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DISCRETE OPTIMIZATION ON A MULTIVARIABLE BOOLEAN LATTICE *

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Consider the problem of finding the minimum value of a scalar objective function whose arguments are the N components of 2^N vector elements partially ordered as a Boolean lattice. If the function is strictly decreasing along any shortest path from the minimum point to its logical complement, then the minimum can be located precisely after sequential measurement of the objective function at $N + 1$ points. This result suggests a new line of research on discrete optimization problems.

1. Introduction

The best method for finding the minimum of a unimodal (unique local minimum) function defined on a simply ordered set of points is known to be the Fibonacci search scheme [4, 5]. This efficient method is defined only for functions of a single variable, a serious limitation in practice, where most problems of interest involve many independent variables. This article considers the case where there are N binary variables generating 2^N cases (or points) of a partially ordered set known as a Boolean lattice [1]. This more complicated situation requires a regularity assumption just as the one variable case does, and the assumption used closely resembles the unimodality hypothesis of the one-dimensional case. The optimal search scheme turns out to be surpri-

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singly simple, involving only $N + 1$ adjacent measurements, and it is further shown that use of this search scheme is equivalent to assuming this multidimensional regularity assumption, called *monotonicity*. Although the article gives an example after the theory has been developed, the reader looking for practical applications may find it worthwhile to read the example first to see if the regularity condition seems plausible.

2. Regularity assumption.

Let B_2^N be a boolean lattice partially ordered by inclusion. That is B_2^N consists of 2^N N-vectors X_i ($i = 1, ..., 2^N$) each of whose N distinct components x_{in} ($n = 1, ..., N$) is either 0 or 1. The fact that the elements are ordered by inclusion means that x_i and x_j are two vectors whose components obey the N inequalities $x_{in} \leq x_{jn}$ ($n = 1, ..., N$), if and only if $x_i \subseteq x_j$ (x_i is contained in x_j). For example, $(0, 1, 0, 1) \subseteq (1, 1, 0, 1)$ and $(1, 0, 0, 0) \subseteq (1, 1, 0, 1)$, but $(0, 1, 0, 1) \nsubseteq (1, 0, 0, 0)$ and $(0, 1, 0, 1) \nsupseteq (1, 0, 0, 0)$.

Two vectors x_i and x_j are said to be *adjacent* if and only if they differ in exactly one component, i.e., $x_{in} = x_{jn}$ for all $n \neq k$ but $x_{ik} \neq x_{jk}$. A *minimal path* between two vectors x_i and x_j differing in exactly K components is any simply ordered set of $K + 1$ adjacent vectors x^k ($k = 0, 1, ..., K$) which differ from $x_i (\equiv x^0)$ in exactly k components.

Let y: $B_2^N \to R$ be a real valued function of the vectors x of B_2^N. It is desired to find the vector x_* where the *objective function* y is minimum, i.e.,

$$y(x_*) < y(x) \quad \text{for all} \quad x \neq x_*.$$

Let x^* be the unique vector in B_2^N differing in all N components from x_*.

$$x_n^* \neq x_{*n} \quad (n = 1, ..., N).$$

Any minimal path (of length N) from x_* to x^* will be called here a *meridian*. The objective function y is said to be *monotonically increasing* if and only if it increases strictly monotonically along any meridian from x_* to x^*.

This regularity condition of monotonicity is motivated by the one-dimensional search problem in which one seeks the maximum of a real function defined on a simply ordered finite set of natural numbers. In that case, if the function is unimodal, that is, if it has a unique local minimum, then the well-known Fibonacci search scheme [4, 5] is optimal in the sense that it uses the least possible number of function evaluations to find the minimum. Since a monotonically increasing function is unimodal on every meridian, one might expect that the best search scheme in this multivariable case would involve some complicated variant of Fibonacci search. However, the next theorem shows that the optimal scheme is, surprisingly enough, much simpler.

Theorem 1A.
Let y be monotonically increasing, and let x_i and x_j, both in B_2^N, be adjacent, with inequality only of the nth components

$$x_{in} \neq x_{jn}$$

$$x_{im} = x_{jm} \quad m \neq n, \quad 1 \leq m \leq N$$

then

$$y(x_i) < y(x_j)$$

if and only if

$$x_{*n} = x_{in} \neq x_{jn}.$$

Proof: There are two cases: x_{jn} and x_{*n} are either unequal or they are equal. If $x_{jn} \neq x_{*n}$, then if P represents the number of components of x_i differing from the corresponding components of the minimum x_*, $P + 1$ will be the number of components of x_j different from those of x_*. Hence any minimal path from x_* to x_i cannot contain x_j among its $P + 1$ vectors. Similarly, any minimal path from x_j to the maximum x^* cannot contain x_i among its $N - P - 1$ elements, since x_i has one more component different from those of x^* than does x_j. The union of these two disjoint minimal paths forms an $N + 1$ vector minimal path from x_* to x^*, which is by definition a meridian x^n ($n = 0, 1, ..., N$), with $x^0 \equiv x_*$, $x^P \equiv x_i$, $x^{P+1} \equiv x_j$, and $x^N \equiv x^*$. Since the objective y is monotonically increasing, it follows that $y(x_i) < y(x_j)$. In the second case, if $x_{jn} = x_{*n}$, then $x_{in} \neq x_{*n}$ and the above argument can be repeated with i and j interchanged, giving $y(x_j) < y(x_i)$ and completing the proof.

Theorem 1A shows that evaluation of a monotonically increasing objective at any pair of adjacent points immediately determines one component of the minimum point x_*. It follows that the minimum x_* can be found after N such comparisons. If this is done by comparing a base point x_i to its N adjacent neighbors, then the minimum can be found after only $N + 1$ evaluations of the objective. Since this method, the simplest imaginable, is often used as a heuristic in practice when little is known about the objective, it is interesting to know that use of this method is equivalent to assuming monotonicity, as the following converse theorem 1B proves.

Theorem 1B (converse).
If at every pair of adjacent points x_i and x_j, equal in all but the nth component, it is true that $y(x_i) < y(x_j)$ if and only if $x_{*n} = x_{in} \neq x_{jn}$, then the function y is monotonically increasing.

Proof: Let x_n $(n = 0, 1, ..., N)$ be a typical point on any meridian, with exactly n components different from the corresponding ones of x_*. Here $x_0 \equiv x_*$ and $x_n \equiv x^*$. Assume, without loss of generality, that adjacent points x_{n-1} and x_n differ only in their nth components $(n = 1, ..., N)$. Then $x_{*n} = x_{n-1} \neq x_{nn}$, and by hypothesis $y(x_{n-1}) < y(x_n)$. Thus y increases strictly monotonically along this, and hence every meridian. Therefore y satisfies the definition of a monotonically increasing function.

3. A 4 variable quality control example.

The practical implications of the monotonicity assumption are illustrated in the following example. Consider a manufacturing process combining four different raw materials. Management has the option of subjecting any or all of them to special quality control testing. The $2^4 = 16$ possible cases can be represented by four component vectors $x \equiv (x_1, x_2, x_3, x_4)$, where $x_n = 1$ if material n is tested, and $x_n = 0$ if it is not. The cases may be arranged in the lattice shown in fig. 1.

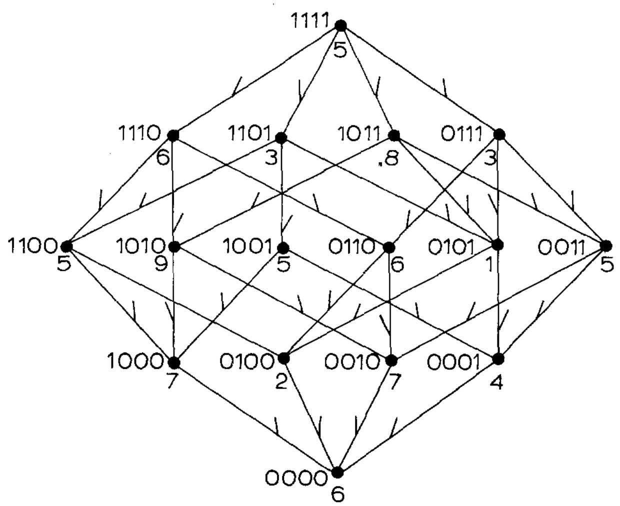

Fig. 1. A 4-variable Boolean lattice.
Values of a monotonically increasing function with minimum a 0101 are shown below each point.

It is desired to find the policy giving minimum total cost, including both the cost of testing and the cost of reprocessing product of inferior quality due to faulty raw material. Thus, if a raw material is of especially high quality, or if it has little effect on the final product, one would expect testing to be unnecessary. On the other hand, testing is justified for possibly shoddy raw materials which can throw the product off grade and cause expensive reprocessing. Thus, the monotonicity assumption seems reasonable in this case, for the larger the deviations from the optimal policy, the greater would be the cost increase. Values of total

Table 1
Search example for four variables

Case No. j	Case $x_j = (x_{j1}, x_{j2}, x_{j3}, x_{j4})$	Objective $y(x_j)$	Comparison	Conclusion
1	(0, 0, 0, 0)	6	–	–
2	(0, 0, 0, 1)	4	$y(x_2) < y(x_1)$	$x_{*4} = 1$
3	(0, 0, 1, 1)	5	$y(x_3) < y(x_2)$	$x_{*3} = 0$
4	(0, 1, 0, 1)	1	$y(x_4) < y(x_2)$	$x_{*2} = 1$
5	(1, 1, 0, 1)	3	$y(x_5) < y(x_4)$	$x_{*1} = 0$

cost for each case are given in fig. 1 to help the reader follow the example, although it is of course assumed that they can only be determined one at a time by a plant study of non-negligible cost. Table 1 shows how the search might proceed, starting from the initial operating policy (0, 0, 0, 0), i.e., no testing whatever. As the theory predicts, exactly 5 cases are examined, one of them, the fourth in this case, being the optimum.

4. Conclusions

A straight forward extension of the unimodality assumption of one variable minimization to the multivariable case leads to a surprisingly simple optimal direct minimization procedure. Moreover, the simple procedure is valid only when this multivariable regularity condition, known as "increasing monotonicity" holds. An article to follow [6] will show that similar results are true when the logic in which the variables are defined has more than two values. The procedure may find practical applications not only in unconstrained discrete optimization problems, but also perhaps in constrained pseudoboolean problems converted into unconstrained problems by the multiplier method of Hammer, Rosenberg, and Rudeanu [2, 3]. The difficulty in practice would be to prove increasing monotonicity. Since the search method is so simple, future research will study how the regularity assumption might be weakened at the expense of requiring more sophisticated search procedures.

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