a_i) in the first step so that the constants Ce are not all zero.

Numerical example 2. To decide if x^0 = origin is efficient in (B, C^*), where B at x^0 is described by $a^{\mathrm{T}} x \leq 0$, $a^{\mathrm{T}} = (1, 1, 2)$ and C^* is the cone $x_3 \geq \sqrt{(x_1^2 + x_2^2)}$. This cone is the intersection of the infinite number of halfspaces described by

$$c_\varphi = \begin{pmatrix} \sin \varphi \\ \cos \varphi \\ 1 \end{pmatrix}, \qquad 0 \leq \varphi < 2\pi .$$

To start, take for instance $\varphi = 0$ so that $c_{\varphi 0}^{\mathrm{T}} = (0, 1, 1)$. Since all elements in c_{φ_0} are nonnegative the t_i $(i = 1, 2, 3)$ constitutes a primal feasible solution and the corresponding $\overline{M}$-matrix is

$$\overline{M}_0 = \left(\begin{array}{ccc|c} 1 & & & \\ & 1 & & \\ & & 1 & \\ \hline \underbrace{1 \quad 1 \quad 1}_{z_0^{\mathrm{T}}} & & & 1 \end{array}\right)$$

In the first step, no pivots are needed for the simplex criterion to be satisfied and $g_{\min}$ is always positive. To choose the new element to be added, we consider

$$\max - z_0^{\mathrm{T}} c_\varphi = \max\,(-\sin\varphi - \cos\varphi - 1)$$
$$\max z_0^{\mathrm{T}} a = 4\ .$$

Irrespective of the value of φ, the latter maximum is greater, so we add $a^{\mathrm{T}} = (1, 1, 2)$. After two pivots the simplex criterion is satisfied and we have $g_{\min} = 1 > 0$, thus more elements must be added. Now, we have $z_1^{\mathrm{T}} = (-1, 1, 0)$, so we consider $\max - z_1^{\mathrm{T}} c_\varphi = \max\,(\sin\varphi - \cos\varphi)$. The maximum is attained for $\varphi_1 = 3\pi/4$ and $c_{\varphi_1} = (2^{-1/2}, -2^{-1/2}, 1)$. We get (4) satisfied by choosing $\alpha = (\sqrt{2} + 1)^{-1}$, then add αc_{φ_1}. After one pivot, we have $g_{\min} = 0.417$ and $z_2^{\mathrm{T}} = (-1, -0.166, 0.583)$. We find $\max -z_2^{\mathrm{T}} c_\varphi$ attained for $\varphi_2 = 1.40$ and $c_{\varphi_2}^{\mathrm{T}} = (0.99, 0.16, 1)$. The next $g_{\min} = 0$ meaning that x^0 (= origin) is efficient.

Problem II. To find an efficient point in B.

Method II.1. Apply theorem 2, that is take any $\lambda_j > 0$ $(1 \leq j \leq m)$ and solve $\max\limits_{x \in B} w^{\mathrm{T}} x$.

Method II.2. This method relies on theorem 3 (which includes theorem 2). The $\lambda_j > 0$ are not fixed in advance as in the former method but are chosen by the algorithm. We use a simplex tableau which includes all the equations defining B and all the objective functions.

	1	$x_1 \quad x_2 \quad \ldots \quad x_r$
$y_1 =$	b_1	
$y_2 =$	b_2	A
.	.	
$y_q =$	b_q	
$f_1 =$	γ_{10}	
.	.	Γ
$f_m =$	γ_{m0}	

(The $y_i \geqq 0$ represent the basic variables and the $x_i \geqq 0$ the nonbasic variables in the present transformation of the program.) The above tableau will be referred to as the "main tableau".

In an ordinary LP problem with only one (transformed) objective function $F = \gamma_0 + \gamma_1 x_1 + \ldots + \gamma_r x_r$, the criterion for optimality (assuming feasibility) says: $x_1 = x_2 = \ldots = x_r = 0$ is an optimal solution if and only if $\gamma_1 \leqq 0, \gamma_2 \leqq 0, \ldots, \gamma_r \leqq 0$. Our criterion is (cf. theorem 3): $x_1 = x_2 = \ldots = x_r = 0$ is an efficient point if and only if there exists a vector $\lambda^T = (\lambda_1, \ldots, \lambda_m)$ with all $\lambda_j > 0$ (describing the separating hyperplane w in the remark following theorem 2) such that

$$\lambda^T \Gamma \leqq 0 \qquad (\text{or } \Gamma^T \lambda \leqq 0)\,. \tag{5}$$

To find such a positive λ satisfying (5) is an LP problem of the same kind as problem I. To solve it, we introduce slack variables $t_k \geqq 0$ $(1 \leqq k \leqq r)$ to take care of the inequalities in (5) and put $\lambda_j - 1 = u_j$ $(1 \leqq j \leqq m)$ whereby we obtain

$$0 = \Gamma^T e + \Gamma^T u + t\,. \tag{5'}$$

Since we already have t, we only have to introduce a vector s to get a phase I LP problem:

$$\begin{aligned} &u \geqq 0\,, \quad t \geqq 0\,, \quad s \geqq 0\,, \\ &0 = \Gamma^T e + \Gamma^T u + t - s \\ &\text{minimize } g = e^T s\,. \end{aligned} \tag{6}$$

If $g_{\min} = 0$, we have $\lambda_j = u_j + 1 > 0$, and the present main tableau corresponds to an efficient point. If $g_{\min} > 0$, some s_k are still in the basis, namely those for which the criterion (5) is not satisfied. Then make an x_k corresponding to an $s_k > 0$ (e.g. the largest s_k) basic in the main tableau, and continue this process. Our suggestion to pivot on the x_k corresponding to the largest s_k is derived from a simple calculation showing that a pivot on x_k increases the function $(\Sigma \lambda_j c_j)^{\mathrm{T}} x$ by $s_k \cdot \Delta x_k$, where Δx_k is the increase of x_k in the pivot. Since all the λ_j are positive (but change from pivot to pivot), also the function $(\Sigma c_j)^{\mathrm{T}} x$ will hopefully but not certainly, increase. Such an increase of a fixed linear function is the guarantee against cycling in LP and in method II.1. Here, the suggested pivot rule is intended to decrease the risk of cycling which may occur in this method.

Numerical example 3. Find an efficient point in the problem:

$$B = \{(x_1, x_2) : 0 \leqq x_1 \leqq 5\,,\ 0 \leqq x_2 \leqq 7\,,\quad x_1 + x_2 \leqq 10\}$$

$$f_1 = 2x_1 - x_2 \qquad\qquad f_2 = -x_1 + 2x_2\,.$$

Introduce nonnegative slack variables x_3, x_4 and x_5, so that the following primal feasible tableau is obtained

	1	x_1	x_2
$x_3 =$	10	−1	−1
$x_4 =$	5	−1	
$x_5 =$	7		−1
$f_1 =$		2	−1
$f_2 =$		−1	2

Is $x_1 = x_2 = 0$ an efficient point? According to method II.2, we shall consider

	1	u_1	u_2	t_1	t_2	s_1	s_2
$0 =$	1	2	−1	1		−1	
$0 =$	1	−1	2		1		−1
$g =$						1	1

A primal feasible solution to this subproblem is

	1	u_1	u_2	t_1	t_2
s_1 =	1	2	−1	1	
s_2 =	1	−1	2		1
g =	2	1	1	1	1

The simplex criterion is satisfied and we have $g_{min} = 2 > 0$, so $x_1 = x_2 = 0$ is not an efficient point. The variables s_1 and s_2 are equal and positive, so either x_1 or x_2 can be made basic. Choose x_1! The new main tableau is

	1	x_4	x_2
x_3 =	5	1	−1
x_1 =	5	−1	
x_5 =	7		−1
f_1 =	10	−2	−1
f_2 =	−5	1	2

Is $x_4 = x_2 = 0$ efficient? Our subproblem is

	1	u_1	u_2	t_1	t_2	s_1	s_2
0 =	−1	−2	1	1		−1	
0 =	1	−1	2		1		−1
g =						1	1

A primal feasible solution of this subproblem is

	1	u_1	u_2	t_2	s_1
$t_1 =$	1	2	−1		1
$s_2 =$	1	−1	2	1	
$g =$	1	−1	2	1	1

After one simplex iteration, we get

	1	u_2	t_2	s_1	s_2
$t_1 =$	3	3	2	1	−2
$u_1 =$	1	2	1		−1
$g =$				1	1

(7)

Now $g_{\min} = 0$, so the main tableau corresponds to an efficient point, namely $x_1 = 5, x_2 = 0$.

Remark 1. In the description of method II.2, we started with a primal feasible solution of the main problem. If one has to find such a solution, an ordinary application of phase I of the simplex method is the usual method. This means that one starts with a problem of the form

$$b = Ax + y$$

$$\text{minimize } h = e^{\mathrm{T}} y \,.$$

The goal is to make all the y_i nonbasic. In each pivot, one usually chooses that x_k to be made basic which has the greatest coefficient in h. We suggest that one already in this phase of the problem also considers the subproblem (5) and confines, if possible, ones choise of the x_k to those which have $s_k > 0$ in the subproblem.

Remark 2. Although method II.2 seems to require more computations than II.1, we do not think that this actually is the case in general. The reason for this is the hope that the number of pivots needed to find *an* efficient point (method II.2) is smaller than the number needed to find *the* efficient point which is predetermined by the choice of the λ_j in method II.1.

Problem III. To decide if a given efficient point is the only one that exists, and if not, find other ones.
As the formulation of the problem indicates, there may be several points in the solution set of a VM problem. It is natural to think that the action to be taken on the basis of the analysis of the problem requires that a single point (combination of activities) be chosen from the solution set. If there are only a few choices, they can be listed by a computer. If there are many choices, they can be made in a "manager-machine" on-line system (method III) or formulated as a type IV problem.

Method III. (Can only be used when the number of objective functions and the number of inequalities describing B are finite.) We use the same kind of tableau as in method II.2. Suppose this tableau corresponds to an efficient point. The subproblem (6) then has a solution with $g_{\min} = 0$, i.e. $s_k = 0$ for all k. The variables t_k have the values of the simplex coefficients for the objective function $f = \Sigma\ \lambda_j c_j^{\mathrm{T}} x$. According to the theory of the simplex method, x^0 is a unique solution of $\max_{x \in B} f$ if and only if all $t_k < 0$. If some $t_k = 0$, the corresponding x_k can be made basic in the main problem and another efficient point y^0 is obtained for which $f(x^0) = f(y^0)$. Every convex combination of x^0 and y^0 is also efficient. Yet, we have only described a situation when another efficient point can easily be found, but not solved problem III. To prove that x^0 is the only efficient point, we must show that we cannot get any $t_k = 0$ for any other choice of the $\lambda_j > 0$. Since the s_k are not basic, and will not be basic, at an efficient point, we just skip them in (6). For each k, we then solve

$$u \geqq 0\,, \qquad t \geqq 0\,,$$

$$0 = \Gamma^{\mathrm{T}} e + \Gamma^{\mathrm{T}} u + t$$

minimize t_k .

If $(t_k)_{\min} > 0$ for all k, x^0 is the only efficient point. If $(t_k)_{\min} = 0$ for some k, the corresponding x_k can be made basic in the main problem and another efficient point is obtained. Note that one can determine whether a positive t_k can be decreased or not, just by checking the signs of the entries in the row representing it.

Numerical example 4. (Continuation of example 3.)
In (7), $t_2 = 0$, so the obtained efficient point is not the only one. Another is obtained if the variable corresponding to t_2, that is x_2, is made basic in the main tableau.

Problem IV. $\qquad \underset{x \in E}{\text{maximize}}\ G^{\mathrm{T}} x$,

where G is a given vector and E is the set of efficient points in a VM problem (B, C^*).

Method IV.1. This method can only be used when G can be written:

$$G = \sum_{j=1}^{m} \lambda_j c_j\,, \qquad \lambda_j > 0 \quad \text{for all } j\,, \qquad c_j \in C\,. \tag{8}$$

Then, by theorem 2, the restriction $x \in E$ can be replaced by $x \in B$ and we have an ordinary LP problem. To find out if there exist such $\lambda_j > 0$, use method I.1 or I.2 to see if there exist $\mu_j \geqq 1$ $(0 \leqq j \leqq m)$ satisfying

$$-\mu_0 G + \sum_{j=1}^{m} \mu_j c_j = 0\,.$$

If no such μ_j exist, method IV.1 cannot be used. If the number of c_j in C is infinite and if we find an expression (8) with a finite number of c_j by method I.2 (this being of course all we can find), then the "cone" $G^{\mathrm{T}} x \geqq 0$ contains the cone C^*, but not strictly. If, in such a case, the solution of the LP problem $\max_{x \in B} G^{\mathrm{T}} x$ has a unique solution x^0, by theorem 1, x^0 is efficient and solves problem IV. If the solution point x^0 is not unique, then although it gives the correct maximum value of problem IV, it is not necessarily efficient.

Method IV.2. This method relies on the methods II and III, so the number of a_i and c_j must be finite. The difficulty in problem IV, when method IV.1 cannot be used is that we have to maximize $G^T x$ over a set that is not convex in general. The present method is a cutting-plane method that masters this difficulty.

Add the G-row at the bottom of the simplex tableau with all the f_j (cf. the beginning of method II.2). Transform this tableau by method II.2 so that an efficient point is obtained. If in the process of finding an efficient point there are alternative pivot choices, make that x_k basic which has the greatest coefficient in the G-row (rather than the one with the greatest $s_k > 0$). When an efficient point is reached, all subsequent iterations shall be done so that they give efficient points. The aim of these subsequent iterations is to increase $G^T x$. If we can get the simplex criterion for $G^T x$ satisfied by doing steps to efficient points (method III), we obtain the solution. If we reach a point x_0 with $G^T x_0 = G_0$ where the simplex criterion for $G^T x$ is not satisfied, and where every step by method III decreases $G^T x$, we add the restriction (cutting plane) $G^T x \geqq G_0$ which only removes uninteresting (efficient) points. Thus, we add the equation

$$x_s = - G_0 + G^T x \ , \quad (x_s \geqq 0) \ ,$$

to the tableau. This makes the system degenerate and we can continue by making x_s nonbasic. If no new efficient points can be reached in this way, $G^T x_0 = G_0$ is the solution. If we find new efficient points that make $G^T x > G_0$, x_s is basic again and does not come into the calculations any more so the x_s-row can be taken away.

Some fictitious efficient points may enter together with the extra equation. These are easily recognized however, since they all lie in the cutting plane. Because the set of efficient points on a polyhedron is simply connected (see e.g. Gerstenhaber in [6]), the described method will lead to the solution

Numerical example 5. (fig. 3)

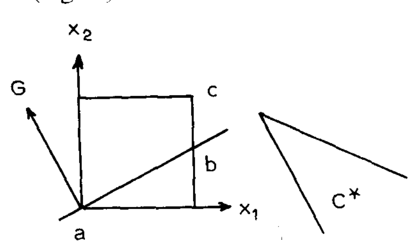

Fig. 3.

Let B be the square defined by $0 \leqq x_1 \leqq 1, 0 \leqq x_2 \leqq 1$. Let

$$C = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} -1 \\ 2 \end{pmatrix} .$$

By introducing slack variables x_3 and x_4 and putting $G^T x = F$, we get a primal feasible main tableau

	1	x_1	x_2
$x_3 =$	1	−1	
$x_4 =$	1		−1
$f_1 =$		−1	−2
$f_2 =$		2	1
$F =$		−1	2

This tableau corresponds to the point $a(x_1 = x_2 = 0)$ in fig. 3 and our first subproblem is to decide if this is an efficient point, and if not to find one. We consider the subproblem (method II.2)

	1	u_1	u_2	t_1	t_2	s_1	s_2
$0 =$	1	−1	2	1		−1	
$0 =$	−1	−2	1		1		−1
$g =$						1	1

A primal feasible solution of the subproblem is

	1	u_1	u_2	t_1	s_2
$s_1 =$	1	−1	1	1	
$t_2 =$	1	2	−1		1
$g =$	1	−1	2	1	1

After one simplex iteration, we get

	1	u_2	t_1	s_1	s_2
u_1 =	1	2	1	−1	
t_2 =	3	3	2	−2	1
g =				1	1

Since $g_{min} = 0$, the point a is efficient. The variable t_1 is zero in the solution so an increase of x_1 would give a new efficient point. Since such an increase would decrease $G^T x$, it is not interesting. We should rather like to increase x_2, so we use method III to investigate if t_2 can be decreased to zero. Thus, we put $s_1 = s_2 = 0$ in the subproblem and consider:

$$\text{minimize } t_2 = 3 + 3u_2 + 2t_1 \ .$$

Since both coefficients are positive, we are at a dead end! We add the condition $G^T x \geqq G_0$, that is we add

$$x_5 \ (= G^T x) = -x_1 + 2x_2 \ , \qquad x_5 \geqq 0$$

to the main tableau and make x_2 basic and x_5 nonbasic. We get

	1	x_1	x_5
x_3 =	1	−1	
x_4 =	1	−1/4	−1/2
x_2 =		1/2	1/2
f_1 =		−2	−1
f_2 =		5/2	1/2
F =			1

We are still in the point a in the diagram but we get a new subproblem, which after one pivot has the form

	1	u_2	t_1	s_1	s_2
$u_1 =$	1/4	5/4	1/2	−1/2	
$t_2 =$	3/4	3/4	1/2	−1/2	1
$g =$				1	1

This shows that $a(x_1 = x_2 = 0)$ is an efficient point as we knew before. We have still $t_1 = 0$ so we can increase x_1 and consider:

$$\text{minimize } t_2 = (3/4) + (3/4)u_2 + (1/2)t_1 \; .$$

Since t_2 is positive and cannot be decreased, x_5 cannot be increased. Now, however, the increase of x_1 which is possible does not decrease $G^T x$, so we make x_1 basic and reach point b in the diagram. From there we can continue to the solution which is point c.

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