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LINEAR INEQUALITIES, MATHEMATICAL PROGRAMMING AND MATRIX THEORY *

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A survey is made of solvability theory for systems of complex linear inequalities. This theory is applied to complex mathematical programming and stability and inertia theorems in matrix theory.

Introduction

This paper is a survey of solvability theory for the following systems of complex linear inequalities.

$$Tx = b,\ x \in S\,. \tag{1}$$

(Section 1, theorem 1)

$$Tx = b,\ x \in \operatorname{int} S\,. \tag{2}$$

(Section 3, theorem 3)

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$$Tx \in \operatorname{int} S_1, \quad x \in \operatorname{int} S_2 \tag{3}$$

(Section 3, theorem 4)

where $T \in C^{m \times n}$, $b \in C^m$ and S, S_1, S_2 are suitable cones.

Theorem 1 is a generalization of the Farkas lemma and of a theorem of Levinson, while theorems 3 and 4 imply generalizations of theorems of the alternative of Gordan and Stiemke respectively.

In section 2, theorem 1 is applied to derive duality theorem of complex linear programming, which generalizes the duality theorem of real linear programming and a duality theorem of Levinson.

In section 4, the solvability theory is applied to matrix spaces with suitable inner products and matrix cones. Theorem 5 is a matrix application of theorems 3 and 4. Other applications are mentioned in the remarks which conclude the paper.

0. Notations and preliminaries

C^n [R^n]	the *n dimensional complex [real] vector space*
$C^{m \times n}$ [$R^{m \times n}$]	the $m \times n$ *complex [real] matrices*
R^n_+	the *nonnegative orthant in* R^n.

For any $x, y \in C^n$:

(x, y)	the *inner product* of x and y
Re x	the *real part* of x.

For any $A \in C^{m \times n}$:

A^T	the *transpose* of A
A^H	the *conjugate transpose* of A
$R(A)$	the *range* of A.
$N(A)$	the *null space* of A.

For $A \in C^{n \times n}$:

$\operatorname{tr}(A)$	the *trace* of A
$\sigma(A)$	the *spectrum* of A
A^{-1}	the *inverse* of A.

For any $S_1, S_2 \subset C^n$:

$S_1 \times S_2$	the *cartesian product* of S_1 and S_2
int S_1	the *interior* of S_1.

A nonempty set S in C^n is
a. a *convex cone* if $S + S \subset S$ and if $\alpha \geq 0 \Rightarrow \alpha S \subset S$.
b. a *polyhedral (convex) cone* if $S = BR^k_+$ for some $B \in C^{n \times k}$.
For any nonempty set S in C^n:
$S^* = \{y \in C^n; x \in S \Rightarrow \mathrm{Re}(y, x) \geq 0\}$ is the *dual* of S.
S^* is a closed convex cone.
The interior of S^* is
$\mathrm{int}\, S^* = \{y \in S^*; 0 \neq x \in S \Rightarrow \mathrm{Re}\,(y, x) > 0\}$.
$S = S^{**}$ if and only if S is a closed convex cone, e.g. [5] theorem 1.5.

1. Linear inequalities over cones

Real linear inequalities can be represented as linear equations over convex cones. For example the system of linear inequalities

$$Au \leqq b$$

with given $A \in R^{m \times k}$ and $b \in R^m$, can be rewritten as

$$Tx = b, \ x \in S \tag{1}$$

where $T = [A, I]$ and $S = R^k \times R^m_+$.

Complex linear inequalities are systems like (1) with complex data, i.e., with $T \in C^{m \times n}$, $b \in C^m$ and S a closed convex cone in C^n.

A characterization of consistency of the (complex) system (1) is given in:

Theorem 1

Let $T \in C^{m \times n}$, $b \in C^m$ and S a closed convex cone in C^n and let TS be closed. Then the following are equivalent:
(a) $Tx = b, x \in S$ is consistent
(b) $T^H y \in S^* \Rightarrow \mathrm{Re}\,(b, y) \geq 0$.

Proof *

(a) $\Leftrightarrow b \in TS$
$\Leftrightarrow b \in (TS)^{**}$ (since TS is a closed convex cone)
$\Leftrightarrow [y \in (TS)^* \Rightarrow \mathrm{Re}\,(b, y) \geq 0]$
$\Leftrightarrow (b)$ □

* Due to R.A. Abrams.

Remarks

1. Two sufficient conditions for TS to be closed are:
   a. S is a polyhedral cone [5]
   b. $N(T) \cap S$ is a subspace [7]
2. Special cases of theorem 1 include the lemma of Farkas [11] (choosing $S = R^n_+$) and a theorem of Levinson [14].

2. Complex linear programming

Definitions

Let $A \in C^{m \times n}$, $b \in C^m$, $C \in C^n$ and let $S_1 \subset C^n$, $S_2 \subset C^m$ be polyhedral cones.

The *primal linear programming problem* is:

(P) minimize $\operatorname{Re}(c, x)$

subject to

$$Ax - b \in S_2, \; x \in S_1 \; . \tag{4}$$

The *dual linear programming problem* is:

(D) maximize $\operatorname{Re}(b, y)$

subject to

$$c - A^{\mathrm{H}} y \in S_1^*, y \in S_2^* \; .$$

A vector $x^0 \in C^n$ is

a. a *feasible solution* of (P) if x^0 satisfies (4)

b. an *optimal solution* of (P) if x^0 is feasible and $\operatorname{Re}(c, x^0) = \min \{\operatorname{Re}(c, x); x \text{ feasible}\}$.

The problem (P) is:

a. *consistent* if it has feasible solutions

b. *unbounded* if consistent and if it has feasible solutions $\{x^k; k = 1, 2 ...\}$ with $\operatorname{Re}(c, x^k) \to -\infty$ as $k \to +\infty$.

Feasible and *optimal* solutions of the dual problem (D), and the *consistency* and *boundedness* of (D), are similarly defined.

The following *duality theorem of complex linear programming* is a symmetric form of theorem 4.6 of [5].

Theorem 2

Let (P) and (D) be the problems defined above. Then exactly one case, of the following four cases, holds:
(a) Both (P) and (D) are consistent, have optimal solutions and their optimal values are equal:

$$\min \{\mathrm{Re}(c, x); Ax - b \in S_2, x \in S_1\} =$$

$$= \max \{\mathrm{Re}(b, y); c - A^{\mathrm{H}} y \in S_1^*, y \in S_2^*\}$$

(b) (P) is inconsistent, (D) is unbounded,
(c) (P) is unbounded, (D) is inconsistent,
(d) Both (P) and (D) are inconsistent.

Proof

Let (P) be inconsistent. This means that $[A, -I] \binom{x}{z} = b$, $\binom{x}{z} \in S_1 \times S_2$ is inconsistent. By theorem 1 there exists a vector y' satisfying

$$A^{\mathrm{H}} y' \in S_1^*, \quad -y' \in S_2^*, \mathrm{Re}(b, y') < 0$$

If (D) is consistent, then for any feasible solution y^0 of (D) and any nonnegative k, the vector

$$y^0 - ky'$$

is also a feasible solution of (D) and $\mathrm{Re}(b, y^0 - ky') \to \infty$ as $k \to \infty$, proving (D) unbounded.

For the case where both (P) and (D) are consistent the reader is refered to lemmas 4.4 and 4.5 of [5].

Examples showing that all the four cases mentioned in the theorem are possible are known from real mathematical programming. □

Remarks

1. Theorem 2 generalizes the *classical duality theorem of real linear programming* (where $A \in R^{m \times n}$, $b \in R^m$, $c \in R^n$, $S_1 = R_+^n$ and $S_2 = R_+^m$) and a duality theorem of Levinson [14].
2. Theorem 1 is also useful in developing a theory of complex nonlinear programming e.g. [1], [2], [3].

3. Linear inequalities over cones with interior

The consistency of the system (2) is characterized in

Theorem 3 [6]

Let $T \in C^{m \times n}$, $b \in C^m$ and let S be a closed convex cone with non-empty interior in C^n. Then the following are equivalent:
(a) $Tx = b, x \in \text{int}\, S$, is consistent
(b) $b \in R(T)$ and $0 \neq T^{\text{H}} y \in S^* \Rightarrow \text{Re}\,(b, y) > 0$.

Proof

Let E denote the manifold $\{x;\, Tx = b\}$. If E is empty then both (a) and (b) are false. Suppose, then, that $E \neq \phi$ and (a) is not true, so that $E \cap \text{int}\, S = \phi$. Then by Mazur's theorem (the geometric version of the Hahn-Banach theorem) [18] p. 69, there is a $z \neq 0$ such that

$$Tx = b \Rightarrow \text{Re}\,(x, z) = c \tag{5}$$

$$x \in \text{int}\, S \Rightarrow \text{Re}\,(x, z) > c \tag{6}$$

(5) $\Rightarrow z \in R(T^{\text{H}})$ that is $z = T^{\text{H}} y$ for some $y \neq 0$.
(6) $\Rightarrow 0 \neq z \in S^*$ and $c < 0$.

$Tx = b \Rightarrow \text{Re}(x, T^{\text{H}} y) \leq 0 \Rightarrow \text{Re}(b, y) \leq 0$ where $0 \neq T^{\text{H}} y \in S^*$ which shows that (b) is false.

Assume now that (b) is false (and $E \neq \phi$). Then there exists a y such that

$$0 \neq T^{\text{H}} y \in S^*, \;\; \text{Re}\,(b, y) \leq 0.$$

Thus $x \in E \Rightarrow \text{Re}\,(Tx, y) \leq 0 \Rightarrow \text{Re}\,(x, T^{\text{H}} y) \leq 0 \Rightarrow x \notin \text{int}\, S$.

Thus (a) is false which completes the proof. □

A consequence of theorem 3 is the following characterization of system (3).

Theorem 4 [7]

Let $T \in C^{m \times n}$ and let S_1 and S_2 be closed convex cones with non-empty interiors in C^m and C^n respectively. Then the following are equivalent:
(a) $Tx \in \text{int}\, S_1$, $x \in \text{int}\, S_2$, is consistent
(b) $-y \in S_1^*$, $T^{\text{H}} y \in S_2^* \Rightarrow y = 0$.

Proof

(a) may be rewritten as

(a′) $[T, -I] \binom{x}{z} = 0$, $\binom{x}{z} \in \operatorname{int} S_2 \times \operatorname{int} S_1 = \operatorname{int}(S_2 \times S_1)$, is consistent

(a′) is equivalent by theorem 3 to

$$(\mathrm{b}')\begin{cases} 0 \in R[T, -I] \\ \text{and} \\ 0 \neq [T^{\mathrm{H}}_{-I}]\, y \in (S_2 \times S_1)^* \Rightarrow \operatorname{Re}(0, y) > 0. \end{cases}$$

which is equivalent to (b).

Remark

Theorems 3 and 4 were used to derive complex theorems of the alternative [6], [7], which generalize classical theorems of Gordan [12] and Stiemke [19].

4. Linear inequalities in matrix theory

The Euclidean inner product in $C^{m \times n}$ is

$$(X, Y) = \operatorname{tr}(XY^{\mathrm{H}}) \tag{7}$$

(7) is reduced in $R^{m \times n}$ to $(X, Y) = \operatorname{tr}(XY^{\tau})$.

Let V be the real space of Hermitian matrices of order n.

(7) reduces in V to:

$$(X, Y) = \operatorname{tr}(XY) \tag{8}$$

Since $C^{m \times n}$ is isomorphic to C^{mn} and V is isomorphic to $R^{n(n+1)/2}$, the theorems of sections 1 and 3 may be applied to matrix spaces where one can derive interesting matrix theorems by choosing appropriate matrix operators and matrix cones.

This is demonstrated in this section by giving a new proof of characterizations of matrices whose all eigenvalues lie in the interior of the unit circle (that is, matrices C such that $C^n \to 0$).

Let PSD denote the closed convex cone in V of the positive semi definite matrices. PSD is self dual (with respect to (8)) and its interior, PD, consists of the positive definite matrices [6].

Theorem 5 (Stein–Taussky)
Let $C \in C^{n \times n}$. Define T: $V \to V$ by $T(X) = X - CXC^{H}$.
Then the following are equivalent:
(a) $|\sigma(C)| < 1$
(b) $TX = I_n$, $X \in$ PD, is consistent
(c) $TX \in PD$, $X \in$ PD, is consistent.

Proof
From (8) it follows that $T^{H} Y = Y - C^{H} YC$. (b) is equivalent, by theorem 3, to

(b′) $I_n \in R(T)$ and, $0 \neq T^{H} Y \in$ PSD $\Rightarrow$ tr $(Y) > 0$,

and (c) is equivalent by theorem 4, to

(c′) $Y \in$ PSD, $C^{H} YC - Y \in$ PSD $\Rightarrow$ $Y = 0$.

(b) $\Rightarrow$ (c) is trivial. We show (a) $\Rightarrow$ (b′) and (c′) $\Rightarrow$ (a).

(a) ⇒ (b′). $T^{H} Y = (I_n - M)Y$ where $I_n : V \to V$ is the identity operator and $M(Y) = C^{H} YC$. Let the eigenvalues of C be γ_i, $i = 1, 2, \ldots, n$. Then the eigenvalues of M are $\gamma_i \bar{\gamma}_j$, e.g. [4] p. 227, and by theorem 3.7 of [21], T^{H} is nonsingular and

$$(T^{H})^{-1} = \sum_{p=0}^{\infty} M^{p}.$$

Thus T is nonsingular and so $I_n \in R(T)$ and $T^{H} Y \in$ PSD $\Rightarrow$ $Y =$

$(T^{H})^{-1} T^{H} Y = \sum_{p=0}^{\infty} (C^{H})^{p} T^{H} YC^{p} \in$ PSD e.g. [16] p. 84. $Y \neq 0$ since $T^{H} Y \neq 0$. Thus tr $(Y) > 0$.

(c′) ⇒ (a′). Suppose (a) is false and there is a $u \neq 0$ such that $C^{H} u = \lambda u$, $|\lambda| \geq 1$. Let $Y = uu^{H}$. Then $0 \neq Y \in$ PSD and $C^{H} YC - Y = \lambda\, uu^{H} \bar{\lambda} - uu^{H} = (|\lambda|^2 - 1)\, uu^{H} \in$ PSD contradicting (c).

Remarks
1. The equivalence of (a) and (c) is a theorem of Stein [18]. More general operators of Schneider [17] and Hill [13] fit the frame of theorem 4. e.g. [17].

2. In [20] Taussky proved (a) ⇔ (b) and pointed out the connection of Stein's theorem with the theorem of Lyapunov [15] characterizing stable matrices (i.e. matrices whose spectrum lie in the open half plane Re $z < 0$). In [6] the theorem of Lyapunov is shown to be a special case of theorem 3.

3. The method demonstrated above may be applied to other matrix cones. The paper is concluded with few examples:

a. Theorem 1, the condition b following it and theorem 3 are used in [7] to prove and relax the assumptions of a well known theorem of Bellman and Fan on linear inequalities in Hermitian matrices.

b. Let K_1 and K_2 be closed convex cones in R^n and R^m respectively and denote $A \in \pi(K_1, K_2)$ if and only if $AK_1 \subset K_2$. $\pi(K_1, K_2)$ is a closed convex cone. Theorem 4 was used in [9] with such cones.

c. The problem, when does the pencil generated by two given Hermitian matrices contain a positive definite matrix, may be treated by using theorem 4. e.g. [8].

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