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MINIMUM COST SPANNING TREE GAMES*

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We consider the problem of cost allocation among users of a minimum cost spanning tree network. It is formulated as a cooperative game in characteristic function form, referred to as a minimum cost spanning tree (m.c.s.t.) game. We show that the core of a m.c.s.t. game is never empty. In fact, a point in the core can be read directly from any minimum cost spanning tree graph associated with the problem. For m.c.s.t. games with efficient coalition structures we define and construct m.c.s.t. games on the components of the structure. We show that the core and the nucleolus of the original game are the cartesian products of the cores and the nucleoli, respectively, of the induced games on the components of the efficient coalition structure.

Key words: Game Theory, Cost Allocation, Spanning Tree.

1. Introduction

Consider a network which is composed of a common supplier connected to a number of geographically separated users by a minimum cost spanning tree graph. An example for such a situation is a cablevision network. The cablevision signals are initiated at a certain location (common supplier), and are being transmitted to the various cities (users) in the system which are connected to the common supplier by a minimum cost spanning tree network, see also [2, 3].

When the cost of a link between any two locations is known, a question that can be naturally raised is how should the total cost be allocated among the various users. Claus and Kleitman [2] initiated the discussion of this problem.

Following [3] we formulate the problem as a cooperative game in characteristic function form, referred to as a minimum cost spanning tree (m.c.s.t.) game. The core of a m.c.s.t. game is defined in fact by one of Claus and Kleitman's cost allocation criteria [2]. We show that the core of a m.c.s.t. game is never empty, and a point in the core can be simply read from any minimum cost spanning tree graph connecting all users with the common supplier. Thus, since there exist extremely efficient algorithms for constructing minimum cost

* This paper is a revision of [4].

spanning tree graphs, there is virtually no bound on the number of users in the network for which a solution in the core (i.e., a satisfactory cost allocation) can be found.

In the last two sections we study m.c.s.t. games with an efficient coalition structure. We define and construct m.c.s.t. games on each one of the components of the structure. We then show that the nucleolus of the original game is the cartesian product of the nucleoli of the m.c.s.t. games defined on the components of the structure. A similar result is obtained for the core of a m.c.s.t. game. These two results can lead to a considerable reduction when attempting to find the complete core or the nucleolus of a m.c.s.t. game. As far as we know, no computationally efficient method to find either all the vertices of the core or the nucleolus of a m.c.s.t. game is known.

1. Game theory formulation

As it should be apparent by now, this paper deals with costs rather then with revenues, which are usually associated with cooperative games. Therefore, traditional inequalities in game theory are reversed. Given a set $N = \{1, \dots, n\}$ of *players* whose power set $2^N \equiv \{S: S \subseteq N\}$ is the set of *coalitions*, a function

$$c: 2^N \to R,$$

with

$$c(\emptyset) = 0$$

is the *characteristic function* of the game $(N; c)$. If the characteristic function c is *monotone* (i.e. $c(S) \leq c(T)$ for $S \subset T \subset N$), then the game $(N; c)$ is *monotone*. The game $(N; c)$ is *proper* if the characteristic function is subadditive (i.e., $c(S) + c(T) \geq c(S \cup T)$ for all $S, T \in 2^N$, $S \cap T = \emptyset$). The *core* of the game $(N; c)$ is the set

$$\left\{(x_1, \dots, x_n) \in R^n: \sum_{i \in N} x_i = c(N), \sum_{i \in S} x_i \leq c(S), \forall S \subset N\right\}.$$

Throughout this paper a *graph* G is denoted by $G = (V, E)$, where V is the *node set* and E is the *edge set*.

The construction of a minimum cost spanning tree game is as follows. Each city (customer, player) i is associated with a node i in the complete graph whose node set is $N \cup \{0\}$. The datum is a non-negative symmetric matrix C whose generic element $c_{ij} = c_{ji}$ is the cost associated with the undirected arc $e_{ij} = e_{ji}$ $(i \neq j)$. The diagonal entries c_{ii} $(i = 0, 1, \dots, n)$ are all zero.

Given the c_{ij}'s we construct $\Gamma_N \equiv (V_N, E_N)$, a minimum cost spanning tree (m.c.s.t.) graph connecting the n cities (i.e. the node set N) with the common supplier (node 0). Similarly, for $S \subset N$ $\Gamma_S \equiv (V_S, E_S)$ is a m.c.s.t. graph whose node set is $S \cup \{0\}$. Of course, $V_N = N \cup \{0\}$.

The characteristic function $c: 2^N \to R$ of the *minimum cost spanning tree* (*m.c.s.t.*) game $(N; c)$ is defined by

$$c(\emptyset) = 0,$$

and

$$c(S) = \sum_{e_{ij} \in E_S} c_{ij} \quad \text{for } \emptyset \neq S \subset N.$$

Note that every m.c.s.t. game is proper, but not necessarily monotone as the following example shows.

Example 1. Suppose the cities served by the common supplier are labeled 1, 2, 3 and the common supplier is labeled 0. The various costs of the possible links are shown in Fig. 1.

In this example $c(\{1\}) = 1$; $c(\{2\}) = c(\{3\}) = 1.7$; $c(\{1,2\}) = c(\{1,3\}) = 2$; $c(\{2,3\}) = 3.4$; $c(\{1,2,3\}) = 3$.

The monotone minimum cost spanning tree (m.m.c.s.t.) game derived from the m.c.s.t. game $(N; c)$ is the game whose characteristic function $\bar{c}$ is

$$\bar{c}(S) = \min_{S \subset T \subset N} c(T), \quad S \subset N. \tag{1}$$

The use of the characteristic function $\bar{c}$ amounts to the assumption that every set of players S is allowed to use nodes in $N \setminus S$ while constructing its separate network. By definition

$$\bar{c}(S) \leq c(S) \quad \text{for all } S \subset N. \tag{2}$$

Furthermore,

$$\bar{c}(S) \leq \bar{c}(T) \quad \text{for all } S \subset T \subset N, \tag{3}$$

i.e. $\bar{c}$ is a monotone set function. In Example 1, $\bar{c}(S) = c(S)$ for all $S \neq \{2, 3\}$; $\bar{c}(\{2,3\}) = 3 < c(\{2,3\})$.

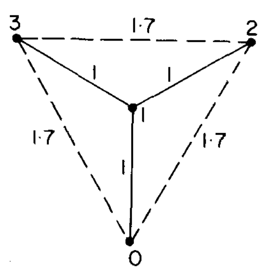

Fig. 1.

To a given characteristic function c of a m.c.s.t. game $(N;c)$ corresponds a collection $Q(c)$ of cost matrices (c_{ij}), such that for each $(c_{ij}) \in Q(c)$ $\sum_{e_{ij} \in E_S} c_{ij} = c(S)$, $\forall S \subseteq N$. The collection $Q(c)$ can be characterized as follows: For each $(c_{ij}) \in Q(c)$

(i) $c_{i0} = c(\{i\})$, $i = 1, \ldots, n$, and

(ii) if $c(\{i,j\}) = c(\{i\}) + c(\{j\})$, then $c_{ij} \geq \max\{c(\{i\}), c(\{j\})\}$; otherwise, if $c(\{i,j\}) < c(\{i\}) + c(\{j\})$, then $c_{ij} = c(\{i,j\}) - \min\{c(\{i\}), c(\{j\})\}$.

Thus $Q(c)$ consists of a unique cost matrix (c_{ij}) (for which $\sum_{e_{ij} \in E_S} c_{ij} = c(S)$, $\forall S \subseteq N$) iff $c(\{i,j\}) < c(\{i\}) + c(\{j\})$ for each pair (i,j). Clearly, if $Q(c)$ is not a singleton it is an infinite collection of cost matrices.

Similarly, to a given characteristic function c of a m.m.c.s.t. game $(N;c)$ corresponds a collection $\bar{Q}(c)$ of cost matrices, such that for each $(c_{ij}) \in \bar{Q}(c)$

$$\operatorname*{Min}_{S \subseteq R} \left\{ \sum_{e_{ij} \in E_R} c_{ij} \right\} = \bar{c}(S), \quad \forall S \subseteq N.$$

One can easily verify that $\bar{Q}(c)$ is a singleton iff the following is satisfied:

(i) $c(\{i\}) < c(\{i,j\})$, $\forall i, j$,

(ii) $c(\{i,j\}) < c(\{i,j,k\})$, $\forall i, j, k$, and

(iii) $c(\{i,j\}) < c(\{i\}) + c(\{j\})$, $\forall i, j$.

Further, let c and $\bar{c}$ denote the characteristic function of a m.c.s.t. game and a m.m.c.s.t. game, resp., derived from the cost matrix (c_{ij}). Then, one can easily show that $Q(c) \subseteq \bar{Q}(\bar{c})$. Finally, let us denote by $(N;c)$ and $(N;c')$ the m.c.s.t. games corresponding to the cost matrices (c_{ij}) and (c'_{ij}), respectively, and assume that for each $S \subseteq N$ $c(S) = c'(S)$. Then, the last observation $(Q(c) \subseteq \bar{Q}(\bar{c}))$ implies that $\bar{c}(S) = \bar{c}'(S)$, $\forall S \subseteq N$, where $\bar{c}(\cdot)$ and $\bar{c}'(\cdot)$ are the characteristic functions in the m.m.c.s.t. games derived from the cost matrices (c_{ij}) and (c'_{ij}), respectively.

2. The existence of the core of a m.c.s.t. game

Claus and Kleitman [2] reviewed the criteria which a cost allocation method among users of a minimum cost spanning tree network must satisfy, and state: "The method must have a stability against system breakup. It should not be an advantage to one or more users to secede from the system. Thus, there are limits to which a method can subsidize one user or class of users at the expense of others."

Formally, let $x = (x_1, \ldots, x_n)$ represent an allocation of the total cost among the cities. In order to comply with the Claus and Kleitman restriction, x must satisfy

$$\sum_{i \in S} x_i \leq c(S) \quad \text{for all } S \subset N, \qquad (4)$$
$$\sum_{i=1}^{n} x_i = c(N).$$

For any vector x that represents a cost allocation scheme and satisfies (4), no subset of cities pays more than it would have paid, had it established its own network. Therefore no subset has an incentive to secede. But the vectors x satisfying (4) are the core of the cooperative game $(N; c)$. Thus, Claus and Kleitman set of satisfactory cost allocations defines precisely the core of a m.c.s.t. game $(N; c)$. A similar definition can be given for the core of the m.m.c.s.t. game $(N; \bar{c})$.

The system (4) may include solutions which are not non-negative, e.g. $(-0.4, 1.7, 1.7)$ is a solution of (4) in Example 1. A negative cost is a payoff, and one may wish to avoid this situation by adding the constraints

$$x_i \geq 0, \quad i = 1, \dots, n \tag{5}$$

to the definition of the core of $(N; c)$. The monotonicity of the characteristic function $\bar{c}$ implies that (5) is redundant for the m.m.c.s.t. game $(N; \bar{c})$. To see this note that if x is in the core, then

$$x_i = \sum_{j\in N} x_j - \sum_{j\in N \setminus \{i\}} x_j = \bar{c}(N) - \sum_{j\in N\setminus\{i\}} x_j \geq \bar{c}(N) - \bar{c}(N \setminus \{i\}) \geq 0.$$

Next, we prove that the core of a m.m.c.s.t. game $(N; \bar{c})$ is nonempty. In fact, we do more than that by showing how points in the core can be simply read from the various minimum cost spanning tree graphs associated with the data c_{ij} $(i, j = 0, 1, \dots, n)$ for any matrix (c_{ij}) in $\bar{Q}(\bar{c})$. Keeping (2) in mind we immediately have the same result for m.c.s.t. games as well.

Given a m.c.s.t. graph $\Gamma_N = (V_N, E_N)$, denote by $e^i \in E_N$ $(i = 1, \dots, n)$ the incident edge to node i which is also in the unique path from node i to node 0 in Γ_N. The vector $(x^1, \dots, x^n)$ is a *minimum cost tree solution* if every x^i is the cost associated with the node e^i. (In Example 1 the minimum cost tree solution is $(1, 1, 1)$). For $S \subset N$, let $X_S = \{e^i : i \in S\}$ and $\bar{X}_S = E_N \setminus X_S$. For any edge set $B \subset E_N$ let $m(B) = \sum_{(i,j)\in B} c_{ij}$. Thus,

$$c(N) = m(X_S) + m(\bar{X}_S). \tag{6}$$

Note that if an edge $(i, j) \in \bar{X}_S$, then either $i \in N \setminus S$ or $j \in N \setminus S$, whereas if $(i, j) \in E_S$, then both $i \in S \cup \{0\}$ and $j \in S \cup \{0\}$. Therefore,

Lemma 1. *The sets E_S and $\bar{X}_S$ are disjoint.*

Next, consider the graph $Y_S = (V_N, E(S))$, where

$$E(S) = E_S \cup \bar{X}_S. \tag{7}$$

From Lemma 1 we have $|E(S)| = n$. We shall prove that Y_S is connected, and therefore a tree.

Lemma 2. *The graph $Y_S = (V_N, E(S))$ is a tree.*

Proof. It is enough to show, in this case, that for each $a_1 \in V_N$ there is a chain in $\bar{X}_S$ (possibly of length 0) emanating from a_1 to some node in V_S.

Let $A_1 = \{e^{a_1}, e^{a_2}, \ldots\}$ denote the unique chain in E_N leading from a_1 to 0, where e^{a_i} denotes the edge (a_i, a_{i+1}), $i = 1, 2, \ldots$. If $A_1 \subset \bar{X}_S$, we are through, since $0 \in V_S$. If $A_1 \not\subset \bar{X}_S$, let e^{a_k} be the first edge of A_1 that is not in $\bar{X}_S$. Then $e^{a_k} \in E_N \setminus \bar{X}_S$, so we have $a_k \in S \subset V_S$, by definition. The chain $\{e^{a_1}, \ldots, e^{a_{k-1}}\}$ therefore provides the desired connection.

The main result of this section is:

Theorem 3. *Given a cost matrix C, every minimum cost tree solution is in the core of both the m.c.s.t. game $(N; c)$ and the m.m.c.s.t. game $(N; \bar{c})$.*

Proof. Let $x = (x^1, \ldots, x^n)$ be the minimum cost tree solution corresponding to a m.c.s.t. graph Γ_N of C. Recalling (2) it suffices to show that x is in the core of $(N; \bar{c})$. By its construction, $\sum_{i\in N} x^i = \bar{c}(N)$.

Now, assume the existence of a coalition R for which

$$\sum_{i\in R} x^i > \bar{c}(R). \tag{8}$$

Let $S \subset N$, and $R \subset S$ such that

$$\bar{c}(R) = c(S). \tag{9}$$

Since $x^i \geq 0$ $(i = 1, \ldots, n)$ we have

$$\sum_{i\in S} x^i > c(S) \tag{10}$$

or equivalently

$$m(X_S) > c(S) \tag{11}$$

which implies, by (6) and (7), that

$$c(N) = m(X_S) + m(\bar{X}_S) > c(S) + m(\bar{X}_S) = m(E(S)). \tag{12}$$

Now, by Lemma 2, Y_S is a tree and thus (12) contradicts the optimality of the m.c.s.t. graph Γ_N. Hence, for each coalition R

$$\sum_{i\in R} x^i \leq \bar{c}(R),$$

which implies that $(x^1, \ldots, x^n)$ is in the core of both $(N; c)$ and $(N; \bar{c})$.

Bird [1] has independently studied the class of m.c.s.t. games, and proved the nonemptiness of the core of $(N; c)$. His proof is, unfortunately, erroneous since he assumes that for every arc (i, k), where (i, k) is that arc in E_S (the edge set of Γ_S) which is on the unique path from i to 0 in Γ_S, the cost of arc (i, k) is always smaller than or equal to x^i. Motivated by [3] and the first version of our paper [4], Megiddo [9] has independently proved the non-emptiness of the core of $(N; \bar{c})$. Further, Megiddo [9] has shown that the core of the game $(N; \hat{c})$, in which $\hat{c}(S)$ is the cost associated with the Steiner tree that spans $S \cup \{0\}$, might be empty. Megiddo [8] and Littlechild [7] have computed the nucleolus of a special class of m.c.s.t. games, see also Granot and Huberman [5].

The core of a game $(N; c)$ is a compact convex polyhedron (possibly empty) in R^n. Any minimum cost tree solution is a point in the polyhedra defined by the cores of $(N; c)$ and $(N; \bar{c})$. In fact, it is a vertex of both polyhedra.

Theorem 4. *Given a cost matrix C, every minimum cost tree solution is a vertex in the core of both the m.c.s.t. game* $(N; c)$ *and the m.m.c.s.t. game* $(N; \bar{c})$.

Proof. The core of the m.c.s.t. game $(N; c)$ contains that of the m.m.c.s.t. game $(N; \bar{c})$, so it suffices to prove the result for the m.c.s.t. game $(N; c)$. Let $x = (x^1, \dots, x^n)$ be a minimum cost tree solution corresponding to a m.c.s.t. graph Γ_N of C. We label the nodes $\{1, \dots, n\}$ of Γ_N so that if $i > j$, then i is not on the (unique) path connecting j and o using edges from E_N. We have to show that there are $n - 1$ linearly independent inequalities in (4) which are satisfied as equalities for x. In fact, the inequalities corresponding to the sets $\{0, 1, \dots, i\}$ $(i = 1, \dots, n - 1)$ are the required inequalities.

In light of the above theorem it is apparent that any minimum cost tree solution is not entirely 'just'. It suggests that users who are directly linked to the common supplier pay the full cost of the link, even though other users indirectly use this link. Thus one is motivated to investigate other solutions, preferably in the core. In the next two sections we suggest certain reductions of the game which can simplify further investigations.

3. Tree decomposition for the core of $(N; c)$ $[(N; \bar{c})]$

We consider the case in which the cost matrix C gives rise to a m.c.s.t. graph Γ_N with more than one (say $p > 1$) edges incident to the common supplier 0. We construct p m.c.s.t. [resp., m.m.c.s.t.] games $(N_i; c^i)$ [resp., $(N_i; \bar{c}^i)$] $(i = 1, \dots, p)$ such that the core of the m.c.s.t. [resp., m.m.c.s.t.] game $(N; c)$ [resp., $(N; \bar{c})$] is the cartesian product of the cores of the m.c.s.t. [resp., m.m.c.s.t.] games $(N_i; c^i)$ [resp., $(N_i; \bar{c}^i)$], $(i = 1, \dots, p)$. A similar result on the nucleolus is obtained in the next section. On the other hand, it is shown that the Shapley value of $(N; c)$ [resp., $(N; \bar{c})$] is not necessarily the cartesian product of the Shapley values of $(N_i; c^i)$ [resp., $(N_i; \bar{c}^i)$].

The statement that Γ_N has p edges incident to the common supplier 0 is equivalent to the existence of an efficient coalition structure $\{N_1, \dots, N_p\}$ in the m.c.s.t. game $(N; c)$, i.e., a partition $\{N_1, \dots, N_p\}$ of N such that

$$c(N) = \sum_{i=1}^{p} c(N_i). \tag{13}$$

Now, one can easily show that $\{N_1, \dots, N_p\}$ is an efficient coalition structure in the m.c.s.t. game $(N; c)$ iff $\{N_1, \dots, N_p\}$ is an efficient coalition structure for $(N; \bar{c})$, i.e. iff

$$\bar{c}(N) = \sum_{i=1}^{p} \bar{c}(N_i). \tag{14}$$

Without loss of generality we assume (unless otherwise stated) that $p = 2$, and

$$N_1 = \{1, \dots, m\}, \qquad N_2 = \{m+1, \dots, n\}. \tag{15}$$

Graphically this means that E_N can be chosen so that there are two nodes, say 1 and $m+1$ such that $\{(0, 1), (0, m+1\} \subset E_N$.

We use (c_{ij}) to construct a cost matrix $C' = (c'_{ij})$ as follows

$$c'_{ij} = \begin{cases} \min\{c_{ij}, \min_{k \in N_2}\{c_{ik}\}\} & j = 0, i \in N_1, \\ \min\{c_{ij}, \min_{k \in N_1}\{c_{ik}\}\} & j = 0, i \in N_2, \\ c_{ij} & \text{otherwise.} \end{cases} \tag{16}$$

The cost matrix $C' = (c'_{ij})$ induces the cost matrices $C^1 = (c^1_{ij})$ and $C^2 = (c^2_{ij})$ on $N_1 \cup \{0\}$ and $N_2 \cup \{0\}$, respectively, i.e. $c^1_{ij} = c'_{ij}$, $(i, j) \in N_1 \cup \{0\}$ and $c^2_{ij} = c'_{ij}$, $(i, j) \in N_2 \cup \{0\}$. Further, we denote by $(N; c')$ [resp., $(N; \bar{c}')$], $(N_1; c^1)$ [resp., $(N_1; \bar{c}^1)$] and $(N_2; c^2)$ [resp., $(N_2; \bar{c}^2)$] the m.c.s.t. [resp., m.m.c.s.t.] games associated with the symmetric cost matrices C', C^1 and C^2, respectively.

Let $\mathcal{B} = \{B_1, \dots, B_r\}$ be a partition of N. The game $(N; c)$ is called decomposable with partition $\mathcal{B}$ if for all $S \subset N$

$$c(S) = \sum_{j=1}^{r} c(B_j \cap S). \tag{17}$$

A game $(N; c)$ is said to be decomposable if there exists a partition $\{B_1, \dots, B_r\}$, $r \geq 2$, satisfying (17).

One can easily verify that the game $(N; c')$ is decomposable with partition $\{N_1, N_2\}$. We have therefore:

Lemma 5. *For all $S_i \subset N_i$ $(i = 1, 2)$*

(a) $\qquad c^1(S_1) + c^2(S_2) \leq c(S_1 \cup S_2)$

and

(b) $\qquad \bar{c}^1(S_1) + \bar{c}^2(S_2) \leq \bar{c}(S_1 \cup S_2).$

Proof.

$$\begin{aligned} c^1(S_1)+c^2(S_2) &= c'(S_1)+c'(S_2) && \text{by definition of } C', C^1, C^2, \\ &= c'(S_1 \cup S_2) && \text{since } (N; c') \text{ is decomposable,} \\ &\le c(S_1 \cup S_2) && \text{by definition of } C'. \end{aligned}$$

To prove the second part we consider the sets R_i, $S_i \subset R_i \subset N_i$ $(i = 1, 2)$ such that $\bar{c}(S_1 \cup S_2) = c(R_1 \cup R_2)$. From part (a) we have $c(R_1 \cup R_2) \ge c^1(R_1) + c^2(R_2)$. From (1) we have $c^i(R_i) \ge \bar{c}^i(S_i)$ $(i = 1, 2)$. Thus,

$$\bar{c}(S_1 \cup S_2) = c(R_1 \cup R_2) \ge c^1(R_1) + c^2(R_2) \ge \bar{c}^1(S_1) + \bar{c}^2(S_2),$$

which yields (b).

Lemma 6. (a) $c^1(S_1) + c^2(N_2) = c(S_1 \cup N_2)$ *and* $\bar{c}^1(S_1) + \bar{c}^2(N_2) = \bar{c}(S_1 \cup N_2)$ *for all* $S_1 \subset N_1$.

(b) $c^1(N_1) + c^2(S_2) = c(N_1 \cup S_2)$ *and* $\bar{c}^1(N_1) + \bar{c}^2(S_2) = \bar{c}(N_1 \cup S_2)$ *for all* $S_2 \subset N_2$.

Proof. In light of Lemma 5 and because of symmetry it suffices to show that

$$c^1(S_1) + c^2(N_2) \ge c(S_1 \cup N_2), \tag{18}$$

and

$$\bar{c}^1(S_1) + \bar{c}^2(N_2) \ge \bar{c}(S_1 \cup N_2). \tag{19}$$

Let $V_{S_1} = \{0\} \cup S_1$, $V_{N_2} = \{0\} \cup N_2$. We start by selecting $(V_{S_1}, E^1_{S_1})$ and $(V_{N_2}, E^2_{N_2})$ which are minimum cost spanning trees with respect to the cost matrices (c^1_{ij}) and (c^2_{ij}), respectively. Since $c^2(N_2) = c(N_2)$ we may assume $E^2_{N_2} = E_{N_2}$. We construct a spanning tree graph of $S_1 \cup N_2$ $(V_{S_1 \cup N_2}, E^{S_1 \cup N_2})$ for which

$$c^1(S_1) + c(N_2) = m(E^{S_1 \cup N_2}), \tag{20}$$

where $m(E^{S_1 \cup N_2})$ is the total cost associated with the edge set $E^{S_1 \cup N_2}$ calculated according to the original cost matrix (c_{ij}) (and therefore $m(E^{S_1 \cup N_2}) \ge c(S_1 \cup N_2)$). The construction algorithm is as follows.

Step 0. Set $\tilde{V} = V_{N_2}$, $\tilde{E} = E_{N_2}$.

Step 1. Pick $(j, k) \in E^1_{S_1}$, such that $j \in S_1$ and $k \in \tilde{V}$. If impossible, stop.

Step 2. If $k \ne 0$ or $c_{j0} = c^1_{j0}$ omit j from S_1 and add it to $\tilde{V}$. Also, omit (j, k) from $E^1_{S_1}$ and add it to $\tilde{E}$. Go to Step 1. Otherwise,

Step 3. Find $l \in N_2$ such that $c_{jl} = c^1_{j0}$. Omit j from S_1 and add it to $\tilde{V}$. Also, omit $(j, 0)$ from $E^1_{S_1}$, add (j, l) to $\tilde{E}$ and go to Step 1.

Upon termination $\tilde{V} = S_1 \cup N_2$ and $\tilde{E} = E^{S_1 \cup N_2}$. Clearly, the tree $(\tilde{V}, \tilde{E})$

satisfies (20), and therefore

$$c^1(S_1)+c^2(N_2)=c(S_1\cup N_2).$$

To prove (19) we consider R_1 such that $S_1\subset R_1\subset N_1$ and $\bar{c}^1(S_1)=c^1(R_1)$. From (18) we have $c^1(R_1)+c^2(N_2)\geq c(R_1\cup N_2)$. By (2) we have $c(R_1\cup N_2)\geq \bar{c}(S_1\cup N_2)$. Thus,

$$\bar{c}^1(S_1)+\bar{c}^2(N_2)=c^1(R_1)+c^2(N_2)\geq c(R_1\cup N_2)\geq \bar{c}(S_1\cup N_2),$$

which completes the proof.

We proceed now to prove the main result of this section.

Theorem 7. *The core of the m.c.s.t. [resp., m.m.c.s.t.] game $(N;c)$ [resp., $(N;\bar{c})$] is the cartesian product of the cores of the m.c.s.t. [resp., m.m.c.s.t.] games $(N_i;c^i)$ [resp., $(N_i;\bar{c}^i)$] $(i=1,\dots,p)$, where $\{N_1,\dots,N_p\}$ is an efficient coalition structure of the m.c.s.t. game $(N;c)$.*

Proof. Without loss of generality we assume $p=2$.

Every vector $(x_1,\dots,x_n)$ in the core of the game $(N;c)$ with an efficient coalition structure $\{N_1,N_2\}$ must satisfy

$$\sum_{j\in N_i} x_j = c(N_i),\quad i=1,2. \tag{20}$$

Let $(x_1,\dots,x_n)$ satisfy $\sum_{j\in S} x_j\leq c(S)$ for all $S\subset N$. In particular

$$\sum_{j\in S_1\cup N_2} x_j\leq c(S_1\cup N_2)\quad \text{for all } S_1\subset N_1. \tag{21}$$

Lemma 6, (20) and (21) imply that

$$\sum_{j\in S_1} x_j\leq c^1(S_1)\quad \text{for all } S_1\subset N_1. \tag{22}$$

Similarly, $\sum_{j\in S_2} x_j\leq c^2(S_2)$ for all $S_2\subset N_2$. The other direction of the proof is a direct corollary of Lemma 5. The proof for the m.m.c.s.t. game $(N;\bar{c})$ is identical.

4. Tree decomposition for the nucleolus of $(N;c)$

We will prove in this section that the presence of an efficient coalition structure in the m.c.s.t. game $(N;c)$ implies that the nucleolus of the m.c.s.t. [resp., m.m.c.s.t.] game $(N;c)$ [resp., $(N;\bar{c})$] is the cartesian product of the nucleoli of the m.c.s.t. [resp., m.m.c.s.t.] games defined on the components of the efficient coalition structure. We do that by resorting to the work of Kopelowitz [6], who constructed an algorithm for computing the nucleolus of a cooperative game $(N; v)$ by solving a sequence of at most 2^n linear programs (where n is the number of players).

Let us briefly review Kopelowitz's algorithm. The first linear program P_1 to be solved in his procedure is:
Problem P_1:

$$\begin{aligned} \min\quad & r, \\ \text{s.t.}\quad & r \geq v(S) - \sum_{k\in S} x_k \quad \text{for all } S \subset N, \\ & \sum_{k=1}^{n} x_k = v(N). \end{aligned}$$

Let r_i denote the optimal value of r in problem P_i; $A_i = \{x\colon x$ is an optimal solution to $P_i\}$; $\xi_i = \{S \subset N;\ r_i = v(S) - \sum_{i\in S} x_i$ for all $x \in A_i\}$. At stage i the linear programming problem P_i to be solved is:
Problem P_i:

$$\begin{aligned} \min\quad & r, \\ \text{s.t.}\quad & r_j = v(S) - \sum_{k\in S} x_k, \quad S \in \xi_j,\ j = 1, 2, \ldots, i-1, \\ & r \geq v(S) - \sum_{k\in S} x_k, \quad S \in 2^N \setminus (\textstyle\bigcup_{j=1}^{i-1} \xi_j), \\ & \sum_{k=1}^{n} x_k = v(N). \end{aligned}$$

The nucleolus is obtained in stage l $(1 \leq l \leq 2^n)$, whenever A_l consists of a single vector.

Given an efficient coalition structure $\{N_1, \ldots, N_p\}$, we construct the m.c.s.t. [resp., m.m.c.s.t.] games $(N_i; c^i)$ [resp., $(N_i; \bar{c}^i)$] $(i = 1, \ldots, p)$ and analyze Kopelowitz's linear programs to show that the nucleolus of the m.c.s.t. [resp., m.m.c.s.t.] game $(N; c)$ [resp., $(N; \bar{c})$] is the cartesian product of the nucleoli of the m.c.s.t. [resp., m.m.c.s.t.] games $(N_i; c^i)$ [resp., $(N_i; \bar{c}^i)$] $(i = 1, \ldots, p)$. Without loss of generality we assume that $p = 2$.

We first outline the proof. At stage i in Kopelowitz's procedure we consider problems (P_i^1), (P_i^2), (P_i) to find the nucleolus of $(N_1; c^1)$, $(N_2; c^2)$, $(N; c)$ [resp., $(N_1; \bar{c}^1)$, $(N_2; \bar{c}^2)$, $(N; \bar{c})$], respectively. We show that the program (P_i^1) and (P_i^2) are relaxations of (P_i). Moreover, we show that the cartesian product of any optimal cost vectors of (P_i^1) and (P_i^2) is feasible to (P_i) and therefore optimal.

Consider the following three linear programs:

$$\begin{aligned} (P_1)\qquad \max\quad & r, \\ \text{s.t.}\quad & r \leq c(S) - \sum_{k\in S} x_k, \quad S \subset N, \end{aligned}$$

$$c(N_1) = \sum_{k \in N_1} x_k,$$

$$c(N_2) = \sum_{k \in N_2} x_k.$$

$$(\mathrm{P}_1^1) \qquad \max \quad r^1,$$

$$\text{s.t.} \quad r^1 \le c^1(S) - \sum_{k \in S} x_k, \quad S \subset N_1,$$

$$c^1(N_1) = \sum_{k \in N_1} x_k.$$

$$(\mathrm{P}_1^2) \qquad \max \quad r^2,$$

$$\text{s.t.} \quad r^2 \le c^2(S) - \sum_{k \in S} x_k, \quad S \subset N_2,$$

$$c^2(N_2) = \sum_{k \in N_2} x_k.$$

Each one of the programs (P_1), (P_1^1), (P_1^2) is the first problem in Kopelowitz's sequence of linear programs that has to be solved to produce the nucleolus of $(N; c)$, $(N_1; c^1)$ and $(N_2; c^2)$, respectively.

Lemma 8. *Problems* (P_1^1) *and* (P_1^2) *are relaxations of* (P_1).

Proof. Every inequality constraint in (P_1^1) is implied by some corresponding constraints in (P_1). Explicitly, by Lemma 6, the constraint

$$r^1 \le c^1(S_1) - \sum_{k \in S_1} x_k, \quad S_1 \subset N_1,$$

in (P_1^1) is implied by the constraints

$$r \le c(S_1 \cup N_2) - \sum_{k \in S_1 \cup N_2} x_k \quad \text{and} \quad \sum_{k \in N_2} x_k = c(N_2)$$

in (P_1). Similarly, every constraint in (P_1^2) is implied by some corresponding constraints in (P_1), which completes the proof.

Let r_1^1 and r_1^2 be the optimal values of (P_1^1) and (P_1^2), respectively, and denote by $r_1^0 = \min\{r_1^1, r_1^2\}$. By Lemma 8, we have that the optimal value of (P_1) cannot exceed r_1^0. We will show that r_1^0 is in fact attainable by (P_1).

Let $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m)$ and $(\bar{x}_{m+1}, \bar{x}_{m+2}, \dots, \bar{x}_n)$ be any pair of optimal cost vectors for (P_1^1) and (P_1^2), respectively. We will show that $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n, r_1^0)$ is feasible to (P_1) and thus optimal.

Let $S = S_1 \cup S_2$, $S_i \subset N_i$ $(i = 1, 2)$. We assume $S_1 \ne N_1$ and the case $S_2 \ne N_2$ will follow by a symmetry argument. From (P_1^1) we have that

$$r_1^0 \le r_1^1 \le c^1(S_1) - \sum_{k \in S_1} \bar{x}_k.$$

Further, since the cores of (N_i, c^i) $(i = 1, 2)$ are not empty, we have that

$$c^2(S_2) - \sum_{k \in S_2} \bar{x}_k \geq 0.$$

Thus,

$$r_1^0 \leq c^1(S_1) - \sum_{k \in S_1} \bar{x}_k + c^2(S_2) - \sum_{k \in S_2} \bar{x}_k \leq c(S_1 \cup S_2) - \sum_{k \in S_1 \cup S_2} \bar{x}_k,$$

where the last inequality follows from Lemma 5. Hence, $(\bar{x}_1, \dots, \bar{x}_n, r_1^0)$ is feasible to (P_1), and thus optimal.

Let

$$A_1^j = \{x: x \text{ is optimal to } (P_1^j)\} \quad (j = 1, 2),$$

$$A_1 = \{x: x \text{ is optimal to } (P_1)\},$$

$$\xi_1^j = \{S \subset N_j: c^j(S) - \textstyle\sum_{k \in S} x_k = r_1^0 \text{ for all } x \in A_1^j\} \quad (j = 1, 2),$$

$$\xi_1 = \{S \subset N: c(S) - \textstyle\sum_{k \in S} x_k = r_1^0 \text{ for all } x \in A_1\}.$$

The above argument showed that $A_1^1 \times A_1^2 \subset A_1$. We may have $\xi_1^1 = \phi$ (if $r_1^0 = r_1^1 < r_1^2$) or $\xi_1^2 = \phi$ (if $r_1^2 < r_1^1$).

The next problems (P_2), (P_2^1), (P_2^2) to be considered are

(P_2)

$$\begin{aligned} \max\quad & r, \\ \text{s.t.}\quad & r_1^0 = c(S) - \sum_{k \in S} x_k, \quad S \in \xi_1, \\ & r \leq c(S) - \sum_{k \in S} x_k, \quad S \in 2^N \setminus \xi_1, \\ & \sum_{k \in N_1} x_k = c(N_1), \\ & \sum_{k \in N_2} x_k = c(N_2). \end{aligned}$$

(P_2^1)

$$\begin{aligned} \max\quad & r^1, \\ \text{s.t.}\quad & r_1^0 = c^1(S) - \sum_{k \in S} x_k, \quad S \in \xi_1^1, \\ & r^1 \leq c^1(S) - \sum_{k \in S} x_k, \quad S \in 2^{N_1} \setminus \xi_1^1, \\ & \sum_{k \in N_1} x_k = c^1(N_1). \end{aligned}$$

(P_2^2)

$$\begin{aligned} \max\quad & r^2, \\ \text{s.t.}\quad & r_1^0 = c^2(S) - \sum_{k \in S} x_k, \quad S \in \xi_1^2, \\ & r^2 \leq c^2(S) - \sum_{k \in S} x_k, \quad S \in 2^{N_2} \setminus \xi_1^2, \\ & \sum_{k \in N_2} x_k = c^2(N_2). \end{aligned}$$

To any equality constraint in problem (P_2^1) or (P_2^2) there corresponds equality constraints in (P_2). Indeed, suppose that

$$r_1^0 = c^1(S_1) - \sum_{k\in S_1} x_k \quad \text{for some } S_1 \in \xi_1^1.$$

Then,

$$r_1^0 = c^1(S_1) - \sum_{k\in S_1} x_k = c^1(S_1) - \sum_{k\in S_1} x_k + c(N_2) - \sum_{k\in N_2} x_k$$
$$= c(S_1 \cup N_2) - \sum_{k\in S_1\cup N_2} x_k.$$

Thus $S_1 \cup N_2 \in \xi_1$.

The above observation implies that (P_2^1) and (P_2^2) are still relaxations of (P_2).

The linear program (P_2) is the second in the sequence of problems that we need to solve in order to produce the nucleolus of $(N; c)$ by Kopelowitz's method. Similarly, if $\xi_1^1 \neq \phi$ [resp., $\xi_1^2 \neq \phi$], then problem (P_2^1) [resp., (P_2^2)] is the second in the sequence of problems that has to be solved in order to produce the nucleolus of $(N_1; c^1)$ [resp., $(N_2; c^2)$].

In general, let

r_i^j be the optimal value of r^j in problem (P_i^j) $(j = 1, 2)$,

$r_i^0 = \min\{r_i^1, r_i^2\}$,

$A_i^j = \{x: x \text{ is an optimal solution to } (P_i^j)\}$ $(j = 1, 2)$,

$A_i = \{x: x \text{ is an optimal solution to } (P_i)\}$,

$\xi_i^j = \{S: S \subset N_j,\ c^j(S) - \sum_{k\in S} x_k = r_i^0 \text{ for all } x \in A_i^j\}$ $(j = 1, 2)$,

$\xi_i = \{S: S \subset N,\ c(S) - \sum_{k\in S} x_k = r_i^0 \text{ for all } x \in A_i\}$

where problems (P_i), (P_i^1), (P_i^2) are of the form

(P_i) $\max r$,

$$r_j^0 = c(S) - \sum_{k\in S} x_k, \quad S \in \xi_j,\ j = 1, \ldots, i-1,$$

$$r \le c(S) - \sum_{k\in S} x_k, \quad S \in 2^N \setminus \left(\bigcup_{j=1}^{i-1} \xi_j\right),$$

$$\sum_{k\in N_1} x_k = c(N_1),$$

$$\sum_{k\in N_2} x_k = c(N_2).$$

(P_i^1) $\max r^1$,

$$r_j^1 = c^1(S) - \sum_{k\in S} x_k, \quad S \in \xi_j^1,\ j = 1, \ldots, i-1,$$

$$r^1 \le c^1(S) - \sum_{k\in S} x_k, \quad S \in 2^{N_1} \setminus \left(\bigcup_{j=1}^{i-1} \xi_j^1\right),$$

$$\sum_{k\in N_1} x_k = c(N_1).$$

(P_i^2) $\quad \max r^2,$

$$r_j^0 = c(S) - \sum_{k \in S} x_k, \quad S \in \xi_j^2, \quad j = 1, \dots, i-1,$$

$$r^2 \le c(S) - \sum_{k \in S} x_k, \quad S \in 2^{N_2} \setminus \left(\bigcup_{j=1}^{i-1} \xi_j^2 \right),$$

$$\sum_{k \in N_2} x_k = c(N_2).$$

The linear programs (P_i), (P_i^1), (P_i^2) have the following properties:

(1) Problem (P_i) is the ith problem in Kopelowitz's procedure to calculate the nucleolus of $(N; c)$. Similar statements can be made with regard to (P_i^1), (P_i^2). In view of Kopelowitz's result, $i \le 2^n$.

(2) $A_i^1 \times A_i^2 \subset A_i$, and $r_i^0 = \min\{r_i^1, r_i^2\}$ is an attainable upper bound for the optimal value of (P_i).

(3) It is possible that for some j, either $\xi_j^1 = \phi$ or $\xi_j^2 = \phi$, but for all j, $\xi_j^1 \cup \xi_j^2 \ne \phi$.

(4) If for some i, A_i^1 [resp., A_i^2] but not A_i^2 [resp., A_i^1] consists of one point, then only problems (P_l) and (P_l^2) [resp., (P_l^1)] should be considered for all $l \ge i+1$.

At the final stage i_0 ($i_0 \le 2^n$) we obtain the sets of $A_{i_0}^1$ and $A_{i_0}^2$ which are the nucleoli of $(N_1; c^1)$ and $(N_2; c^2)$, respectively. Since $A_{i_0}^1 \times A_{i_0}^2 \subset A_{i_0}$, and because of the uniqueness of the nucleolus, $A_{i_0} = A_{i_0}^1 \times A_{i_0}^2$ is the nucleolus of $(N; c)$.

The same discussion can be applied to the nucleoli of $(N; \bar{c})$, $(N_1; \bar{c}^1)$ and $(N_2; \bar{c}^2)$, and therefore a similar result holds. We therefore have:

Theorem 9. *The nucleolus of the m.c.s.t. [resp., m.m.c.s.t.] game* $(N; c)$ *[resp.,* $(N; \bar{c})$*] is the cartesian product of the nucleoli of the m.c.s.t. [resp., m.m.c.s.t.] games* $(N_i; c^i)$ *[resp.,* $(N_i; \bar{c}^i)$*]* $(i = 1, \dots, p)$, *where* $\{N_1, \dots, N_p\}$ *is an efficient coalition structure of the m.c.s.t. game* $(N; c)$.

Example 2. Consider the m.c.s.t. game $(\{1, \dots, 5\}; c)$ which is determined by the cost matrix C, where

$C =$

	1	2	3	4	5
0	2	5	7	2	3
1		3	6	3	4
2			4	4	3
3				6	5
4					2

A m.c.s.t. graph for C is described in Fig. 2. The corresponding spanning tree solution is $(2, 3, 4, 2, 2)$.

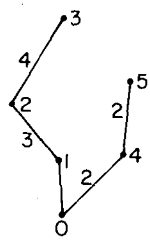

Fig. 2.

An efficient coalition structure of $\{\{1, \dots, 5\}; c\}$ is $\{\{1, 2, 3\}; \{4, 5\}\}$. The m.c.s.t. games $(\{1, 2, 3\}; c^1)$ and $(\{4, 5\}; c^2)$ are determined by the following cost matrices

$$C^1 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 0 & 2 & 3 & 5 \\ 1 & & 3 & 6 \\ 2 & & & 4 \end{array} \qquad C^2 = \begin{array}{c|cc} & 4 & 5 \\ \hline 0 & 2 & 3 \\ 4 & & 2 \end{array}$$

Minimum cost spanning tree graphs corresponding to C^1 and C^2 are described in Fig. 3.

The game $(\{1, 2, 3\}; c^1)$ also has an efficient coalition structure $\{\{1\}, \{2, 3\}\}$. Therefore it can be further tree decomposed into $(\{1\}; c^3)$ and $(\{2, 3\}; c^4)$. The final tree decomposition is described in Fig. 4.

The nucleoli of $(\{1\}; c^3)$, $(\{2, 3\}; c^4)$ and $(\{4, 5\}; c^2)$ are (2), (2.5, 4.5) and (1.5, 2.5), respectively. Therefore by Theorem 9 the nucleolus of the original m.c.s.t. game $(\{1, 2, 3, 4, 5\}; c)$ is (2, 2.5, 4.5, 1.5, 2.5). Using the same decomposition we see that the core of the original m.c.s.t. game is the convex hull of (2, 3, 4, 2, 2), (2, 3, 4, 1, 3), (2, 2, 5, 2, 2) and (2, 2, 5, 1, 3).

As one can see from the above example, a tree decomposition of a m.c.s.t. game $(N; c)$ to $(N_i; c^i)$ $(i = 1, \dots, p)$ might further enable a tree decomposition of the induced games $(N_i; c^i)$. This leads to a further simplification in finding the core or the nucleolus of $(N; c)$.

A result similar to that of Theorem 9 is not valid for the Shapley value [10] of a m.c.s.t. game as the following example shows.

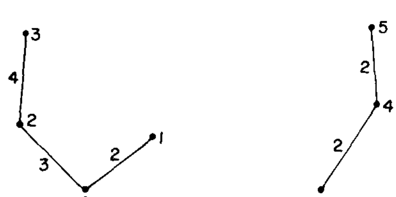

Fig. 3.

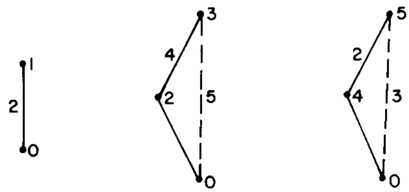

Fig. 4.

Example 3. Consider the m.c.s.t. game $(\{1,2,3\}; c)$ where c is determined by the following cost matrix C

$$C = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 0 & 2 & 3 & 2 \\ 1 & & 2 & 3 \\ 2 & & & 2 \end{array}$$

Fig. 5a describes a minimum cost spanning tree graph for the game $(\{1,2,3\}; c)$ and Fig. 5b describes a final tree decomposition of the game.

The Shapley value of the game $(\{1,2,3\}; c)$ is $(\frac{11}{6}, \frac{14}{6}, \frac{11}{6})$, while the Shapley value of each one of the induced games is 2.

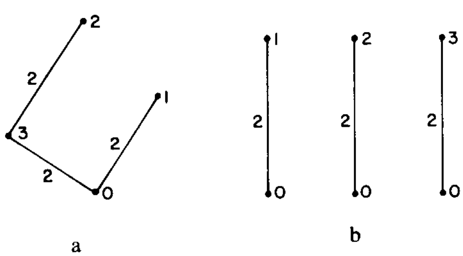

Fig. 5.

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