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OPTIMALITY CONDITIONS FOR QUADRATIC PROGRAMMING

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This paper establishes a set of necessary and sufficient conditions in order that a vector x^* be a local minimum point to the general (not necessarily convex) quadratic programming problem:

minimize $p^{\mathrm{T}}x + \frac{1}{2}x^{\mathrm{T}}Qx$, subject to the constraints $Hx \geqq h$.

Introduction

In the last two decades much effort has been devoted to the problem of optimality conditions in nonlinear programming. Since John's paper [2] and that of Kuhn and Tucker [3] in which they established their famous conditions for the convex case, which are at the same time necessary local criteria for the general case, a number of papers appeared on the subject. Recently Orden [5], Ritter [6], McCormick [4] and Fiacco [1] gave new insight into the problem by extending the classical methods using second derivatives.

A rough summary of the results is as follows:

(1) If a feasible point x^* is a local minimum point and at this point certain regularity assumptions hold, then there exist nonnegative Lagrange multipliers so that the Kuhn-Tucker conditions hold and in addition to that the Hessian matrix of the Lagrangian function must be positive semidefinite on the polyhedral cone which is the intersec-

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tion of the polar cone of the set of feasible directions at x^* with the orthogonal subspace to the normals of the strictly binding constraints [4].

(2) If at x^* in addition to certain regularity the Kuhn-Tucker conditions hold and the Hessian matrix of the Lagrangian function is positive definite on the subspace which is the orthogonal complement of the normals to the strictly binding constraints at x^*, or positive semidefinite on some extension of the latter set then the point is a local minimum point [1, 4].

A binding constraint is said to be strictly binding if the corresponding Lagrange multiplier is positive.

The only essential difference between the necessary and the sufficient conditions is the set on which the nonnegativity of the Hessian matrix is required, and by using second order derivatives only we may not hope to fill this gap. In this paper we show that in the case of quadratic programming we are able to get a stronger result than can be obtained by specializing Fiacco's theorem [1], by showing that the conditions which are only necessary in the general case are sufficient ones, as well.

Quadratic programming

Let H be an m by n real matrix and assume that $x^* \in \mathcal{H} = \{x \mid Hx \geqq h\}$. Denoting by H_i the i-th row of H, we define the set of subscripts I^* by $I^* = \{i \mid H_i x^* = h_i\}$. Let finally Q be any given symmetric n by n real matrix, then $q(x) = p^{\mathrm{T}} x + \frac{1}{2}\, x^{\mathrm{T}} Q x$ defines a general /not necessarily convex/ quardatic function.

Theorem. Necessary and sufficient conditions that x^* be a local minimum point to the quadratic programming problem

$$\min\ q(x)$$

$$x \in \mathcal{H} \tag{1}$$

are that there exists an m-vector u^* such that the Kuhn-Tucker conditions

$$p + Qx^* - H^{\mathrm{T}} u^* = 0, \tag{2}$$

$$-h + Hx^* \geqq 0, \tag{3}$$

$$u^* \geqq 0, \tag{4}$$

$$(-h + Hx^*)^{\mathrm{T}} u^* = 0 \tag{5}$$

hold, and such that for every n-vector w where

$$H_i w = 0 \quad \text{for all} \quad i \in J^* = \{i \mid u_i^* > 0\} \tag{6}$$

$$H_i w \geqq 0 \quad \text{for all} \quad i \in I^* - J^* \tag{7}$$

it follows that

$$w^{\mathrm{T}} Q w \geqq 0. \tag{8}$$

Proof. The necessity of the conditions can be obtained by specializing McCormick's theorem [4, Theorem 4].

Before proving the sufficiency we mention two lemmas.

Lemma 1. Assume that $c \neq 0$ and

$$\lim_{x \to x_0} \frac{c^{\mathrm{T}} x}{x^{\mathrm{T}} Q x} = 0.$$

Then

$$x_0^{\mathrm{T}} Q x_0 \neq 0.$$

Proof. Clearly $c^{\mathrm{T}} x_0 = 0$. Thus from the relation $x_0^{\mathrm{T}} Q x_0 = 0$ it follows that

$$\frac{c^{\mathrm{T}} x}{x^{\mathrm{T}} Q x} = \frac{c^{\mathrm{T}} (x - x_0)}{2x_0^{\mathrm{T}} Q (x - x_0) + (x - x_0)^{\mathrm{T}} Q (x - x_0)}.$$

Let specially $x = x_0 + tc$ where t is a real number. $t \to 0$ implies $x \to x_0$ i.e.

$$\lim_{t \to 0} \frac{c^2}{2x_0^{\mathrm{T}} Q c + t c^{\mathrm{T}} Q c} = \frac{c^2}{2x_0^{\mathrm{T}} Q c} = 0.$$

contradicting to $c \neq 0$.

Lemma 2. Let us assume that the conditions (2)-(8) hold and let

$$W = \{w \mid \|w\| = 1, H_i w \geqq 0 \;\; i \in I^*, \;\; w^{\mathrm{T}} Q w < 0\}.$$

Then we have

$$\epsilon = \inf_{w \in W} \frac{2(u^*)^{\mathrm{T}} H w}{-w^{\mathrm{T}} Q w} > 0.$$

Proof. Clearly $\epsilon \geqq 0$. Assuming that $\epsilon = 0$ we have a sequence $\{w_k\} \subset W$ with the property that

$$\lim_{k \to \infty} \frac{2(u^*)^{\mathrm{T}} H w_k}{-w_k^{\mathrm{T}} Q w_k} = 0.$$

As the sequence $\{w_k\}$ is bounded, it has a density point w_0. For this the following relations hold

$$\|w_0\| = 1 \tag{13}$$

$$H_i w_0 \geqq 0 \quad \text{for all} \quad i \in I^* \tag{14}$$

$$2(u^*)^{\mathrm{T}} H w_0 = 0 \tag{15}$$

$$w_0^{\mathrm{T}} Q w_0 \leqq 0. \tag{16}$$

By Lemma 1 and by the inequality (16) we know that

$$w_0^{\mathrm{T}} Q w_0 < 0. \tag{17}$$

From (4), (14) and (15) it follows that

$$H_i w_0 = 0 \;\; \text{for all} \;\; i \in J^*. \tag{18}$$

The relations (14), (18) and (17) contradict to the assumed validity of (6), (7) and (8).

Now we are able to prove the sufficiency of our optimality conditions.

Let us assume that all the conditions (2)–(8) hold. Using (2) we get

$$q(x) - q(x^*) = (u^*)^{\mathrm{T}} H(x-x^*) + \tfrac{1}{2}(x-x^*)^{\mathrm{T}} Q(x-x^*). \quad (19)$$

Define ϵ by

$$\epsilon = \inf_{w \in W} \frac{2(u^*)^{\mathrm{T}} Hw}{-w^{\mathrm{T}} Qw}.$$

Let us choose any $x \in \mathcal{H}$ such that

$$\|x - x^*\| < \epsilon. \quad (20)$$

Clearly

$$H_i(x-x^*) \geqq 0 \quad \text{for all} \quad i \in I^* \quad (21)$$

whence

$$(u^*)^{\mathrm{T}} H(x-x^*) \geqq 0 \quad (22)$$

and

$$(u^*)^{\mathrm{T}} H(x-x^*) = 0 \quad (23)$$

holds if and only if

$$H_i(x-x^*) = 0 \quad \text{for all} \quad i \in J^*. \quad (24)$$

Assuming this to be the case, (19) yields

$$q(x) - q(x^*) = \tfrac{1}{2}(x-x^*)^{\mathrm{T}} Q(x-x^*).$$

We see that $w = x - x^*$ satisfies (21) and (24) therefore it satisfies (6) and (7) whence we have by that

$$q(x) \geqq q(x^*).$$

In the opposite case we have

$$(u^*)^{\mathrm{T}} H(x-x^*) > 0.$$

Now we use (19) in the following slightly modified form:

$$q(x) - q(x^*) = \|x-x^*\| \left[(u^*)^{\mathrm{T}} H \frac{x-x^*}{\|x-x^*\|} + \right.$$

$$\left. + \frac{\|x-x^*\|}{2} \frac{(x-x^*)^{\mathrm{T}}}{\|x-x^*\|} Q \frac{x-x^*}{\|x-x^*\|} \right]$$

If $(x-x^*)^{\mathrm{T}} Q(x-x^*) \geqq 0$ then the right hand side is positive. Finally if $(x-x^*)^{\mathrm{T}} Q(x-x^*) < 0$ then by Lemma 2 and (20) we know that

$$\frac{2(u^*)^{\mathrm{T}} H \dfrac{x-x^*}{\|x-x^*\|}}{-\dfrac{(x-x^*)^{\mathrm{T}}}{\|x-x^*\|} Q \dfrac{x-x^*}{\|x-x^*\|}} \geqq \epsilon > \|x-x^*\| .$$

From this we get immediately

$$q(x) \geqq q(x^*),$$

and the proof is complete.

As an important special case we mention the following

Corollary. Necessary and sufficient conditions that x^* be a local minimum point to the general quadratic programming problem: minimize $q(x)$ subject to the constraints $Ax \geqq b$, $x \geqq 0$, are that there exist vectors y^*, u^*, v^*, such that the conditions

$$\begin{pmatrix} u^* \\ v^* \end{pmatrix} = \begin{pmatrix} Q & -A^{\mathrm{T}} \\ A & 0 \end{pmatrix} \begin{pmatrix} x^* \\ y^* \end{pmatrix} + \begin{pmatrix} p \\ -b \end{pmatrix} ,$$

$$\begin{pmatrix} u^* \\ v^* \end{pmatrix} \geqq 0, \qquad \begin{pmatrix} x^* \\ y^* \end{pmatrix} \geqq 0,$$

$$\begin{pmatrix} u^* \\ v^* \end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix} x^* \\ y^* \end{pmatrix} = 0$$

hold and such that for every n-vector w where

$$A_i w = 0 \quad \text{if} \quad y_i^* > 0,$$

$$A_i w \geqq 0 \quad \text{if} \quad v_i^* = y_i^* = 0,$$

$$w_j = 0 \quad \text{if} \quad u_j^* > 0,$$

$$w_j \geqq 0 \quad \text{if} \quad x_j^* = u_j^* = 0,$$

it follows that

$$w^{\mathrm{T}} Q w \geqq 0 .$$

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