Mathematical Programming 2 (1972) 32–64. North-Holland Publishing Company

ENUMERATIVE INEQUALITIES IN INTEGER PROGRAMMING *

Claude-Alain BURDET

Carnegie-Mellon University, Pittsburgh, Pennsylvinia, U.S.A.

Received 13 November 1970
Revised manuscript received 13 August 1971

For a linear integer programming problem, the *local information* contained at an optimal solution $\bar{x}$ of the continuous linear programming extension stems from the theory of L.P. solutions. This paper proposes the use of *environmental information* (of a global nature but pertaining to the discrete vicinity of $\bar{x}$), in order to isolate the set of integer solutions which may be considered as true candidates for the optimum. The concept of *enumerative inequalities* is introduced and it is shown how it can be obtained in the context of the convex outer-domain theory of Balas, Young, et al.

Generally speaking, enumerative inequalities can be made arbitrarily strong (deep), but at the cost of an increasing amount of work (i.e. enumeration) for their construction. In particular cases, however, very little global information can produce enumerative inequalities stronger than any *valid* cut.

0. Introduction

For a *discrete* mathematical optimization *problem* DP, one often considers *continuous* approximating extensions CP; the feasible solutions of DP are then contained in the set of feasible solutions of CP. For a linear integer programming problem, the *local information* contained at an optimal solution $\bar{x}$ (of the continuous extension) can be grouped in the following way:

- characteristics of $\bar{x}$ (with respect to the continuous problem): feasibility, optimality.
- integrality requirements and especially the two sets:

  N_I: set of the integer-constrained variables of the problem in its original formulation;

* This paper was presented at the 7th Mathematical Programming Symposium 1970, The Hague, The Netherlands.

$\bar{N}_I$: set of the integer-constrained non-basic variables at the optimum x;

– algebraic properties which can be derived from the numerical values in the optimal tableau $\bar{A}$ and the integral properties of the variables (group structure, for instance).

This paper proposes the use of *environmental information* (global, but in the vicinity of x) in order to isolate the set of integer solutions which may be considered as true candidates for the optimum. The concept of *enumerative inequalities* is introduced; it is shown how they can be obtained in the context of the convex outer-domain theory of Balas, Young, Glover, et al.

The mixed-integer cutting planes due to Gomory can also be obtained as intersection cuts; however their derivation only makes use of the local information. Following the approach of [3] and [4] enumerative inequalities can be obtained which generalize the Gomory planes but are no longer *valid*. It is shown, however, that they are very intimately related to the latter and possess similar characteristics.

Generally speaking, enumerative inequalities can be made arbitrarily strong (deep) but at the cost of an increasing amount of work for their construction. In particular cases, however, very little global information can produce enumerative inequalities *stronger than any valid cut*. Two examples for the construction of such inequalities are given in the appendix; in a way they illustrate extreme cases in the use of diamond-polytopes as convex outer-domains to generate enumerative cuts.

1. Convex outer-domains and intersection inequalities

1.1. *The problems*

Consider the linear programming problem

$$\text{maximize } x_0 = cx \,. \tag{1a}$$

$$\text{subject to } Ax \leqslant b \tag{1b}$$

$$x \geqslant 0 \tag{1c}$$

where x and c are n-vectors, b is an m-vector and A a matrix with m rows and n columns. All the constraints (1b and c) can be expressed by the non-negativity conditions $\tilde{x} \geqslant 0$ if one defines the slack variables $\tilde{x}_{n+1}, \tilde{x}_{n+2}, \dots, \tilde{x}_{n+m}$:

$$\tilde{x}_k = b_k - \sum_{i \in N} a_{ki} x_i \geqslant 0, \quad k \in M = \{n+1, n+2, \ldots, n+m\}$$
$$\tilde{x}_i = x_i \geqslant 0, \quad i \in N = \{1, 2, \ldots, n\}. \tag{2}$$

Let us now suppose that only the solutions

– which are feasible with respect to (2)

and

– which satisfy the integrality requirements (3)

$$\tilde{x}_i \equiv 0 \; (\text{mod } 1), \qquad \forall i \in N_{\text{I}} \subset N \tag{3}$$

are of interest in the original linear program (1). The conditions (3 change drastically the nature of the problem.
Note that when $N_{\text{I}} = N$, one often deals with slack variables $\tilde{x}_k$, $k \in M_{\text{I}} \subset M$, which are (automatically) *integer valued*; this happens whenever th k-th row of the matrix A contains only integer coefficients, i.e.

$$\left.\begin{aligned} a_{ki} &\equiv 0 \\ b_k &\equiv 0 \end{aligned}\right. \quad (\text{mod } 1), \;\; \forall i \in N.$$

Hence we may legitimately replace (3) by the *stronger* condition:

$$\tilde{x}_j \equiv 0 \; (\text{mod } 1), \;\; \forall j \in (N_{\text{I}} \cup M_{\text{I}}) \tag{4}$$

When $(N_{\text{I}} \cup M_{\text{I}}) = (N \cup M)$, the problem is called *all-integer*; in all othe cases where

$$\emptyset \neq (N_{\text{I}} \cup M_{\text{I}}) \subset (N \cup M)$$

one speaks of *mixed-integer* problems.

In the approach adopted here the distinction between continuous an discrete variables is made in a particular way; the problem (2, 4) is systematically embedded in the continuous analogon (2); then the integrality requirements (4) are translated into extreme point properties o polyhedral sets (hyper-cubes or -prisms) in the n-dimensional space $\mathbb{R}^n_x$

$$\mathbb{R}^n_x = \{x = (x_1, x_2, \ldots, x_n) | x_i \in \mathbb{R}, \forall i \in N\},$$

the original vector space in which problem (1) is formulated.

1.2. *Local characterization of the continuous optimum*

Suppose that the problem (1) has been solved (say with the simplex method) and let $\bar{x}$ be an optimal (feasible) basic solution. It is assumed that there exists such an $\bar{x}$.

Denote by

$\bar{N}$ = the index set of the non-basic variables called t_j $(j \in \bar{N})$

$\bar{A}$ = the matrix of the optimal tableau,

and assume (of course) that $\bar{x}$ does not solve the integer constrained problem (1, 3).

The *local* information which can be called for at the optimum $\bar{x}$ has a twofold nature: continuous and discrete. The continuous information stems from the linear programming optimality criteria:

a) All the feasible solutions (of the continuous and hence also of the discrete problem) lie within the polyhedral cone $\bar{C}$ defined by

$$t_j \geqslant 0 , \quad j \in \bar{N} . \tag{5}$$

b) There exists no feasible solution in $\bar{C}$ which furnishes a larger value of the objective function than $\bar{x}$ (i.e., $t_j = 0$, $\forall j \in \bar{N}$).

c) The optimal basis delivers a correspondence between the original variables $x_1, x_2, ..., x_n$ and the current non-basic variables; it reads

$$x_i = \bar{x}_i - \sum_{j \in \bar{N}} \bar{a}_{ij} t_j \geqslant 0 , \quad i \in N . \tag{6}$$

Now the discrete information is still expressed by the integrality requiremants (3) but in terms of the non-basic variables $t_j \in \mathbb{R}$, $\forall j \in \bar{N}$, i.e.

$$x_i - \sum_{j \in \bar{N}} \bar{a}_{ij} t_j \equiv 0 \ (\mathrm{mod}\ 1), \qquad i \in N_{\mathrm{I}} . \tag{7}$$

Furthermore it may well happen that some of the variables t_j are themselves integer-valued, i.e., when

$$\bar{N}_{\mathrm{I}} = \bar{N} \cap (N_{\mathrm{I}} \cup M_{\mathrm{I}}) \neq \emptyset . \tag{8}$$

Clearly one wishes to make use of as much local information as available in order to derive strong criteria (sharp inequalities) for the characterization of the integer solutions. For the sake of completeness, one could finally mention a third source of local information which lies in the (algebraic) structure of the matrix $\bar{A}$ combined with the conditions (7). Indeed the coefficients $\bar{a}_{ij}$ possess divisibility properties which imply that the variables t_j may only appear in certain combinations of one another; this algebraic information allows one to impose conditions (inequality constraints) on the t_j which reflect, in part or in extenso, the group structure of the modulo constraints (7). This is, in essence, the aim of algebraic studies like [8].

Whatever local information one finds (more or less readily available) in the optimal tableau $(\bar{A}, \bar{x})$ it is not the only one which can be efficiently employed. The main purpose of this study is to propose ways of exploring the vicinity of $\bar{x}$, and in particular to check the feasibility of certain *neighbouring integer solutions*, in order to sharpen the inequalities furnished by the local analysis. From a practical point of view, it often turns out that such a global search in the vicinity of $\bar{x}$ can be made at a relatively low cost and yields a better overall efficiency than if only local information were used.

1.3. *An outline of the convex outer-domain theory*

Suppose that the simplex method has delivered the continuous optimum $\bar{x}$ and the corresponding tableau $\bar{A}$ is in Tucker format (as custo mary in the exposition of a cutting plane algorithm using successive dual-simplex iterations). The relevant information lies in the column vectors $\bar{x}$ and $\bar{a}_j$, $j \in \bar{N}$ defined by

$$\bar{x} = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_i, ..., \bar{x}_n) = \text{column of constants}$$

$$\bar{a}_j = (\bar{a}_{1j}, \bar{a}_{2j}, ..., \bar{a}_{ij}, ..., \bar{a}_{nj})$$

with $i \in N$

on one hand, and, on the other, in the sets of indices $N_I \subset N$, $\bar{N}$, and $\bar{N}_I$

$$\bar{N}_I = \bar{N} \cap (N_I \cup M_I) \subset \bar{N}.$$

Geometrically speaking, the half-lines u^j, $j \in \bar{N}$

$$u^j = \bar{x} - t_j \bar{a}_j, \quad t_j \geqslant 0 \tag{9}$$

may be viewed in $\mathbb{R}^n_x$ as the edges of the polytope (cone) $\bar{C}$ previously defined, with t_j as (half-)line parameter. All this is delivered by the linear programming theory.

1.3.1. *The inequality*

$$\sum_{j \in \bar{N}} \alpha_j t_j \geqslant 1 \qquad (10a)$$

where

$$\infty > \alpha_j \geqslant 0, \qquad \forall j \in \bar{N} \qquad (10b)$$

is not satisfied by the optimal solution x (generated by setting $t_j = 0$, $\forall j \in \bar{N}$), and the system (5, 10) defines a truncated cone $C \subset \bar{C} \subset \mathbb{R}$ such that $\bar{x} \notin C$. An inequality of this type (10) is called *valid* if it is satisfied by every feasible integer solution of the original problem (1, 3); it is *conditionally valid* if there exists a set $S \neq \emptyset$ of points which are feasible with respect to (1, 3) but do *not* satisfy (10). A conditionally valid inequality is also called *enumerative*; one can only use it "with a clear conscience" once the set S has been properly searched (typically by implicit enumeration) to determine and store away those elements of S which are candidates for the optimal solution of (1, 3); the algorithmic implementation of enumerative inequalities is described in more detail further below.

1.3.2. *The basic tool* for generating an enumerative inequality is simple; one merely has to observe that for any convex subset D of $\mathbb{R}^n_x$ containing $\bar{x}$, we can generate a cutting plane of the type (10) which is determined by the intersection points of the n extreme rays of the cone C (see fig. 1).

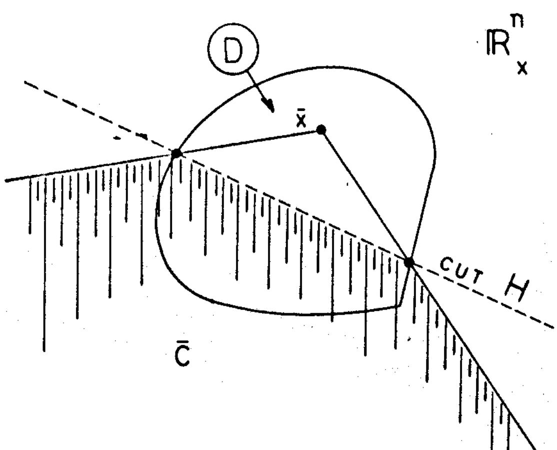

Fig. 1.

In general, according to the size and shape of D, there will be integer feasible solutions of the problem (1) within the interior of the set D; but clearly, the convexity of D implies that no other integer feasible solution can be cut off by the cut H than those lying in the set $\bar{C} \cap \mathrm{Int}(D)$, or, a fortiori, in $\mathrm{Int}(D)$. If we call S a set of integer solutions which contains all the integer solutions lying in $\bar{C} \cap \mathrm{Int}(D)$, it then becomes obvious that we may implement the cut H, just as a valid cutting plane, adding a further restraint to problem (1, 3); provided the set S is properly enumerated (i.e. after finding among all the elements of S an integer feasible solution, if there exists one, which delivers the best value of the objective function); naturally much of the enumeration can be done implicitly to improve the overall efficiency of the algorithm.

In our previous notations, the construction of an enumerative inequality can be made as follows:

1.3.3. Define a *convex outer-domain* $D(\bar{x}, S)$ with the properties

- $D(\bar{x}, S)$ is a closed convex subset of the n-dimensional space $\mathbb{R}^n_x$ of the variables x_i, $i \in N$.
- $\bar{x}$ is an interior point of $D(\bar{x}, S)$.
- $S = \{x \mid x \in D(\bar{x}, S), x_i \equiv 0 \pmod 1), \forall i \in N_1\}$

Then intersect the ray u^j

$$u^j = \bar{x} - \lambda^j \bar{a}_j\,, \quad \lambda^j \geqslant 0\,, \quad j \in \bar{N}$$

with the boundary of $D(\bar{x}, S)$, and let the intersection point be $\bar{u}^j = \bar{x} - \bar{\lambda}^j \bar{a}_j$. One easily shows that $\bar{\lambda}^j > 0$ (see [3] for instance). Finally the intersection inequality defined by

$$\sum_{j \in \bar{N}} \alpha_j t_j \geqslant 1 \tag{11}$$

where

$$0 \leqslant \alpha_j = \frac{1}{\bar{\lambda}^j} < \infty\,, \qquad \forall j \in \bar{N}$$

and it has the following properties

- the continuous optimum $\bar{x}$ (corresponding to $t_j = 0$, $\forall j \in \bar{N}$) does not satisfy (11)
- all feasible integer solutions of the original problem (1,3) satisfy (11) except possibly those which lie in the set S.

Clearly this intersection inequality corresponds to our previous definition of *enumerative* inequalities with the possible exception that the set S may be empty, i.e., the inequality may be *valid*. Of course, there remains the practical construction, i.e., the computation of the coefficients α_j and the characterization of the set S. Many types of outer-domains have been proposed in the literature for generating valid [1, 2, 3, 6] and conditionally valid [4] inequalities. The classical mixed-integer cutting planes due to Gomory [6] seem to enjoy particular properties (see [8]), also in the all-integer case; in section 2, we present enumerative inequalities which are intimately related to the above mentioned cutting planes and may be regarded as their generalization. Paradoxically however, their derivation was obtained in an effort to generalize the intersection cut approach of Balas [1].

1.3.4. *Strengthening an intersection cut with the help of integer-valued non-basic variables t_j, $j \in \bar{N}_\mathrm{I}$.*

At this point, the convex outer-domain theory makes no use of the integrality property (8) of some non-basic variables. We now show how (8) can be used to generate (often significantly) deeper intersection cuts. *

Consider $\forall j \in \bar{N}_\mathrm{I}$ the rays

$$u^j = \bar{x} - \lambda^j \bar{a}_j \quad , \qquad \lambda^j \geqslant 0$$

and

$$\tilde{u}^j = \bar{x} - \tilde{\lambda}^j f_j \quad , \qquad \tilde{\lambda}^j \geqslant 0$$

with **

$$f_{ij} = \begin{cases} \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor, \text{ if } \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor \leqslant g_i = \bar{x}_i - \lfloor \bar{x}_i \rfloor & i \in N_\mathrm{I} \\ \bar{a}_{ij} - \lceil \bar{a}_{ij} \rceil, \text{ if } \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor \geqslant g_i & j \in \bar{N}_\mathrm{I} \\ \bar{a}_{ij}, \ \forall i \in N - N_\mathrm{I} \ \text{ and/or } \ j \in \bar{N} - \bar{N}_\mathrm{I}. \end{cases}$$

* The use of the non-basic variables to obtain stronger cuts is also proposed in Glover [7]. A more general study of this strengthening procedure can be found in Balas [13].

** For a number $q \in \mathbb{R}$, we use the following notations:

$\lceil q \rceil$ denotes the smallest integer $\geqslant q$,

$\lfloor q \rfloor$ denotes the largest integer $\leqslant q$,

Clearly $\lfloor q \rfloor = \lceil q \rceil$ when q is integer.

Lemma 1: By definition, one has

$$\lfloor \bar{x}_i \rfloor < \bar{x}_i - f_{ij} \leqslant \lceil \bar{x}_i \rceil \ , \ \forall i, j \ .$$

Proof: by inspection of the definition.
Geometrically, the points $(\bar{x} - f_j)$, $j \in \bar{N}_\mathrm{I}$ are seen to lie in the cubic hyperprism $U^0(\bar{x}) = \{x \in \mathbb{R}^n_x \mid \lfloor \bar{x}_i \rfloor \leqslant x_i \leqslant \lceil \bar{x}_i \rceil, \forall i \in N_\mathrm{I}$ $x_i \in \mathbb{R}, \forall i \in N - N_\mathrm{I}.\}$

Lemma 2: The systems of equations (A) and (F) below have the same set of integer solutions, in the sense that: for every solution t with $t_j \equiv 0$, $\forall j \in \bar{N}_\mathrm{I}$ one has

$$x_i \equiv 0 \ \text{ iff } \ y_i \equiv 0 \ \ i \in N_\mathrm{I}$$

The system (A) stems from problem (1) directly, i.e.,

$$\text{(A)} \qquad x_i = \bar{x}_i - \sum_{j \in \bar{N}} \bar{a}_{ij} t_j \ , \ i \in N_\mathrm{I}$$

and the system (F) is derived from (A) by the above-mentioned rules, i.e.,

$$\text{(F)} \qquad y_i = g_i - \sum_{j \in \bar{N}_\mathrm{I}} f_{ij} t_j - \sum_{j \in (\bar{N} - \bar{N}_\mathrm{I})} \bar{a}_{ij} t_j \ , \quad i \in N_\mathrm{I} \ .$$

Proof: By construction $x_i - y_i \equiv 0$ (mod. 1), $\forall t$, with $t_j \equiv 0$, $\forall j \in \bar{N}_\mathrm{I}$.
q.e.d.

Proposition: If $D(\bar{x}, S) \supset U^0(\bar{x})$ and if the matrix $F = (f_{ij})$ has rank n then the intersection points of the rays $\tilde{u}^j$ with the boundary of $D(\bar{x}, S)$ correspond to values $\tilde{\lambda}^j \geq 1$. Furthermore, they generate a legitimate (S - conditionally valid) cut.

Proof: The intersection points $(\bar{x} - \tilde{\lambda}^j f_j)$ are not interior to $U^0(\bar{x})$ by hypothesis; hence, by lemma 1, $\tilde{\lambda}^j \geq 1$. By hypothesis the vectors f_j are independent and hence $\Sigma_{j \in \bar{N}} \ t_j/\tilde{\lambda}_j \leq 1$ defines a half-space (cut) in $\mathbb{R}^n$. From lemma 2 we know that no integer solution to problem (1,3) has been cut off except possibly those belonging to S. q.e.d.

Remark 1: Since the reduced F-problem is used to generate an intersection cut, we consider a domain $D(\bar{x}, S) \subset \mathbb{R}^n_y$ and the elements of S are points with integer y_i values ($i \in N_\mathrm{I}$). The strengthened cut (generated by $\tilde{N}$) therefore requires at most the same amount of enumeration after; as before the reduction; of course, one only needs to search, among the elements of S, for those which satisfy $t_j \equiv 0$, $\forall j \in \bar{N}_\mathrm{I}$. Figures 2a, b and c illustrate a strengthening of this type; in 2a one can see that the reduction improves both the *depth and the amount* of enumeration of a cut.

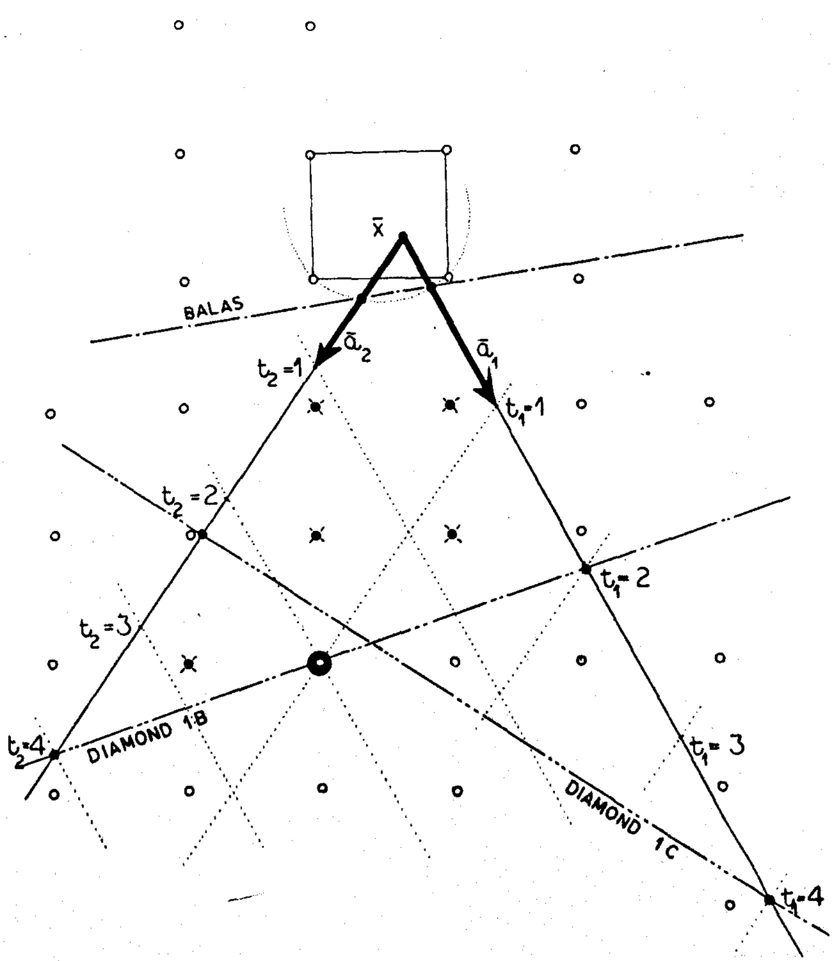

Fig. 2a.

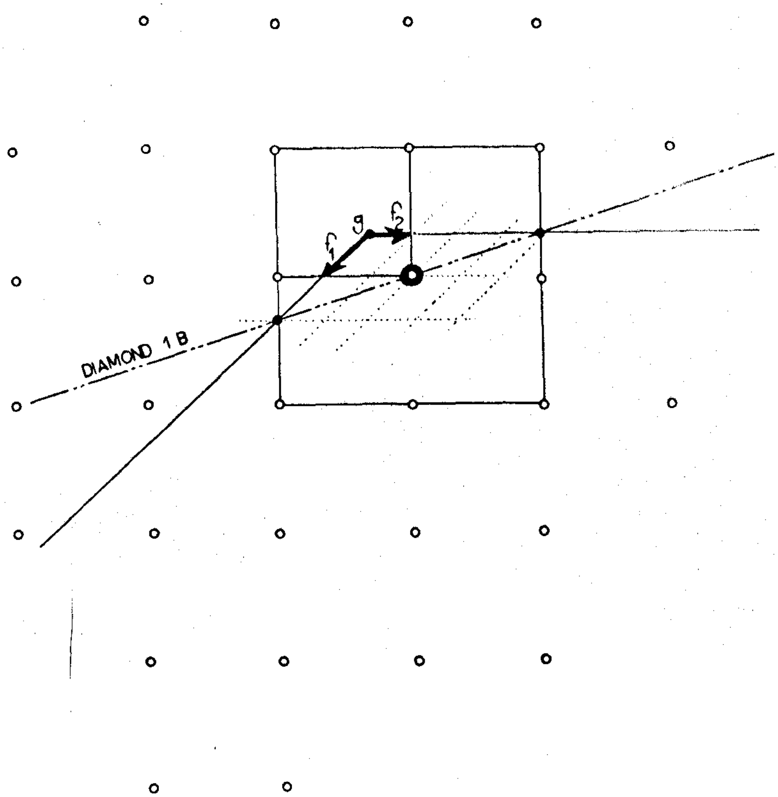

Fig. 2b.

Remark 2: In general it will be inconvenient to test the rank of F. This reduction may therefore appear questionable from a practical point of view for arbitrary outer-domains. Fortunately, however, we shall see that F is not required to have rank n for the diamond cuts of section 2.

1.4. *Direct search or cutting planes?*

Originally, the basic idea underlying the construction of *enumerative inequalities* was to use implicit enumeration as an accessory device to improve cutting planes. But it also leads to improvements for the implicit enumeration algorithms by means of a "trimming" device which reduces the size and amount of the tree search.

The improvement of a cutting plane algorithm is conceptually simple: one engages a direct search code in the finding of the best feasible solution of the set S attached to each cut (see section 1.3 for the definition of S).

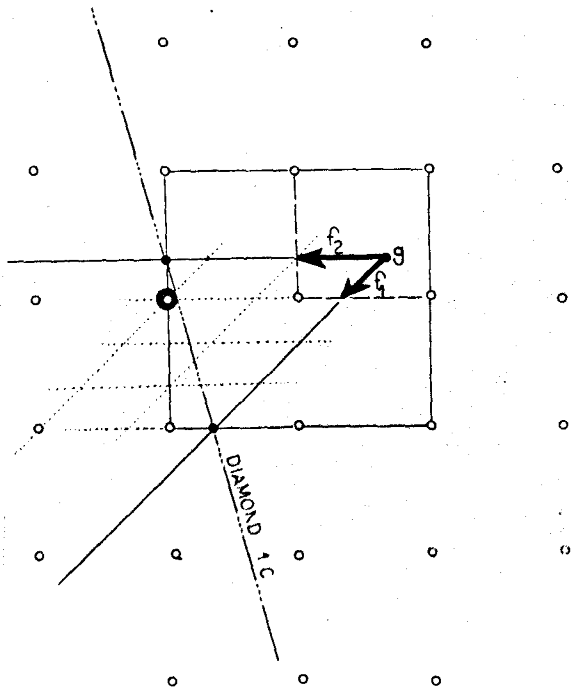

Fig. 2c.

Fig. 2. An illustration of the strengthening procedure of section 1.3.4.

In fig. 2a a 2-dimensional all-integer example for problem (A) is shown. The points marked by small white circles are integer solutions in $\mathbf{R}_x^2$. The feasible region (i.e., the cone $\bar{C}$) is delimited by the vectors $\bar{a}_1$ and $\bar{a}_2$. The integrality of the non-basic variables t_1 and t_2 can be represented geometrically by the dotted grid. A feasible integer solution of the problem (1, 3) therefore can only consist of those points in $\bar{C}$ which lie both on the x-grid and the t-grid. One such point is marked with a large bold face circle. The Balas-cut generated from the unit sphere around $\bar{x}$ is shown. The diamond cuts 1B and 1C are obtained by the strengthening procedure and are constructed in figs. 2b and 2c respectively. In this example, one has $\lceil \bar{a}_{ij} \rceil - \bar{a}_{ij} = g_i$ and one may therefore choose $f_{ij} = \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor$ or $f_{ij} = \bar{a}_{ij} - \lceil \bar{a}_{ij} \rceil$ indifferently. Figures 2b and 2c show the reduced problems obtained for two different cases of the vector f_2. In fig. 2b one obtains $\bar{\lambda}_1 = 2$ and $\bar{\lambda}_2 = 4$. From the points $(t_1 = \bar{\lambda}_1, t_2 = 0)$ and $(t_1 = 0, t_2 = \bar{\lambda}_2)$, in fig. 2a one obtains the valid cut called diamond 1B. Note that the set S is defined for the reduced problem (i.e., on fig. 2b) and not in the original problem. Thus only the point marked by a bold face circle has to be enumerated. If the set S has been defined in the original formulation, all the points marked with a crossed black circle (and probably many others, depending on the chosen convex outer-domain $D(x, S)$) would have had to undergo enumeration. This illustrates how integrality properties of the non-basic variables t_j can ***both increase the depth of an enumerative cut and reduce the amount of*** (explicit or implicit) ***enumeration.*** In fig. 2c note that no points need be enumerated (i.e., $S = \emptyset$ and the cut is therefore ***valid***) because the center of the cube U^1 does not lie on the dotted grid (i.e. $t_2 \equiv 0 \pmod 1$). Here one has $\bar{\lambda}_1 = 4$ and $\bar{\lambda}_2 = 7/3$; this generates the valid cut called diamond 1C on fig. 2a.

The improvement of branch and bound algorithms concerns the so-called backtracking phase, where a new partial solution P_ν is chosen for enumeration: one chooses a set S of solutions ($S \subset P_\nu$) and defines the corresponding enumerative cut in such a way that a bound B_ν can be obtained (by dual simplex optimization, for instance) which allows us to disregard P_ν from further consideration.

The set S is always smaller than P_ν: this is the reward for the computations required by the construction of the cut. The appendix I presents such a constructive procedure; for the 0–1 case one notes that the definition of S (I.12 and 13) corresponds to a particular partition of P_ν; its implementation into existing codes, which usually make use of a variety of similar optimality tests, therefore presents no difficulty.

2. Diamond cuts

Let us define the parallelotope $U(\bar{x}, \Delta^+, \Delta^-)$ as the set of points x in the n-dimensional space $\mathbf{R}^n_x$ of the variables x_i ($i \in N$) satisfying the following conditions (see fig. 3):

$$-\Delta_i^- \leqslant x_i - \tilde{x}_i \leqslant \Delta_i^+ , \qquad \forall i \in N_{\mathrm{I}}$$

$$x_i \text{ arbitrary} \qquad \forall i \in (N - N_{\mathrm{I}})$$

where $\tilde{x}$ is a point satisfying the integrality requirements (3) and otherwise arbitrary, for instance

$$\tilde{x}_i = \bar{x}_i - g_i , \qquad \forall i \in N_{\mathrm{I}}$$

where

$$\text{either } g_i = \bar{x}_i - \lfloor \bar{x}_i \rfloor$$

$$\text{or} \quad g_i = \bar{x}_i - \lceil \bar{x}_i \rceil .$$

It is furthermore assumed that $\bar{x}$ lies in the interior of $U(\bar{x}, \Delta^+, \Delta^-)$ and the quantities

$$\left.\begin{array}{ll} \Delta_i^+ \geqslant 0 , & \Delta_i^+ \equiv 0 \pmod 1 \\ \Delta_i^- \geqslant 0 , & \Delta_i^- \equiv 0 \pmod 1 \end{array}\right\} \quad \forall i \in N_{\mathrm{I}}$$

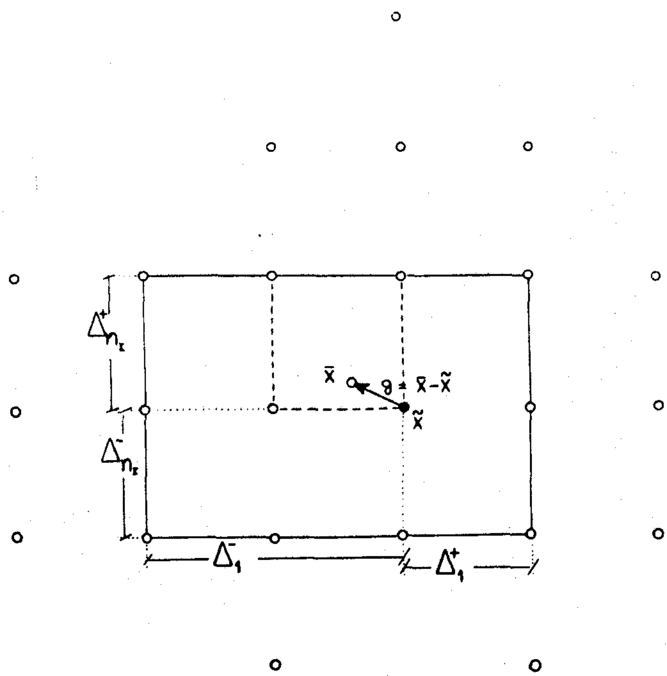

Fig. 3.

are subject to

$$\left.\begin{array}{l} -\Delta_i^- < g_i < \Delta_i^+ \\ \Delta_i^+ + \Delta_i^- \geqslant 1\,. \end{array}\right. \qquad \forall i \in N_\mathrm{I}$$

For simplicity, we shall only consider the following two typical examples in this section.

Example 1: $U^0(\bar{x})$. (see fig. 4a)

Let $\quad g_i = \bar{x}_i - \lfloor \bar{x}_i \rfloor$

and $\quad \Delta_i^+ = 1$

$\Delta_i^- = 0\,, \qquad \forall i \in N_\mathrm{I}$,

then $U^0(\bar{x})$ contains no point satisfying the integrality requirements (3) in its interior. It is in a way, the smallest desirable parallelotope $U(\bar{x})$.

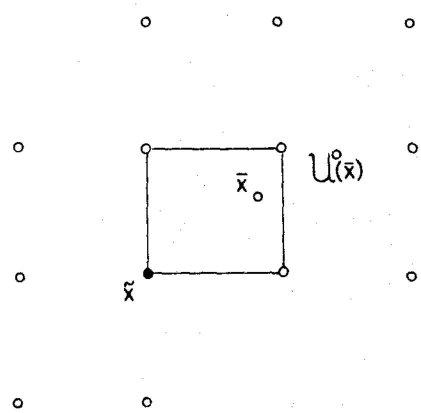

Fig. 4a.

Example 2. $U^1(\bar{x})$. (see fig. 4b)

Let

$$g_i = \bar{x}_i - \lfloor \bar{x}_i \rfloor \ \text{ if } \ \bar{x}_i - \lfloor \bar{x}_i \rfloor \leqslant 1/2 \qquad i \in N_\mathrm{I}$$

and

$$g_i = \bar{x}_i - \lceil \bar{x}_i \rceil \ \text{ otherwise .}$$

$$\Delta_i^+ + \Delta_i^- = 1 \,, \qquad i \in N_\mathrm{I} \tag{12}$$

The parallelotope $U^1(\bar{x})$ generated in this manner contains exactly one $(n-n_\mathrm{I})$-dimensional linear subspace R which satisfies the conditions (3), namely

$$R = \left\{ x \,\middle|\, \begin{array}{l} x_i = \bar{x}_i - g_i = \tilde{x}_i \,, \quad \forall i \in N_\mathrm{I} \\ x_i \in \mathbb{R} \,, \quad \forall i \in (N - N_\mathrm{I}) \,. \end{array} \right\} \tag{13}$$

In a two dimensional all-integer situation the parallelograms $U^0(\bar{x})$ and $U^1(\bar{x})$ are shown in fig. 4. Note incidentally that $U^1(\bar{x})$ has a volume $V = 2^{n_\mathrm{I}}$ (in n_I-space) while containing only one point x in its interior for any dimension n_I. (On the other hand, the volume of $U^0(\bar{x})$ is only 1 for all n_I!) In view of the intersection cut theory, this may serve as an indication that $U^1(\bar{x})$ is a better convex outer-domain to begin with than $U^0(\bar{x})$.

We now turn to the construction of the convex polytopes $D(\bar{x}, k, \Delta^+, \Delta^-)$ called diamonds in [4], which contain the parallelotope $U(\bar{x}, \Delta^+, \Delta^-)$. These sets $D(\bar{x}, k, \Delta^+, \Delta^-)$ are meant to be used as convex outer

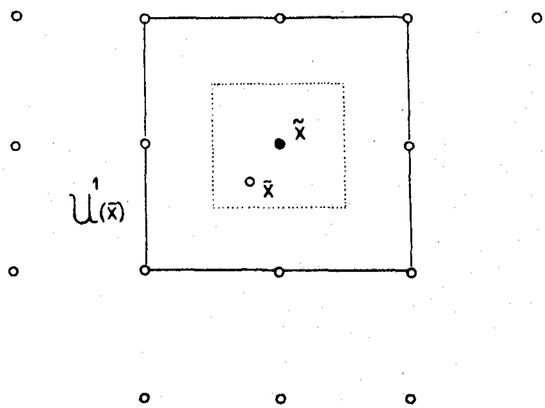

Fig. 4b.

domains, and their primary motivation therefore probably lies in the fact that the coefficients α_j of the intersection inequality are relatively easy to obtain numerically. Also it turns out that the sets S corresponding to $D(\bar{x}, k, \Delta^+, \Delta^-)$ can be conveniently identified, in a quite appropriate way for implicit enumeration.

And finally, let us mention that diamond cuts reflect algebraic properties of the matrix $\bar{A}$, in very much the same way as the mixed-integer Gomory cuts [6]; in fact much of the recent developments of the mixed-integer theory [8] can be applied in this context.

After these few lines of justification, let us define $D(\bar{x}, k, \Delta^+, \Delta^-)$. First one constructs the section $SD(\bar{x}, k, \Delta^+, \Delta^-)$ of $D(\bar{x}, k, \Delta^+, \Delta^-)$ with the n_I-dimensional manifold obtained by setting

$$x_i = \bar{x}_i\,, \qquad \forall i \in N - N_I \tag{14}$$

in the space $\mathbb{R}^n_x$. This manifold is the space of the integer constrained variables x_i $(i \in N_I)$, and $D(\bar{x}, k, \Delta^+, \Delta^-)$ is then the prismal extension of $SD(\bar{x}, k, \Delta^+, \Delta^-)$ obtained by letting the remaining variables x_i, take arbitrary values: $x_i \in \mathbb{R}$, $i \in (N-N_I)$. For brevity we shall define here the polyhedron $SD(\bar{x}, k, \Delta^+, \Delta^-)$, called *diamond*, in geometrical terms; another definition can be found in [4].

For any given integer k ($1 \leqslant k \leqslant n_{\mathrm{I}}$) and the parallelopiped $SU(\bar{x}, \Delta^+, \Delta^-)$ which is the section of $U(\bar{x}, \Delta^+, \Delta^-)$ according to (14), one considers the following $(2^{n_{\mathrm{I}}} + 2n_{\mathrm{I}})$ integer points: (see fig. 5) first, the $2^{n_{\mathrm{I}}}$ vertices of $SU(\bar{x}, \Delta^+, \Delta^-)$ characterized by

$$x_i = \begin{cases} = \tilde{x}_i + \Delta_i^+ \\ \text{or} \\ = \tilde{x}_i - \Delta_i^- \end{cases} , \forall i \in N_{\mathrm{I}} \tag{15}$$

and then, the $2n_{\mathrm{I}}$ points lying on the coordinate axis through x_i at the ordinate

$$\left.\begin{array}{lll} & \bar{x}_i + k(\Delta_i^+ - g_i) & (> \bar{x}_i) \\ \text{or} & \bar{x}_i - k(\Delta_i^- + g_i) & (< \bar{x}_i) \end{array}\right\} , i \in N_{\mathrm{I}} \tag{16}$$

the other components being $x_s = \bar{x}_s$, $s \in N_{\mathrm{I}} - \{i\}$.
$SD(\bar{x}, k, \Delta^+, \Delta^-)$ is then the convex hull of the above-defined is $(2^{n_{\mathrm{I}}} + 2n_{\mathrm{I}})$ points (see fig. 5). Since $1 \leqslant k \leqslant n_{\mathrm{I}}$ there are clearly n_{I} diamonds SD. For instance, the two 2-dimensional diamonds constructed with $U^{\mathrm{I}}(\bar{x})$ (see fig. 4) are shown in fig. 6.

2.1. *The cuts*

The detailed algebraic characterization of the faces of the diamonds SD and the formulae for the coefficients α_j, $j \in \bar{N}$ (10) of the diamonds intersection inequalities can be written down in a straightforward manner, and are presented in [4]. Instead we choose here to derive expressions for α_j directly from the inequalities (18) (described below), which turn out to be identical with the diamond cuts generated in the mixed-integer case from the polytopes $D(\bar{x}, k, \Delta^+, \Delta^-)$.

2.1.1: *The mixed-integer Gomory cutting planes and their enumerative extensions*:

Consider the i-th row of $\bar{A}$, $i \in N_{\mathrm{I}}$

$$x_i = \bar{x}_i - \sum_{j \in \bar{N}} \bar{a}_{ij} t_j \, . \tag{17}$$

One may then derive the inequality (for more details see [4], [6])

$$\sum_{j \in \bar{N}} \sigma_i^j \bar{a}_{ij} t_j \geqslant (\Delta_i^+ - g_i)(\Delta_i^- + g_i) \tag{18}$$

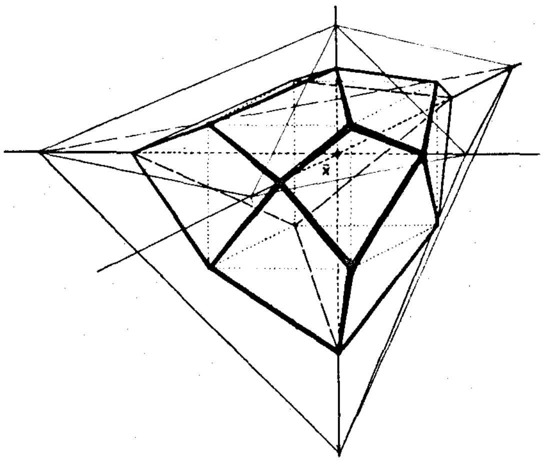

Fig. 5a. (k = 2).

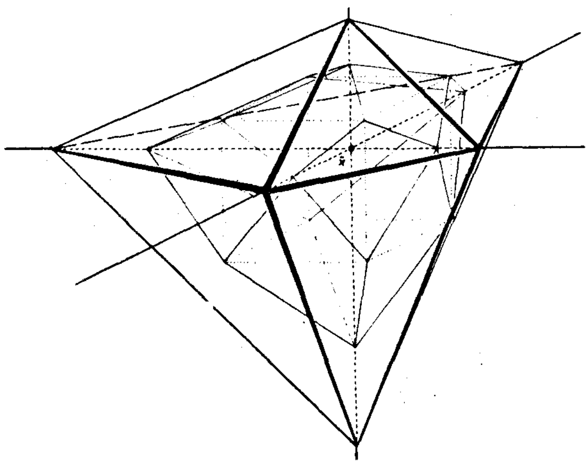

Fig. 5b. (k = 3).

Fig. 5. Example of the three dimensional diamond polyhedra $SD(\bar{x}, k, \Delta^+, \Delta^-)$ based on $U^1(\bar{x})$, i.e. with $\Delta_i^+ = \Delta_i^- = 1, \quad i = 1, 2, 3$.

where

$$g_i = \bar{x}_i - \lfloor \bar{x}_i \rfloor \text{ , positive fractional part of } \bar{x}_i$$

$$\Delta_i^+ \geqslant 1, \ \Delta_i^- \geqslant 0 \text{ and } \Delta_i^+, \Delta_i^- \equiv 0 \ (\text{mod } 1)$$

$$\sigma_i^j = \begin{cases} = (\Delta_i^+ - g_i) & \text{if } \bar{a}_{ij} \geqslant 0 \\ = -(\Delta_i^- + g_i), & \text{if } \bar{a}_{ij} < 0 \end{cases}$$

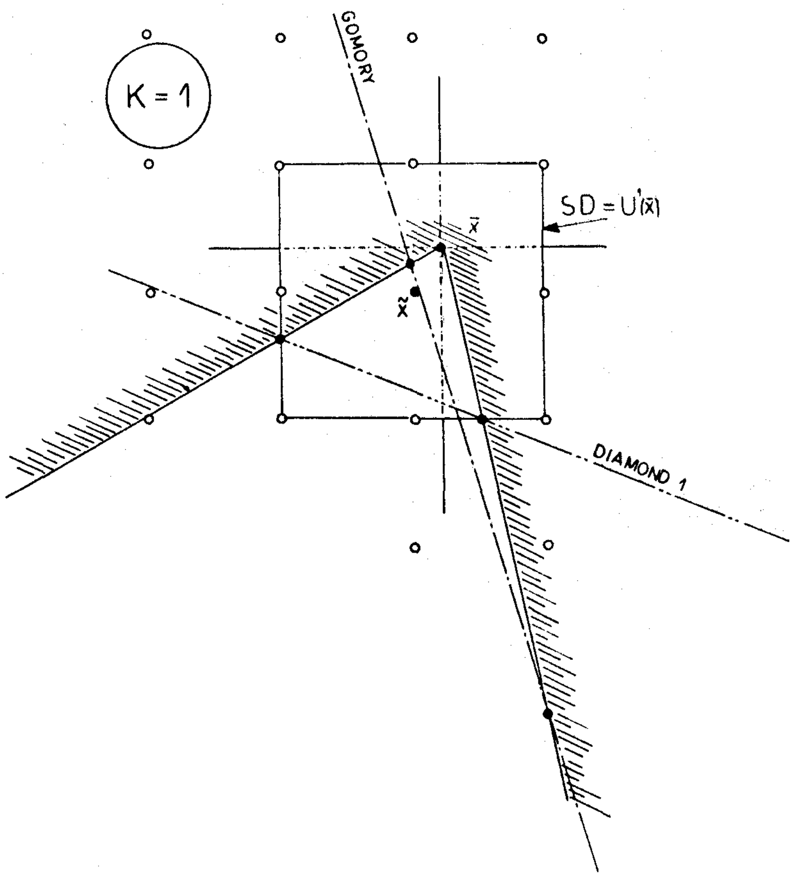

Fig. 6a. The case $k = 1$. The $2n_I$ points (16) are not extreme, as they lie on the faces of $U^1(x)$; this is always the case for any n_I when $k = 1$.

Fig. 6. A comparison of diamond and Gomory cuts in the two dimensional case. As in fig. 5, the diamond polyhedra *SD* here are based on $U^1(\bar{x})$.

The set S is defined by

$$S = \left\{ x \equiv (x_1, x_2, ..., x_n) \middle| \begin{array}{l} x \in C\,,\ x_i \equiv 0 \text{ (mod 1)} \\ \lfloor \bar{x}_i \rfloor - \Delta_i^- + 1 \leqslant x_i \leqslant \lfloor \bar{x}_i \rfloor + \Delta_i^+ - 1 \end{array} \right\}.$$

Clearly $S = \emptyset$ when $\Delta_i^+ + \Delta_i^- = 1$ and (18) then corresponds to a Gomory cut [6]; it is shown in section 2.1.3 how the coefficients $\bar{a}_{ij}$ can

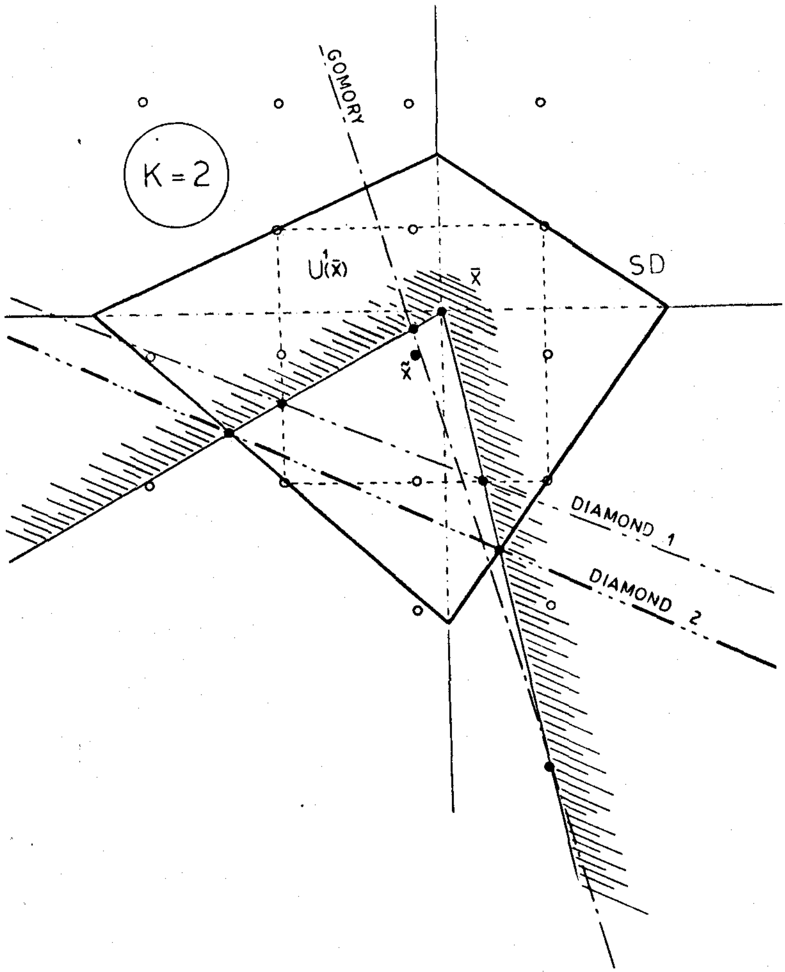

Fig. 6b. The case $k = 2$. The 2^{n_I} points (15) are not extreme, as they lie on the faces of SD; this is always the case for any n_I when $k = n_I$.

be replaced by their fractional parts f_{ij}, as for the conventional Gomory mixed-integer cut.

2.1.2. *Diamond cuts*

Suppose one generates all the n_I inequalities (18) which can also be written

$$\sum_{j \in N} q_{ij} t_j \geq 1\,, \quad \forall i \in N_I \tag{19}$$

with

$$q_{ij} = \begin{cases} = \dfrac{-\bar{a}_{ij}}{(\Delta_i^+ - g_i)}, & \text{if } \bar{a}_{ij} \leqslant 0 \\[2ex] = \dfrac{\bar{a}_{ij}}{(\Delta_i^- + g_i)}, & \text{if } \bar{a}_{ij} > 0 . \end{cases} \tag{20}$$

Now, for each $j \in \bar{N}$, let us take the *arithmetic mean of the* k ($1 \leqslant k \leqslant n_{\mathrm{I}}$) *largest coefficients* q_{ij} (which are all $\geqslant 0$) and thus set

$$\alpha_j = k^{-1} \sum_{i \in N^j} q_{ij} , \quad j \in \bar{N} \tag{21}$$

where

$$N^j = \{ i \in N_{\mathrm{I}} \mid q_{ij} \geqslant q_{sj} , \ \forall s \in N_{\mathrm{I}} \} . \tag{22}$$

Clearly (19, 20, 21, and 22) imply the inequality

$$\sum_{j \in \bar{N}} \alpha_j t_j \geqslant 1 \tag{23}$$

and, on the other hand, this is the diamond cut generated from $D(\bar{x}, k, \Delta^+, \Delta^-)$, as can be verified by comparison of (21) with the corresponding result in [4].
Because of the parameters Δ_i^+ and Δ_i^-, diamond inequalities can be made arbitrarily sharp (deep) and in particular deeper than the classical mixed-integer Gomory cuts; but one shouldn't forget that they require some additional work since there still remains to check the feasible integer solutions possibly contained in the set $S(k, \Delta^+, \Delta^-)$ which belongs to $D(\bar{x}, k, \Delta^+, \Delta^-)$.

2.1.3. *Proposition*: Replacing the quantities $\bar{a}_{ij}$ for $i \in N_{\mathrm{I}}$ and $j \in \bar{N}_{\mathrm{I}}$ for f_{ij} according to the following rule

$$f_{ij} = \begin{cases} = \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor , & \text{when} \quad \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor \leqslant g_i \\ = 0 , & \text{if} \quad \bar{a}_{ij} \equiv 0 \ (\mathrm{mod}\ 1) \\ = \bar{a}_{ij} - \lceil \bar{a}_{ij} \rceil , & \text{when} \quad \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor \geqslant g_i \\ = \bar{a}_{ij} , & \text{when either } i \notin N_{\mathrm{I}} \text{ or } j \notin \bar{N}_{\mathrm{I}} \text{ (or both)} \end{cases} \tag{24}$$

with $g_i = \bar{x}_i - \lfloor \bar{x}_i \rfloor \neq 0$ one produces a diamond inequality which is *uniformly stronger* then (21, 23).

A rigorous proof is not presented here, but it is, in fact, sufficient to remark that the Gomory cuts are known to become uniformly stronger when (24) is used; hence the same will hold for the arithmetic means (21). Note that no assumption concerning the matrix F (see section 1.3.4) is necessary here; this is due to the fact that the proof is algebraic in nature.

2.2. *Checking the validity of a cut*

2.2.1. Let us first describe a *very general procedure* for generating and checking enumerative inequalities. In many ways it is conceptually related to the approach given in [9]. In the n-dimensional space $\mathbb{R}_x^n$ consider the simplex defined by the following system

$$x_i = x_i - \sum_{j \in N} a_{ij} t_j \geqslant 0 \,, \quad i \in N \tag{25a}$$

$$\sum_{j \in \bar{N}} \alpha_j t_j \geqslant 1 \tag{26b}$$

The above simplex contains all the integer solutions which are eventually cut off by the cut (25b); it may therefore be taken as the set S which has to be enumerated in order to check the (conditional) validity of the cut (see section 1.3). From a practical point of view it is often preferable for this enumeration to have the integer-valued variables as non-basics and one may therefore decide to exchange the $t_j, j \in (\bar{N} - \bar{N}_{\mathrm{I}})$ for some of the x_i, $i \in N_{\mathrm{I}}$ in a few pivot steps; alternately one can simply go back to the original tableau A (choosing the constraints corresponding to $j \in \bar{N}$), and transform the coefficients α_j of the cut into

$$\tilde{\alpha}_i = \sum_{j \in \bar{N}} a_{ij} \,, \qquad i \in N \tag{26}$$

(25) then become

$$\sum_{i \in N} \tilde{\alpha}_i x_i \geqslant 1 \,.$$

At any rate, whatever definition of the simplex one uses, the idea is now to use one of the existing implicit enumeration schema either to determine that there is no integer solution or to compute the integer solution which maximizes the objective function in the simplex. Of course one can argue that this is, in principle, a problem which is of the same type as the original one (1), but, in fact, the size of the enumeration remains small here because of the cut (25b or 26). Obviously there is a trade-off here between the depth of the cut and the amount of computation involved in the enumeration. The main advantage of this approach is that it is completely general: there is no need for the definition of a convex outer-domain since the parameters α_j can be chosen arbitrarily $\forall j \in \bar{N}$ such that $\infty > \alpha_j \geqslant 0$.
In contrast to the previous general procedure, we now want to focus our attention on the convex outer-domain (used to generate the cut) in order to characterize the set S in a more precise and direct way. Here again we only consider diamond polytopes $D(\bar{x}, k, \Delta^+, \Delta^-)$ as outer-domains (other examples are given in [4]).

Naturally the first case which one wants to deal with is the case where the convex outer-domain $D(\bar{x}, S)$ is known (by construction) to contain no feasible integer points in its interior (i.e., $S = \emptyset$).

2.2.2. *Some valid inequalities*

Starting with the parallelotope $U^0(\bar{x})$ (and the corresponding parameter Δ^+, Δ^-) one considers the diamond polytope $D(\bar{x}, n)$. It is shown in [3] that the intersection inequalities generated in this manner are

$$\sum_{j \in \bar{N}} \alpha_j t_j \geqslant 1 \tag{27a}$$

with

$$\alpha_j = \sum_{i \in N_1} \delta_i \sigma_i^j \tag{27b}$$

$$\sigma_i^j = \begin{cases} = \dfrac{\bar{a}_{ij}}{g_i}, & \bar{a}_{ij} \geqslant 0 \\[2ex] = \dfrac{-\bar{a}_{ij}}{(g_i - 1)}, & \bar{a}_{ij} < 0 \end{cases} \tag{27c}$$

where $g_i = \bar{x}_i - \lfloor \bar{x}_i \rfloor$,

$$\sum_{i \in N_I} \delta_i g_i (1 - g_i) = 1 \qquad (27d)$$

$$\delta_i \geqslant 0 , \quad \forall i \in N_I .$$

Thus the cuts (27) obtained in this manner are arbitrary *convex combinations* of the mixed integer Gomory cutting planes, generated from the rows i $\forall i \in N_I$. A procedure for improving the diamond cuts (27) is given in the appendix I.

2.2.3. *Conditionally valid inequalities*

While the previous section 2.2.2 was concerned with the construction of inequalities which are satisfied by all the feasible integer solutions of the original problem (1, 3), let us now turn to enumerative inequalities of another kind, namely where the enumeration is used to detect those solutions to (1, 3) which do not satisfy the new inequality. For simplicity of the exposition let us consider the particular parallelotope $U^1(\bar{x})$, and the corresponding diamond polytopes $D(\bar{x}, k)$ obtained by setting (12) into (15, 16). Since we are primarily concerned with the discrete variables in the enumeration procedure, it is sufficient to examine the sections SU^1 and $SD(x, k)$ defined by (14). (See fig. 5). For $k = 1$ one has $D(\bar{x}, 1) = U^1(\bar{x})$ and it has exactly one interior point $\tilde{x}: \tilde{x}_i = \bar{x}_i - g_i$, $i \in N_I$ (see (13)). If $k \geqslant 2$, for each $(n-k)$-dimensional face $F^{(n-k)}$, the points lying on the border of $U^1(\bar{x})$ but in the interior of $F^{(n-k)}$ will lie in the interior of $D(\bar{x}, l)$, $\forall l \geqslant k$. Algebraically one can characterize all the integer points z which lie in the interior of $D(\bar{x}, k)$ by the condition

$$\sum_{i \in N(k)} q_i < k$$

where $N(k)$ is an arbitrary subset of N_I, containing k elements;

$$q_i = \begin{cases} = \dfrac{\bar{x}_i - z_i}{\Delta_i^- + g_i} , & \text{for} \quad z_i \leqslant \bar{x}_i \\[2ex] = \dfrac{\bar{x}_i - z_i}{g_i - \Delta_i^+} , & \text{for} \quad z_i > \bar{x}_i . \end{cases}$$

By definition $q_i \geq 0$ and even $q_i \geq 1$ if $z \notin \text{int } U^1$. It is interesting to note that the number of points in the interior of $D(\bar{x}, k)$ grows exponentially with k but *not with the dimension* n_I. For instance $D(\bar{x}, 2)$ has at most $4n+1$ inner integer points, (this number grows with n as the $(k-1)$ power of n, $n^{(k-1)}$, for $D(\bar{x}, k)$) among which only those which are feasible are of interest.
Numerical experiments on small problems have shown a clear gain in overall efficiency when diamond inequalities were used in a cutting plane algorithm as compared to, say, the classical mixed-integer Gomory cuts [6] (see 2.1.1) or the intersection cuts [1]. A sample of small non-structured randomly generated problems was selected with the following characteristics:

a) Less than 10 integer constrained variables and less than 10 constraints. (Some problems were of the knapsack type.)

b) Gomory's cutting plane method did not give the solution in over a hundred cuts.

c) Balas' spherical cut did not give the solution in over a hundred cuts.

Various types of enumerative cuts were then used to solve these problems in order to study the comparative strength of enumerative inequalities of different types. All problems were solved in less than 10 cuts with an enumeration consisting of less than a hundred integer solutions. One may be encouraged (by the above argumentation) to use enumerative cuts for large problems as well (n_I large) but additional computational results are needed. Also for conclusive remarks on their efficiency in a branch and bound approach, further experimentation is necessary.

Appendix I

Example of an enumerative cut *

Let us consider the inequalities (18)

$$\sum_{j \in \bar{N}} q_{ij} t_j \geq 1, \quad i \in N_\text{I} \tag{I.1}$$

* The approach in this procedure bears several resemblances to the "cut-search" approach proposed by Glover in [12].

where

$$g_i = \bar{x}_i - \lfloor \bar{x}_i \rfloor \tag{I.2}$$

$$q_{ij} = \begin{cases} = \dfrac{\bar{a}_{ij}}{\Delta_i^- + g_i} & \text{if} \quad \bar{a}_{ij} \geq 0 \quad (I.3) \\[2ex] = \dfrac{\bar{a}_{ij}}{g_i - \Delta_i^+} & \text{if} \quad \bar{a}_{ij} < 0\,, \quad (I.4) \end{cases}$$

From (I.1–4) one may derive the (weaker) inequality

$$\sum_{j \in \bar{N}} \alpha_j t_j \geq 1 \tag{I.5}$$

where

$$\alpha_j = \max_{i \in N_I} q_{ij} \tag{I.6}$$

which is the diamond cut corresponding to $k = 1$. The algorithm below starts with the system (I.1) of Gomory cuts (i.e., $\Delta_i^+ = 1$, $\Delta_i^- = 0$) and derives the cut (I.5) which is a weaker inequality implied by (I.1). However, by increasing, the parameters $\Delta_i^{\pm}$ in a prescribed manner, the inequality (I.5) can be made arbitrarily strong.

Step 0: Set $\Delta_i^+ = 1$ and $\Delta_i^- = 0$, $\forall i \in N_I$.

Step 1: Generate the quotients q_{ij} (I.3–4).
Compute α_j, $\forall j \in N$ according to (I.6).
Call $i(j)$ the index $i \in N_I$ for which $\alpha_j = q_{ij}$ (in case there are several possibilities, just take one of them). If one desires a deeper cut yet, then go to 2, else stop.

Step 2: Determine $j_0 \in \bar{N}$ such that

$$\bar{c}_{j_0}/\alpha_{j_0} \leq \bar{c}_j/\alpha_j\,, \quad \forall j \in \bar{N} \tag{I.7}$$

where $\bar{c}_j$ are the coefficients of the objective function (topmost row).

Step 3: Define $i_0 = i(j_0)$ and set

$$\Delta^+_{i_0} := \Delta^+_{i_0} + 1 \quad \text{if} \quad \bar{a}_{i_0 j_0} < 0$$

or (I.8)

$$\Delta^-_{i_0} := \Delta^-_{i_0} + 1 \quad \text{if} \quad \bar{a}_{i_0 j_0} \geqslant 0 .$$

At any stage of the algorithm, the number of integer grid points contained in $D(\bar{x}, 1)$ is

$$s = \prod_{i \in N_I} (\Delta^+_i + \Delta^-_i - 1) . \tag{I.9}$$

The above algorithm is conceived for integrality requirements of the form

$$x_i \equiv 0 \pmod 1) , \qquad i \in N_I$$

If they read $x_i = 0$, or 1 ($i \in N_I$) however, then an improvement can be made in step 3;

Step 3': Define $i_0 = i(j_0)$ and set

$$\Delta^+_{i_0} = +\infty , \quad \text{if} \quad \bar{a}_{i_0 j_0} < 0 \tag{I.10}$$

or

$$\Delta^-_{i_0} = +\infty , \quad \text{if} \quad \bar{a}_{i_0 j_0} \geqslant 0 .$$

At any stage of this algorithmic construction, the cut may be geometrically represented as an intersection cut with a convex outer-polytope P (often unbounded) generated by some of the unit cube constraints

$$\left.\begin{array}{l} x_i \geqslant 0 \\ x_i \leqslant 1 \end{array}\right. , \quad i \in N_I \tag{I.11}$$

In fact the algorithm begins with P = unit cube (I.11);

then, setting $\left\{ \text{or} \begin{array}{l} \Delta^+_i = \infty \\ \Delta^-_i = \infty \end{array} \right\}$ corresponds to deleting the constraint $\left\{ \text{or} \begin{array}{l} x_i \leqslant 1 \\ x_i \geqslant 0 \end{array} \right\}$

Thus the outer-domain P becomes step-wise larger and larger. At any stage the set S consists of those vertices of the unit cube (I.11) which are in the interior of P; algebraically these points are characterized by

$$x_i = 1 \quad , \text{ if } \quad \Delta_i^+ = 1$$

$$x_i = 0 \quad , \text{ if } \quad \Delta_i^- = 0 \qquad \text{(I.12)}$$

$$x_i = 0 \text{ or } 1 \ , \text{ if } \quad \Delta_i^+ = \Delta_i^- = \infty \ . \qquad \text{(I.13)}$$

Note that $S = \emptyset$ whenever there exists at least one $i \in N_1$ with $\Delta_i^+ = 1$ and $\Delta_i^- = 0$. Thus the number s of elements in the set S is

$$s = 0, \quad \text{if} \quad N_0 = \{i \in N_1 \mid \Delta_i^+ = 1 \text{ and } \Delta_i^- = 0\} \neq \emptyset$$

$$s = 2^{n_1} \text{ if a) } N_0 = \emptyset$$

$$\text{and b) } N_1 = \{i \in N_1 \mid \Delta_i^+ = \Delta_i^- = \infty\}$$

where n_1 is the number of elements in N_1.

Practically, it was found that this procedure generates cuts which become rapidly deeper than the original Gomory cuts, also when the number s is restricted to small values. Figure 6 is an illustration of this fact. Furthermore the following remarks speak in favor of the use of such cuts in a branch and bound procedure (as an elimination or "trimming" device):

- The cuts have, by construction, the tendency to become fairly parallel to the objective function. (Thus the enumeration is concerned with relevant points, and the cuts give good bounds.)
- The sets S(I.12, 13) represent a way of choosing some of the variables which are not yet fixed (at the node considered) in a particular manner which improves the bound at the node whenever a new element of S (feasible or infeasible solution) is enumerated.
- This set S possesses exactly the same structure as the partial solutions defined in branch and exclude algorithms [11]; it is in fact, a new (smaller) partial solution defined by (I.12–13). The impelementation of the above procedure in existing branch and bound codes therefore presents no particular difficulty.

Appendix II

Another example for the construction of enumerative cuts

Consider the optimal tableau $\bar{A}$

$$x_i = \bar{x}_i - \sum_{j \in \bar{N}} \bar{a}_{ij} t_j , \quad i \in (N \cup M) . \tag{II.1}$$

One has $\forall i \in N_{\mathrm{I}} \subset N$:

$$\bar{x}_i - x_i = \begin{Bmatrix} g_i \\ g_i - 1 \end{Bmatrix} = \sum_{j \in \bar{N}} \bar{a}_{ij} t_j = \sigma_i^{-1} g_i (1 - g_i) \tag{II.2}$$

where $\sigma_i = -g_i$ or $(1 - g_i)$
(the vector σ characterizes a vertex of $U^0(\bar{x})$ and vice versa). One establishes that the diamond $SD(x, n_{\mathrm{I}})$ defined on $SU^0(x)$ is the set of points satisfying the $2^{n_{\mathrm{I}}}$ following inequalities

$$\sum_{i \in N_{\mathrm{I}}} \sum_{j \in \bar{N}} \delta_i \sigma_i \bar{a}_{ij} t_j \leqslant 1 \tag{II.3}$$

with

$$\sum_{i \in N_{\mathrm{I}}} \delta_i g_i (1 - g_i) = 1 , \quad \forall i \in N_{\mathrm{I}}$$
$$\delta_i \geqslant 0 . \tag{II.4}$$

The plane (II.3) with equality sign passes through the vertex of $U^0(\bar{x})$ which corresponds to σ, and therefore guarantees that this vertex does not lie in the interior of $D(\bar{x}, n_{\mathrm{I}})$. The intersection cut generated by the diamond $D(\bar{x}, n_{\mathrm{I}})$ is *valid* and reads

$$\sum_{j \in \bar{N}} \alpha_j t_j \geqslant 1$$

with

$$\alpha_j = \sum_{i \in N_{\mathrm{I}}} \delta_i \sigma_i^j \bar{a}_{ij} \tag{II.5}$$

where

$$\sigma_i^j = \begin{cases} = (1 - g_i), & \text{if} \quad \bar{a}_{ij} \geqslant 0, \quad i \in N_{\mathrm{I}}, \quad j \in \bar{N} \\ \\ = -g_i & \text{, if} \quad \bar{a}_{ij} < 0 \end{cases} \tag{II.6}$$

(it is a convex combination of Gomory cuts).

If one now decides to check a few of the $2^{n_{\mathrm{I}}}$ vertices of $U^0(\bar{x})$ directly, there is no need any longer for the presence of the corresponding inequalities in the system (II.3); these selected (few) vertices then become interior to the outer-domains defined by the reduced system (II.3). In particular, if one checks the vertex corresponding to σ^j for a given $j \in \bar{N}$, then (II.5) becomes

$$\alpha_j = \sum_{i \in N_{\mathrm{I}}} \delta_i \sigma_i^j \bar{a}_{ij} - \Delta_j(\delta) \tag{II.7}$$

where

$$\Delta_j(\delta) = \min_{\text{all } \tilde{\sigma} \neq \sigma^j} \left(\sum_{i \in \tilde{N}} \delta_i \, |\bar{a}_{ij}| \geqslant 0 \right) \tag{II.8}$$

$$\tilde{N} = \{i \,|\, \tilde{\sigma}_i \neq \sigma_i^j\} . \tag{II.9}$$

Thus (II.7) indicates a possible way to make the cut (II.5) deeper provided that

a) the vertex σ^j is checked independently,

b) the minimum over all possible $\tilde{\sigma}$ (II.8) has been determined.

To avoid the difficulty of b) one chooses δ in the following way:

$$\begin{aligned} &\delta_i^j = [\,|\bar{a}_{ij}|\, \Sigma^j\,]^{-1} \geqslant 0\,, \quad i \in N^j \\ &\Sigma^j = \sum_{i \in N^j} |\bar{a}_{ij}|^{-1} \, g_i(1 - g_i) \\ &N^j = \{i \in N_{\mathrm{I}} \,|\, \bar{a}_{ij} \neq 0\} \subset N_{\mathrm{I}} \\ &\delta_i^j = 0\,, \quad \forall i \in (N_{\mathrm{I}} - N^j) \end{aligned} \tag{II.10}$$

(the property II.4 is easily verified).

Then

$$\delta_i^j |\bar{a}_{ij}| = (\Sigma^j)^{-1} , \qquad \forall i \in N^j$$

and

$$\Delta_j(\delta^j) = K^j/\Sigma^j \tag{II.11}$$

where K^j is the number of elements contained in $\widetilde{N}$.

This provides now for a simple way of computing Δ_j namely;

- "check" if the vertex (i.e., integer solution) corresponding to σ^j is feasible; if yes, evaluate the objective function at this feasible integer point (and update the current best solution whenever appropriate).
- Then set $K^j = 1$.

Furthermore, one may set $K^j = 2$ if one chooses also to "*check*" also the $\binom{n}{1} = n$ vertices (neighbours) which differ from σ^j exactly by one component; frequently this requires no computation because these neighbours have been "checked" in previous steps. In general checking

$$\sum_{r=0}^{R} \binom{n}{r} \text{ vertices implies that one may set } K^j = R + 1.$$

Formulae for the construction of a cut

$$\sum_{j \in \bar{N}} \alpha_j t_j \geqslant 1$$

with coefficient $\alpha_{j_0} = \alpha_{j_0}(K^{j_0})$ for $K^{j_0} = 0, 1, 2, \ldots, n$

$$\alpha_{j_0} = \Sigma^{-1} \left(\sum_{i \in N^{j_0}} |\sigma_i^{j_0}| - K^{j_0} \right) \tag{II.12}$$

$$\alpha_j = \Sigma^{-1} \left(\sum_{i \in N^{j_0}} |\sigma_i^j| \, q_{ij}^{j_0} - s_j^{j_0}(K^j) \right) \tag{II.13}$$

$$N^{j_0} = \{ i \in N_{\mathrm{I}} \,|\, \bar{a}_{ij_0} \neq 0 \} \tag{II.14}$$

$$\Sigma = \sum_{i \in N^{j_0}} |\bar{a}_{ij_0}|^{-1} g_i(1 - g_i) \tag{II.15}$$

$$q_{ij}^{j_0} = \left| \frac{\bar{a}_{ij}}{\bar{a}_{ij_0}} \right|, \qquad i \in N^{j_0} \tag{II.16}$$

$s_j^{j_0}(K^j) \geqslant 0$ is a parameter which depends on $K^j (j \neq j_0)$. One may set, for instance

$$s_j^{j_0}(K^j) = \text{sum of the } K^j \text{ smallest values } q_{ij}^{j_0}, \ i \in N^{j_0} .$$

(A more detailed construction of these cuts is given in [3].)

Remarks

1) Since in a cut one has $\alpha_j \geqslant 0$, there is no need to increase the parameter K^j beyond the point where (II.7) becomes negative.

2) The amount of work required by the enumerative checking grows with

$$\sum_{r=0}^{R} \binom{n}{r} \qquad \text{where} \qquad R = K^j - 1 .$$

As a function of n this amount grows like n^R (i.e., *not* exponentially).

3) The above analysis has led to the improvement of α_j for one (previously chosen) $j \in \bar{N}$. Naturally the same can be done $\forall j \in \bar{N}$; one may generate n improved inequalities in this manner. There are many conceivable strategies whicn can be applied here:

- generate all n inequalities systematically, with for instance $K^j = K$, $\forall j \in \bar{N}$ ("cutting-polyhedron" approach).
- A choice rule to determine K^j. $\forall j \in \bar{N}$; typically a step-wise improvement increasing one K^j at a time, as to improve the depth of the cut (in the direction of the objective function gradient) with relatively little enumeration.

4) in the mixed-integer case where $N_{\mathrm{I}} \neq N$, the explicit enumerative checking of a vertex σ amounts to the solution of a linear program with $n-n_{\mathrm{I}}$ (non-basic) variables.

5) The enumerative checking of the vertices σ of $U^0(\bar{x})$ can be made explicitely (one after the other) or implicitly (direct search in the set of selected vertices). Because there are ways to accomplish the explicit computations in a very economical manner, it is not clear at this point, which one of these two alternatives yields the best overall efficiency.

6) Basically the example of appendix II makes use of the diamond of order n, $D(\bar{x}, n)$ whereas the example of appendix I builds on the diamond $D(\bar{x}, 1)$. There are many possible ways to think of intermediate versions based on diamonds $D(\bar{x}, k)$, $1 \leqslant k \leqslant n$. This is an open research area for the construction of inequalities with the best overall efficiency.

Acknowledgement

I wish to thank Egon Balas and Ellis Johnson for our many discussions and their helpful criticism.

References

[1] E. Balas, "The intersection cut – A new cutting plane for integer programming", *Management Science Research Report No. 187* (1969), Carnegie-Mellon University.

[2] E. Balas, "Intersection cuts from maximal extensions of the ball and the octahedron", 7th Mathematical Programming Symposium, The Hague (1970), *Mathematical Programming* to be published.

[3] C.-A. Burdet, "A class of cuts and related algorithms in integer programming", *Management Science Research Report No. 220* (1970) Carnegie-Mellon University.

[4] C.-A. Burdet, "The method of enumerative cuts", IBM, RC 3174 (1970) Watson Research Center, New York.

[5] A. Geoffrion, "Integer programming by implicit enumeration and Balas' method", *SIAM Review* 9 (1967) 178–190.

[6] R. Gomory, "An algorithm for integer solutions to linear programs", in: *Recent advances in mathematical programming*, Eds. R.L. Graves and P. Wolfe (McGraw-Hill, 1963).

[7] F. Glover, "Convexity cuts", University of Texas, Austin, Dec. 1969. *Operations Research*, (1971) to appear.

[8] R. Gomory and E. Johnson, "Some continuous functions related to corner polyhedra", 7th Mathematical Programming Symposium, The Hague (1970), *Mathematical Programming* to appear.

[9] F. Hillier, "A bound-and-scan algorithm for pure integer linear programming with general variables", *Operations Research* 17, No. 3 (1969).

[10] Hoang Tuy, "Concave programming under linear constraints", *Soviet Mathematics* (1964) 1437–40.

[11] A. Land and A. Doig, "An automatic method of solving discrete programming problems", *Econometrica* 28 (1960).

[12] F. Glover, "Cut-search methods in integer programming", 7th Mathematical Programming Symposium, The Hague (1970), *Mathematical Programming* to appear.

[13] E. Balas, "Integer programming and convex analysis", Management Science Report # 246 (April 1971), Carnegie-Mellon University.